GEOMETRICAL EXPANSION OF AN OPERATOR EQUATION

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Abstract. Lawson-Lim [9] had given a generalization of Karcher equation and the equations determining power means. We formulate these operator equations by simpler forms, which are geometrically meaningful.

Let $\mathbb{A}(A_1, \cdots, A_n)$ be positive operators and $\omega = \{\omega_1, \cdots, \omega_n\}$ be a weight. Then the operator equation

$$0 = \sum_{i=1}^{n} \omega_i T_r(X|A_i)$$

has a unique positive solution for each $r \in [-1, 1]$, where $T_r(X|A) = \dfrac{X \natural_r A - X}{r}$ is the Tsallis relative operator entropy and $A \natural_r B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}}$ for $r \in \mathbf{R}$. We show the exact form to the unique solution of the above operator equation in the case $n = 2$.

1 Introduction.

Let $\mathbb{P}^+$ be the set of all positive invertible operators acting on a Hilbert space. Lawson-Lim [9] showed the Karcher equation for $A_1, \cdots, A_n$ $X \in \mathbb{P}^+$ and a weight $\{\omega_1, \cdots, \omega_n\}$

$$\textbf{(KE)} \qquad 0 = \sum_{i=1}^{n} \omega_i \log(X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}})$$

has a unique positive invertible solution $X = G_K(\omega_i; A_i)$ which is called (*weighted n-variable*) Karcher mean. In [2], we introduced the relative operator entropy; for $A, B \in \mathbb{P}^+$,

$$S(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = \lim_{t \to 0} \frac{A \sharp_t B - A}{t},$$

that is, $\dfrac{d}{dt} A \sharp_t B|_{t=0} = S(A|B)$, where $A \sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}},\ t \in [0, 1]$. From this viewpoint, the equation **(KE)** is represented by multiplying $X^{\frac{1}{2}}$ on both sides as follows: For $A_1, \cdots, A_n, X \in \mathbb{P}^+$

$$(*) \qquad 0 = \sum_{i=1}^{n} \omega_i S(X|A_i).$$

The path $A \sharp_t B$ $(t \in [0, 1])$ is geodesic in $\mathbb{P}^+$ under the Thompson metric, i.e., $d(A, B) = \| \log A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \|$, and $S(A|B) = \dfrac{d}{dt} A \sharp_t B \,|_{t=0}$. Therefore geometrical meaning of $(*)$ is clarified, see Figure 1.

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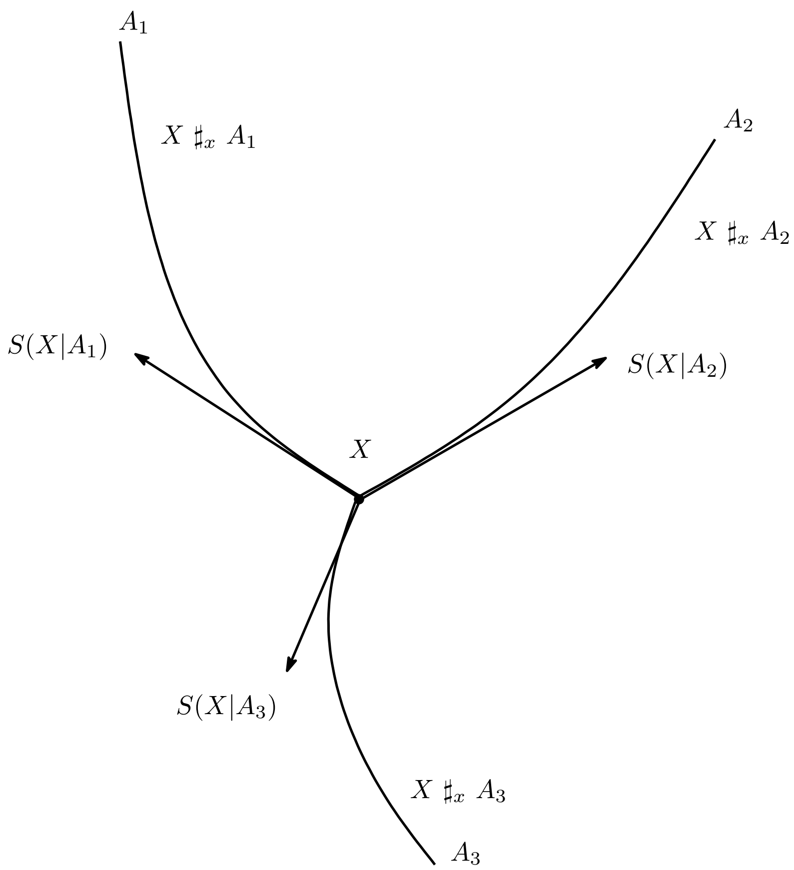


$$\omega_1 S(X|A_1) + \omega_2 S(X|A_2) + \omega_3 S(X|A_3) = 0$$

Figure 1

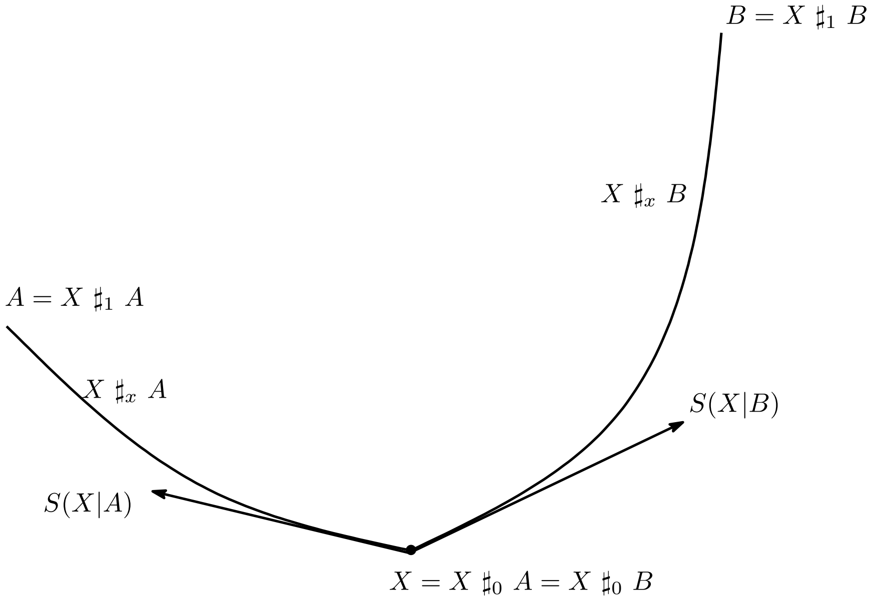
In the case $n = 2$, the equation $(*)$ is

$$(1-\alpha)S(X|A) + \alpha S(X|B) = 0, \quad \alpha \in [0,\ 1]$$

for $A,\ B,\ X \in \mathbb{P}^+$. It has the unique solution $X = A \sharp_\alpha B$, this is given in section 3 as Corollary 3 (cf. [6],[9]). Figure 2 shows if a path combining A and B is not geodesic, then X can't be a solution of the operator equation $(*)$. The Figure 3 gives an image that any X on the geodesic path combining A and B can be the solution of the operator equation $(*)$, where the notations $A \sharp_t B \supset X \sharp_y B$ and $X \sharp_x A \subset A \sharp_t B$ in Figure 3 mean that the path $A \sharp_t B$ contains the paths $X \sharp_y B$ and $X \sharp_x A$ as subdivisions of itself.
In the final section, we show that the above equation can be more generalized as

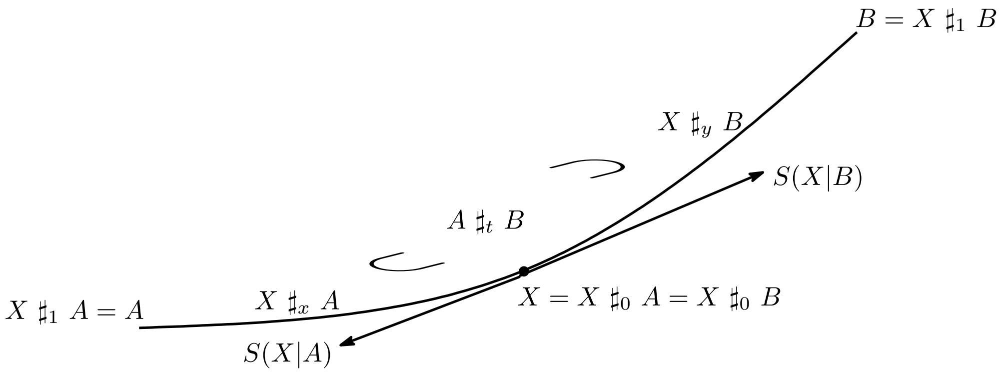
$$\alpha S(X|A) + \beta S(X|B) = 0, \quad \alpha,\ \beta \in \mathbf{R}$$

and this has the unique solution $X = A \natural_{\frac{\beta}{\alpha+\beta}} B$ also under some reasonable conditions.



$$(1-\alpha)S(X|A) + \alpha S(X|B) \neq 0$$

Figure 2



$$(1-\alpha)S(X|A) + \alpha S(X|B) = 0$$

Figure 3

In [9], Lawson-Lim defined the ω-weighted power mean $P_t(\omega : \mathbb{A})$ of order t of $\mathbb{A} = (A_1, \cdots, A_n)$ as the solution of

$$\textbf{(LLE)} \qquad X = \sum_{i=1}^{n} \omega_i (X \natural_t A_i), \quad \text{for} \quad t \in (0,\ 1],$$

for the weighted geometric operator mean [8] and $\omega = \{\omega_1, \cdots, \omega_n\}$ is a weight (cf.[10]). We show this equation has a geometrical structure.

As an approximation to the relative operator entropy, the Tsallis relative operator entropy is defined in [11] by

$$T_r(X|A) = \frac{X \natural_r A - X}{r}, \quad \text{for a fixed} \quad r \neq 0,$$

where $X \natural_r A = X^{\frac{1}{2}}(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r X^{\frac{1}{2}}$, which is not a Kubo-Ando mean but still the geodesic in the CPR-geometry [1]. We note that $S(X|A) = \lim_{r\to 0} T_r(X|A)$ (cf.[3],[4],[5]).

In section 3, we show in the following: For $\alpha \in [0,\ 1]$ and $r \in \mathbf{R}$, the equation

$$X = (1-\alpha)X \natural_r A + \alpha X \natural_r B$$

has the unique solution $X = A \sharp_{\alpha,r} B = A^{\frac{1}{2}}\{(1-\alpha)I + \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}}A^{\frac{1}{2}}$. We propose the following operator equation:

$$(**) \qquad 0 = \sum_{i=1}^{n} \omega_i T_r(X|A_i), \quad r \in \mathbf{R},$$

especially if $r = 0$, $(**)$ is understood as $(*)$. For the 2 variable equations of $(**)$ and $(*)$, we can give the solutions.

2 Expanded LL equation.

For $t \in (0,\ 1]$, the operator equation **(LLE)** is understood as

$$(***) \qquad 0 = \sum_{i=1}^{n} \omega_i T_t(X|A_i),$$

whose unique solution is just $P_t(\omega;\mathbb{A})$, the ω-weighted power mean of order t of $\mathbb{A}$.
On the other hand, for $t \in [-1,\ 0)$, we show the power mean is also the solution of $(***)$. It has a parallelism with the Karcher equation $(*)$ by the use of the relative operator entropy.

For $t \in [-1,\ 0)$, Lawson-Lim [9] give $P_t(\omega;\mathbb{A})$ as the unique positive definite solution of the operator equation

$$(\mathbf{LLE}*) \qquad X = \Big(\sum_{i=1}^{n} \omega_i (X \sharp_{-t} A_i)^{-1}\Big)^{-1} \quad \text{or} \quad X^{-1} = \sum_{i=1}^{n} \omega_i (X^{-1} \sharp_{-t} A_i^{-1}),$$

and $P_t(\omega;\mathbb{A}) = P_{-t}(\omega;\mathbb{A}^{-1})^{-1}$ where $\mathbb{A}^{-1} = (A_1^{-1},\cdots,A_n^{-1})$. It is rewritten with no use of $\sharp_{-t}$ as follows: $(\mathbf{LLE}*)$ is equivalent to

$$I = \sum_{i=1}^{n} \omega_i (X^{\frac{1}{2}}A_i^{-1}X^{\frac{1}{2}})^{-t} = \sum_{i=1}^{n} \omega_i (X^{-\frac{1}{2}}A_iX^{-\frac{1}{2}})^{t}.$$

Furthermore it is given with the use of the Tsallis relative operator entropy; for n positive operators $\mathbb{A} = (A_1,\cdots,A_n)$ and a weight $\{\omega_1,\cdots,\omega_n\}$, we can propose the operator equation $0 = \sum_{i=1}^{n} \omega_i T_r(X|A_i)$, for $r \in \mathbb{R}$. Lawson-Lim [9] teaches us this equation has a unique positive solution if $r \in [-1,\ 1]$.

Theorem 1. *Let $\mathbb{A} = (A_1,\cdots,A_n)$ be positive operators and $\{\omega_1,\cdots,\omega_n\}$ be a weight. Then for each $r \in [-1,\ 1]$, the operator equation*

$$(**) \qquad 0 = \sum_{i=1}^{n} \omega_i T_r(X|A_i)$$

has a unique positive solution.

3 On 2 variable expanded Karcher equation. Let A and B be positive invertible operators on a Hilbert space. The operator power mean we use in this note is the following:

$$A \sharp_{\alpha,r} B = A^{\frac{1}{2}}\{(1-\alpha)I + \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}}A^{\frac{1}{2}} = A \natural_{\frac{1}{r}} \{A \nabla_\alpha (A \natural_r B)\},$$

where $0 \le \alpha \le 1$ and $r \in \mathbf{R}$. We regard this as a path combining A and B for each $r \in \mathbf{R}$, and $A \sharp_{\alpha,1} B = A \nabla_\alpha B = (1-\alpha)A + \alpha B$, weighted arithmetic operator mean, $A \sharp_{\alpha,0} B = A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$, weighted geometric operator mean and $A \sharp_{\alpha,-1} B = A \Delta_\alpha B = 2(A^{-1}+B^{-1})^{-1}$, weighted harmonic operator mean.
Related to the representing function $1 \sharp_{\alpha,r} x$, we consider for a fixed $x > 0$ and $r \in \mathbb{R}$, the function $\psi(\alpha) = (1-\alpha+\alpha x^r)^{\frac{1}{r}}$, $\alpha \in [0,\ 1]$, and

$$\frac{d}{d\alpha}\psi(\alpha) = \left(1-\alpha+\alpha x^r\right)^{\frac{1}{r}-1}\frac{x^r-1}{r} = (1-\alpha+\alpha x^r)^{\frac{1}{r}}(1-\alpha+\alpha x^r)^{-1}\frac{x^r-1}{r}.$$

So we gave in [4] the relative operator entropy along this path as follows:

$$(\heartsuit) \qquad S_{\alpha,r}(A|B) = \left(A \sharp_{\alpha,r} B\right)\left(A \nabla_\alpha (A \natural_r B)\right)^{-1}T_r(A|B),$$

especially $S_{0,r}(A|B) = T_r(A|B)$.

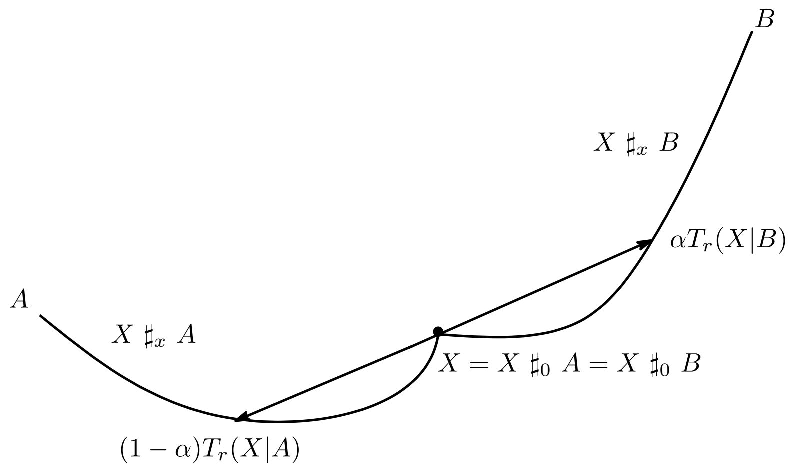
$T_r(A|B)$ has a property that for $t \in [0,\ 1]$, $tT_r(A|B) = T_r(A|A \sharp_{t,r} B)$ ([4],[5],[7]).

Theorem 2. *For $A,\ B,\ X \in \mathbb{P}^+$ and $\alpha \in [0,\ 1]$, $r \in \mathbf{R}$, the operator equation*

$$(1-\alpha)T_r(X|A) + \alpha T_r(X|B) = 0$$

has a unique solution $X = A \sharp_{\alpha,r} B$.

We give the follwing Figure 4 as an image of this theorem.



$(1-\alpha)T_r(X|A) + \alpha T_r(X|B) = 0$

Figure 4

Proof of Theorem 2. The following equivalent relations lead us to the conclusion.

$$
\begin{aligned}
& (1-\alpha)T_r(X|A) + \alpha T_r(X|B) = 0 \\
\Longleftrightarrow\quad & (1-\alpha)(X \natural_r A - X) + \alpha(X \natural_r B - X) = 0 \\
\Longleftrightarrow\quad & (1-\alpha)(X \natural_r A) + \alpha(X \natural_r B) = X \\
\Longleftrightarrow\quad & (1-\alpha)(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r + \alpha(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^r = I \\
\Longleftrightarrow\quad & \alpha(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^r = I - (1-\alpha)(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r \\
\Longleftrightarrow\quad & B = \alpha^{-\frac{1}{r}}X^{\frac{1}{2}}\big(I-(1-\alpha)(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r\big)^{\frac{1}{r}}X^{\frac{1}{2}} \\
\Longleftrightarrow\quad & A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = \alpha^{-\frac{1}{r}}A^{-\frac{1}{2}}X^{\frac{1}{2}}\big(I-(1-\alpha)(X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{-r}\big)^{\frac{1}{r}}X^{\frac{1}{2}}A^{-\frac{1}{2}} \\
& \qquad\qquad\quad \overset{(*)}{=} \alpha^{-\frac{1}{r}}\big(I-(1-\alpha)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{-r}\big)^{\frac{1}{r}}A^{-\frac{1}{2}}XA^{-\frac{1}{2}} \\
\Longleftrightarrow\quad & (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = \alpha^{-1}\big(I-(1-\alpha)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{-r}\big)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r \\
\Longleftrightarrow\quad & \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r - (1-\alpha)I \\
\Longleftrightarrow\quad & (1-\alpha)I + \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r \\
\Longleftrightarrow\quad & A^{-\frac{1}{2}}XA^{-\frac{1}{2}} = \{(1-\alpha)I + \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}} \\
\Longleftrightarrow\quad & X = A^{\frac{1}{2}}\{(1-\alpha)I + \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}}A^{\frac{1}{2}} = A \sharp_{\alpha,r} B. \ \Box
\end{aligned}
$$

The equation $\overset{(*)}{=}$ holds because $Yf(Y^*Y) = f(YY^*)Y$ for a continuous function f on an interval containing spectra of Y^*Y and YY^*.

Since $T_0(A|B) = \lim_{r\to 0} T_r(A|B) = S(A|B)$ and $A \sharp_{\alpha,0} B = \lim_{r\to 0} A \sharp_{\alpha,r} B = A \sharp_\alpha B$, we have the following if $r=0$ (cf.[6],[9]):

Corollary 3. *For $A, B, X \in \mathbb{P}^+$ and $\alpha \in [0,\ 1]$, the operator equation*

$$(1-\alpha)S(X|A) + \alpha S(X|B) = 0$$

has a unique solution $X = A \sharp_\alpha B$.

4 Generalizations of Theorem 2 and Corollary 3.

In this section, we point out that Theorem 2 has more general form.

Theorem 4. *Let $A,\ B,\ X \in \mathbb{P}^+$ and $\alpha,\ \beta \in \mathbf{R}$ such that $\alpha+\beta \neq 0$ and $\alpha\beta \neq 0$. If the operator equation*

$$\beta T_r(X|A) + \alpha T_r(X|B) = 0$$

holds, then $\frac{\beta}{\alpha+\beta}I + \frac{\alpha}{\alpha+\beta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r \geq 0$ and this equation has the unique solution

$$X = A^{\frac{1}{2}}\Big\{\frac{\beta}{\alpha+\beta}I + \frac{\alpha}{\alpha+\beta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\Big\}^{\frac{1}{r}}A^{\frac{1}{2}}.$$

Proof. The following equation $\overset{(*)}{=}$ is led by the same reason in the proof of Theorem 2.

$$\begin{aligned}
& \beta T_r(X|A) + \alpha T_r(X|B) = 0 \\
\iff\ & \beta(X \natural_r A - X) + \alpha(X \natural_r B - X) = 0 \\
\iff\ & \beta(X \natural_r A) + \alpha(X \natural_r B) = (\alpha+\beta)X \\
\iff\ & \beta(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r + \alpha(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^r = (\alpha+\beta)I \\
\iff\ & \alpha(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^r = (\alpha+\beta)I - \beta(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r \\
\iff\ & (X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^r = \frac{\alpha+\beta}{\alpha}I - \frac{\beta}{\alpha}(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r \geq 0 \\
\iff\ & B = X^{\frac{1}{2}}\big(\frac{\alpha+\beta}{\alpha}I - \frac{\beta}{\alpha}(X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^r\big)^{\frac{1}{r}}X^{\frac{1}{2}} \\
\iff\ & A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = A^{-\frac{1}{2}}X^{\frac{1}{2}}\big(\frac{\alpha+\beta}{\alpha}I - \frac{\beta}{\alpha}(X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}})^{-r}\big)^{\frac{1}{r}}X^{\frac{1}{2}}A^{-\frac{1}{2}} \\
& \qquad \overset{(*)}{=} \big(\frac{\alpha+\beta}{\alpha}I - \frac{\beta}{\alpha}(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{-r}\big)^{\frac{1}{r}}A^{-\frac{1}{2}}XA^{-\frac{1}{2}} \\
\iff\ & (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = \big(\frac{\alpha+\beta}{\alpha}I - \frac{\beta}{\alpha}(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^{-r}\big)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r \\
\iff\ & \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = (\alpha+\beta)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r - \beta I, \\
\iff\ & \beta I + \alpha(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = (\alpha+\beta)(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r, \\
\iff\ & \frac{\beta}{\alpha+\beta}I + \frac{\alpha}{\alpha+\beta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r = (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^r \geq 0 \\
\iff\ & A^{-\frac{1}{2}}XA^{-\frac{1}{2}} = \{\frac{\beta}{\alpha+\beta}I + \frac{\alpha}{\alpha+\beta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\}^{\frac{1}{r}} \\
\iff\ & X = A^{\frac{1}{2}}\Big\{\frac{\beta}{\alpha+\beta}I + \frac{\alpha}{\alpha+\beta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\Big\}^{\frac{1}{r}}A^{\frac{1}{2}}
\end{aligned}$$

The next theorem is a modification of Corollary 3, which is given as the case $r = 0$ in Theorem 4. But we here pose an independent proof of Theorem 4.

Theorem 5. *For $\alpha,\ \beta \in \mathbf{R}$ such that $\alpha+\beta \neq 0$ and $\alpha\beta \neq 0$, the operator equation*

$$\beta S(X|A) + \alpha S(X|B) = 0$$

has the unique solution $X = B \natural_{\frac{\beta}{\alpha+\beta}} A = A \natural_{\frac{\alpha}{\alpha+\beta}} B$.

We recall $rS(A|B) = S(A|A \natural_r B)$ for $r \in \mathbf{R}$, (cf.[4],[5],[7]), and prepare the next lemma.

Lemma 6. (cf.[9]) *Let $A,\ B,\ X \in \mathbb{P}^+$, then the following hold:*

$$S(X|A) + S(X|B) = 0 \quad \text{if and only if} \quad X = A \sharp B$$

Proof of Theorem 5. Since

$$\beta S(X|A) + \alpha S(X|B) = S(X|X \natural_\beta A) + S(X|X \natural_\alpha B) = 0, \quad \alpha, \beta \in \mathbf{R},$$

we have $X = (X \natural_\beta A) \sharp (X \natural_\alpha B)$ by the above lemma, which is equivalent to

$$I = (X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^\beta \sharp (X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^\alpha.$$

Since $C \sharp D = I \iff C = D^{-1}$ for $C,\ D \in \mathbb{P}^+$, we have

$$(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^\alpha = (X^{-\frac{1}{2}}AX^{-\frac{1}{2}})^{-\beta},$$

that is,

$$A = X^{\frac{1}{2}}(X^{-\frac{1}{2}}BX^{-\frac{1}{2}})^{-\frac{\alpha}{\beta}}X^{\frac{1}{2}} = X \natural_{-\frac{\alpha}{\beta}} B = B \natural_{\frac{\alpha+\beta}{\beta}} X.$$

Hence $B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = (B^{-\frac{1}{2}}XB^{-\frac{1}{2}})^{\frac{\alpha+\beta}{\beta}}$, we have $(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{\beta}{\alpha+\beta}} = B^{-\frac{1}{2}}XB^{-\frac{1}{2}}$, and

$$X = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{\beta}{\alpha+\beta}}B^{\frac{1}{2}} = B \natural_{\frac{\beta}{\alpha+\beta}} A = A \natural_{\frac{\alpha}{\alpha+\beta}} B.$$

So we have

$$\beta S(A \natural_{\frac{\alpha}{\alpha+\beta}} B|A) + \alpha S(B \natural_{\frac{\beta}{\alpha+\beta}} A|B) = 0. \qquad \square$$

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