THE STRUCTURE OF PROJECTION METHODS FOR VARIATIONAL INEQUALITY PROBLEMS AND WEAK CONVERGENCE THEOREMS

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Abstract. In this paper, we study the structure of projection methods for variational inequality problems and then prove weak convergence theorems which generalize Takahashi and Toyoda [W. Takahashi and M. Toyoda, Weak convergence theorems for nonepxansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003), 417–428] and Nadezhkina and Takahashi [N. Nadezhkina and W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 128 (2006), 191-201]. Our proofs are different from them. Furthermore, using these weak convergence theorems, we obtain some new results.

1. Introduction

Throughout this paper, we denote by R the set of real numbers and by N the set of positive integers. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a non-empty subset of H. Let T be a mapping of C into H. We denote by $F(T)$ the set of fixed points of T and by $A(T)$ the set of attractive points [23] of T, i.e.,

$$F(T) = \{u \in C : Tu = u\},$$
$$A(T) = \{u \in H : \|Tx - u\| \leq \|x - u\|, \ \forall x \in C\}.$$

A mapping $T : C \to H$ is said to be k-Lipschitz continuous if there exists $k > 0$ such that $\|Tx - Ty\| \leq k\|x - y\|$ for all $x, y \in C$. If a mapping $T : C \to H$ is 1-Lipschitz continuous, it is said to be nonexpansive, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - v\| \leq \|x - v\|$ for all $x \in C$ and $v \in F(T)$. We note that the condition $F(T) \subset A(T)$ always holds if T is quasi-nonexpansive. We denote by I the identity mapping on H. A mapping $A : C \to H$ is said to be monotone if $\langle x-y, Ax-Ay \rangle \geq 0$ for all $x, y \in C$. Let $\alpha > 0$. A mapping $A : C \to H$ is said to be α-inverse strongly monotone if $\langle x-y, Ax-Ay \rangle \geq \alpha\|Ax-Ay\|^2$ for all $x, y \in C$. It is obvious that if A is α-inverse strongly monotone, then A is monotone and $1/\alpha$-Lipschitz continuous. In the case $a \in (0, 2\alpha]$, it is known that $I - aA$ is nonexpansive. In fact, we have that for any $x, y \in C$

$$\|(I - aA)x - (I - aA)y\|^2 \leq \|x - y\|^2 - a(2\alpha - a)\|Ax - Ay\|^2;$$

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see, for instance, [21]. Assume that C is non-empty, closed and convex. In this case, for each $x \in H$, there exists a unique $x_0 \in C$ such that $\|x - x_0\| = \min\{\|x - y\| : y \in C\}$. The mapping P_C defined by $P_C x = x_0$ for $x \in H$ is called the metric projection of H onto C. Let C be a subset of a Hilbert space H and let A be a mapping of C into H. We denote by $VI(C, A)$ the set of solutions of the variational inequality for A, i.e.,

$$VI(C, A) = \{x \in C : \langle y - x, Ax\rangle \geq 0, \ \forall y \in C\}.$$

Let C be a closed and convex subset of a n-dimensional Euclidean space R^n. Let A be a monotone and k-Lipschitz continuous mapping of C into R^n with $VI(C, A) \neq \emptyset$. For $a \in (0, 1/k)$, let V_a and U_a be a self–mappings on C defined by

$$V_a x = P_C(I - aA)x, \quad U_a x = P_C(I - aAV_a)x, \quad \forall x \in C.$$

Let $x_1 \in C$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C such that $y_n = V_a x_n$ and $x_{n+1} = U_a x_n$ for all $n \in N$. This iterative procedure called the extragradient method was introduced by Korplevich [8]. Under these conditions, he proved that both sequences $\{x_n\}$ and $\{y_n\}$ converge to the same point in $VI(C, A)$. In 2003, Takahashi and Toyoda [24] proved the following theorem; also see [7].

Theorem 1.1. *Let C be a closed and convex subset of a Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H. Let $\{a_n\}$ be a sequence in $[c_1, d_1]$ as $0 < c_1 \leq d_1 < 2\alpha$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n} x = P_C(I - a_n A)x$ for all $x \in C$. Let S be a nonexpansive mapping of C into itself. Assume that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[c_2, d_2]$ as $0 < c_2 \leq d_2 < 1$. Let $x_1 \in C$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined by*

$$y_n = V_{a_n} x_n, \quad x_{n+1} = \alpha_n S V_{a_n} x_n + (1 - \alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $u \in F(S) \cap VI(C, A)$.

In 2006, Nadezhkina and Takahashi [17] also proved the following theorem.

Theorem 1.2. *Let C be a closed and convex subset of a Hilbert space H and A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{a_n\}$ be a sequence in $[c_1, d_1]$ as $0 < c_1 \leq d_1 < 1/k$. For each $n \in N$, let V_{a_n} and U_{a_n} be mappings of C into itself defined by*

$$V_{a_n} x = P_C(I - a_n A)x, \quad U_{a_n} x = P_C(I - a_n A V_{a_n})x, \quad \forall x \in C.$$

Let S be a nonexpansive mapping of C into itself. Assume that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[c_2, d_2]$ as $0 < c_2 \leq d_2 < 1$. Let $x_1 \in C$ and let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in C defined by

$$y_n = V_{a_n} x_n, \quad z_n = U_{a_n} x_n, \quad x_{n+1} = \alpha_n S U_{a_n} x_n + (1 - \alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge weakly to a point $u \in F(S) \cap VI(C, A)$.

Motivated by Takahashi and Toyoda [24] and Nadezhkina and Takahashi [17], we study properties of projection methods for variatinal inequality problems and then prove weak convergence theorems which generalize Theorems 1.1 and 1.2. Though almost all techniques in this paper are in Takahashi and Toyoda [24] and Nadezhkina and Takahashi [16, 17], our proofs are different from them. Our techniques depend on the structure of projection methods for variatinal inequality problems and our class of nonlinear mappings S in Theorems 1.1 and 1.2 is a broad class including nonexpansive mappings. Furthermore, using these weak convergence theorems, we obtain some new results.

2. Preliminaries

Let H be a Hilbert space. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. From [21] we have that for $x, y \in H$ and $\lambda \in R$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

We also know that for $x, y, u, v \in H$

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

A Hilbert space satisfies Opial's condition [18], that is,

$$\liminf_{n\to\infty} \|x_n - u\| < \liminf_{n\to\infty} \|x_n - v\|$$

if $x_n \rightharpoonup u$ and $u \neq v$; see [18]. Let C be a non-empty subset of H. A mapping $T : C \to H$ is called firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in C$. If a mapping T is firmly nonexpansive, then it is nonexpansive. If $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex; see [21]. We also know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$. This inequality is equivalent to

$$\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2 \tag{2.3}$$

for all $x \in H$ and $y \in C$; see, for instance, [20]. Recently, many researchers considered broad classes of nonlinear mappings which contain nonexpansive mappings. Kocourek, Takahashi and Yao [9] introduced a class of mappings called generalized hybrid. Let C be a non-empty subset of a Hilbert space H. Then a mapping $T : C \to H$ is called generalized hybrid if there exist $\alpha, \beta \in R$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$; see also [1]. Such a mapping T is also called (α, β)-generalized hybrid. A (1,0)-generalized hybrid mapping is nonexpansive. A (2,1)-generalized hybrid mapping is nonspread; see [10, 11]. It is also hybrid in the sense of [22] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. Suzuki [19] also introduced a new class of nonlinear mappings. A mapping T of C into itself is said to satisfy Condition (C) if for any $x, y \in C$

$$\tfrac{1}{2}\|x - Tx\| \leq \|x - y\| \quad \Rightarrow \quad \|Tx - Ty\| \leq \|x - y\|.$$

It is obvious that if T is nonexpansive, then T satisfies Condition (C). Motivated by these mappings, Takahashi and Takeuchi [23] considered a class of mappings which satisfies the following condition:

$$F(T) \subset A(T). \tag{2.4}$$

Falset, Fuster and Suzuki [6] also considered the following class of mappings: There exists $s \in [0, \infty)$ such that

$$\|x - Ty\| \leq s\|x - Tx\| + \|x - y\|, \quad \forall x, y \in C. \tag{2.5}$$

We note that a nonexpansive mapping and a mapping satisfying Condition (C) satisfy (2.5) as $s = 1$ and $s = 3$, respectively. We also note that (2.5) is stronger than (2.4). In fact, if (2.5) holds and $u \in F(T)$, then we have that $\|u - Ty\| \leq \|u - y\|$ for all $y \in C$. A mapping T is quasi-nonexpansive if T satisfies $F(T) \neq \emptyset$ and (2.4). We finally note that a generalized hybrid mapping satisfies (2.4). Let C be a non-empty subset of H and let S be a mapping of C into H. $I - S$ is called demiclosed at 0 if a sequence $\{x_n\}$ in C converges weakly to $u \in C$ and $\lim_n \|Sx_n - x_n\| = 0$, then $u \in F(S)$. The following lemma was proved by Takahashi, Wong and Yao [25].

Lemma 2.1 ([25].)**.** *Let C be a non-empty subset of a Hilbert space H and let S be a generalized hybrid mapping of C into itself. Let $\{x_n\}$ be a sequence in C which converges weakly to $u \in H$ and satisfies $\lim_n \|Sx_n - x_n\| = 0$. Then $u \in A(S)$. In addition, if C is closed and convex, then $u \in F(S)$.*

The following lemma was essentially proved in [19].

Lemma 2.2. *Let C be a closed and convex subset of a Hilbert space H and let S be a mapping of C into itself which satisfies (2.5). Let $\{x_n\}$ be a sequence in C which converges weakly to $u \in C$ and satisfies $\lim_n \|Sx_n - x_n\| = 0$. Then $u \in F(S)$.*

Proof. Assume $u \neq Su$. Since $\{x_n\}$ converges weakly to u, from the Opial property we have $\liminf_n \|x_n - u\| < \liminf_n \|x_n - Su\|$. We also have that there exists $s \in [0, \infty)$ such that

$$\|x_n - Su\| \leq s\|x_n - Sx_n\| + \|x_n - u\|, \quad \forall n \in N.$$

By $\lim_n \|Sx_n - x_n\| = 0$, this implies that $\liminf_n \|x_n - Su\| \leq \liminf_n \|x_n - u\|$. We have a contradiction. This completes the proof. □

Let C be a non-empty subset of a Hilbert space H. For a mapping A of C into H, we define the set $vi(C, A)$ as follows:

$$vi(C, A) = \{v \in C : \langle z - v, Az \rangle \geq 0, \ \forall z \in C\}.$$

From [20, Lemma 7.1.7] we have the following:

Lemma 2.3. *Let C be a convex subset of a Hilbert space H. Let A be a mapping of C into H. Then the following hold:*

1. *If A is continuous, then $vi(C, A) \subset VI(C, A)$.*
2. *If A is monotone then $\langle y - u, Ay \rangle \geq \langle y - u, Au \rangle \geq 0$ for $u \in VI(C, A)$ and $y \in C$. That is, if A is monotone then $VI(C, A) \subset vi(C, A)$.*
3. *If A is monotone and continuous, then $VI(C, A) = vi(C, A)$.*

3. Lemmas

In this section, we present some lemmas which are connected with properties of projection methods. The following lemma is well-known. For the sake of completeness, we give the proof.

Lemma 3.1. *Let C be a non-empty, closed and convex subset of a Hilbert space H. Let A be a mapping of C into H. Let $a \in (0, \infty)$ and let V_a be a mapping of C into itself defined by $V_a x = P_C(I - aA)x$ for all $x \in C$. Then $F(V_a) = VI(C, A)$.*

Proof. Let $u \in F(V_a)$. Then $u = P_C(I - aA)u$. From the property of P_C we have that for any $y \in C$

$$0 \leq \langle y - u, u - (u - aAu) \rangle = \langle y - u, aAu \rangle = a \langle y - u, Au \rangle .$$

From $a > 0$ we have that $\langle y - u, Au \rangle \geq 0$ for all $y \in C$. This implies $u \in VI(C, A)$. The reverse is similar. □

Lemma 3.2. *Let $c, k > 0$ and $\{a_n\} \subset [c, \infty)$. Let C be a non-empty, closed and convex subset of a Hilbert space H and let A be a monotone and k-Lipschitz continuous mapping of C into H with $VI(C, A) \neq \emptyset$. Let $\{V_{a_n}\}$ be a sequence of mappings on C defined by $V_{a_n} x = P_C(I - a_n A)x$ for all $x \in C$ and $n \in N$. Let $\{x_n\}$ be a bounded sequence in C. If $\lim_n \|V_{a_n} x_n - x_n\| = 0$, then the weak limit of any weakly convergent subsequence of $\{x_n\}$ is in $VI(C, A)$.*

Proof. Let $y_n = V_{a_n} x_n$ for all $n \in N$. Since $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergent subsequence. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $u \in C$. By $\lim_n \|V_{a_n} x_n - x_n\| = 0$, we also have that $\{y_{n_j}\}$ converges weakly to u. We first show $\langle z - u, Az \rangle \geq 0$ for all $z \in C$. Take $z \in C$. Since A is monotone, we have that $\left\langle z - y_{n_j}, Az - Ay_{n_j} \right\rangle \geq 0$ for all $j \in N$, that is,

$$\left\langle z - y_{n_j}, Az \right\rangle \geq \left\langle z - y_{n_j}, Ay_{n_j} \right\rangle . \tag{3.1}$$

Using $y_{n_j} = P_C(x_{n_j} - a_{n_j} Ax_{n_j})$ and $z \in C$, we also have from the property of P_C that

$$0 \geq \left\langle z - y_{n_j}, (x_{n_j} - a_{n_j} Ax_{n_j}) - y_{n_j} \right\rangle .$$

From $a_{n_j} > 0$ we have that

$$0 \geq \frac{1}{a_{n_j}} \left\langle z - y_{n_j}, x_{n_j} - y_{n_j} \right\rangle - \left\langle z - y_{n_j}, Ax_{n_j} \right\rangle . \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\left\langle z - y_{n_j}, Az \right\rangle \geq \frac{1}{a_{n_j}} \left\langle z - y_{n_j}, x_{n_j} - y_{n_j} \right\rangle + \left\langle z - y_{n_j}, Ay_{n_j} - Ax_{n_j} \right\rangle .$$

Since $1/a_{n_j} \leq 1/c$ and A is k-Lipschitz continuous, we have that

$$\left\langle z - y_{n_j}, Az \right\rangle \geq -\frac{1}{c} \|z - y_{n_j}\| \, \|x_{n_j} - y_{n_j}\| - k\|z - y_{n_j}\| \, \|y_{n_j} - x_{n_j}\|. \tag{3.3}$$

Since $\{y_{n_j}\}$ converges weakly to u, we have that $\langle z - u, Az \rangle \geq 0$. Since $z \in C$ is arbitrary, we have that $\langle z - u, Az \rangle \geq 0$ for all $z \in C$. By the continuity of A and Lemma 2.3 (1), we have $u \in VI(C, A)$. □

Remark 1. The inequality (3.3) is essential in the proof of Lemma 3.2. In the case $\lim_j a_{n_j} = 0$, we cannot prove the result. This problem appears when we deal with Halpern's type iterations with extragradient methods. We really know that there are some articles which have mathematical errors for this problem.

The following lemma plays crucial roll in the proof of Theorem 4.1.

Lemma 3.3. *Let C be a non-empty, closed and convex subset of a Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H with $VI(C, A) \neq \emptyset$. Let $\{a_n\}$ be a sequence in $[c, d]$ as $0 < c \leq d < 2\alpha$. Let $\{V_{a_n}\}$ be a sequence of mappings on C defined by $V_{a_n} x = P_C(I - a_n A)x$ for $x \in C$. If $\{x_n\}$ is a sequence in C such that $\lim_n \|x_n - u\| = \lim_n \|V_{a_n} x_n - u\|$ for some $u \in VI(C, A)$, then $\lim_n \|V_{a_n} x_n - x_n\| = 0$.*

Proof. Set $y_n = V_{a_n} x_n = P_C(I - a_n A)x_n$ for all $n \in N$. By Lemma 3.1, we have that $F(V_{a_n}) = VI(C, A)$ for $n \in N$. By our assumptions, $\{x_n\}$ and $\{y_n\}$ are bounded. Since $u \in VI(C, A)$ and A is α-inverse strongly monotone, we have

$$\begin{aligned}\|y_n - u\|^2 &= \|P_C(I - a_n A)x_n - P_C(I - a_n A)u\|^2 \\ &\leq \|(I - a_n A)x_n - (I - a_n A)u\|^2 \\ &\leq \|x_n - u\|^2 - a_n(2\alpha - a_n)\|Ax_n - Au\|^2\end{aligned}$$

for $n \in N$. From $a_n \in [c, d] \subset (0, 2\alpha)$, it follows that for $n \in N$

$$c(2\alpha - d)\|Ax_n - Au\|^2 \leq a_n(2\alpha - a_n)\|Ax_n - Au\|^2 \leq \|x_n - u\|^2 - \|y_n - u\|^2.$$

By $c(2\alpha - d) > 0$ and $\lim_n \|x_n - u\| = \lim_n \|y_n - u\|$, we have $\lim_n \|Ax_n - Au\| = 0$. Since P_C is firmly nonexpansive and $I - a_n A$ is nonexpansive, we have

$$\begin{aligned}2\|y_n - u\|^2 &= 2\|P_C(I - a_n A)x_n - P_C(I - a_n A)u\|^2 \\ &\leq 2\langle P_C(I - a_n A)x_n - P_C(I - a_n A)u, (I - a_n A)x_n - (I - a_n A)u\rangle \\ &= 2\langle y_n - u, (I - a_n A)x_n - (I - a_n A)u\rangle \\ &= \|y_n - u\|^2 + \|(I - a_n A)x_n - (I - a_n A)u\|^2 \\ &\qquad - \|(y_n - u) - ((I - a_n A)x_n - (I - a_n A)u)\|^2 \\ &\leq \|y_n - u\|^2 + \|x_n - u\|^2 \\ &\qquad - \|(y_n - x_n) + a_n(Ax_n - Au)\|^2 \\ &= \|y_n - u\|^2 + \|x_n - u\|^2 \\ &\qquad - \|y_n - x_n\|^2 - 2a_n\langle y_n - x_n, Ax_n - Au\rangle - a_n^2\|Ax_n - Au\|^2\end{aligned}$$

for all $n \in N$. Thus it follows that for $n \in N$

$$\begin{aligned}\|y_n - x_n\|^2 &\leq \|x_n - u\|^2 - \|y_n - u\|^2 \\ &\qquad - 2a_n\langle y_n - x_n, Ax_n - Au\rangle - a_n^2\|Ax_n - Au\|^2.\end{aligned}$$

By $\lim_n \|x_n - u\| = \lim_n \|y_n - u\|$ and $\lim_n \|Ax_n - Au\| = 0$, we have

$$\lim_n \|y_n - x_n\| = \lim_n \|V_{a_n} x_n - x_n\| = 0.$$

This completes the proof. □

Let $\{a_n\}$ be a sequence in $(0, \infty)$. Let C be a non-empty, closed and convex subset of a Hilbert space H. Let A be a mapping of C into H such that $VI(C, A) \neq \emptyset$. Let $\{V_{a_n}\}$ be a sequence of mappings on C defined by $V_{a_n} x = P_C(I - a_n A)x$ for all $x \in C$ and let $\{W_n\}$ be a sequence of mappings on C such that $F(W_n) \subset A(W_n)$ for all $n \in N$. Then $\{W_n\}$ said to satisfy Condition (E) with $\{V_{a_n}\}$ if there exist $M_1, M_2 > 0$ such that for any $n \in N$

(E_1) $\|W_n x - x\| \leq M_1\|V_{a_n} x - x\|, \quad \forall x \in C;$

(E_2) $\|x - V_{a_n} x\|^2 \leq M_2(\|x - u\|^2 - \|W_n x - u\|^2), \quad \forall x \in C,\ u \in VI(C, A).$

We note that $F(W_n) \subset A(W_n)$ and $F(W_n) \neq \emptyset$ if and only if W_n is quasi-nonexpansive.

Lemma 3.4. *Let $\{a_n\}$ be a sequence in $(0, \infty)$. Let C be a non-empty, closed and convex subset of a Hilbert space H. Let A be a mapping of C into H with $VI(C, A) \neq \emptyset$. Let $\{V_{a_n}\}$ be a sequence of mappings on C defined by $V_{a_n} x =$*

$P_C(I - a_n A)x$ for $x \in C$. If $\{W_n\}$ is a sequence of mappings on C which satisfies Condition (E) with $\{V_{a_n}\}$, then for each $n \in N$

$$F(V_{a_n}) = F(W_n) = VI(C, A).$$

Proof. Fix $n \in N$ arbitrarily. We already know that $F(V_{a_n}) = VI(C, A)$. Let $v \in F(V_{a_n}) = VI(C, A)$. From (E_1) we have

$$\|W_n v - v\| \le M_1 \|V_{a_n} v - v\| = 0.$$

Then $\emptyset \ne F(V_{a_n}) \subset F(W_n)$. Let $u \in VI(C, A)$ and $w \in F(W_n)$. From (E_2) we have

$$\|w - V_{a_n} w\|^2 \le M_2(\|w - u\|^2 - \|W_n w - u\|^2) = M_2(\|w - u\|^2 - \|w - u\|^2) = 0.$$

Then $F(W_n) \subset F(V_{a_n})$. Thus $F(V_{a_n}) = F(W_n) = VI(C, A)$ for all $n \in N$. □

The following lemma is a result to simplify the proof of Lemma 3.6.

Lemma 3.5. *Let C be a non-empty, closed and convex subset of a Hilbert space H. Let $k > 0$ and let A be a monotone and k-Lipschitz continuous mapping of C into H such that $VI(C, A) \ne \emptyset$. Let $a \in (0, 1/k]$. Let $x \in C$, $y = P_C(x - aAx)$, $z = P_C(x - aAy)$ and $u \in VI(C, A)$. Then the following hold:*

(1) $\langle y - z, aAy\rangle \ge \langle u - z, aAy\rangle$;
(2) $\|x - z\|^2 + 2\langle z - y, aAy\rangle \ge (1 - a^2k^2)\|x - y\|^2 + (ak\|x - y\| - \|y - z\|)^2 \ge 0$;
(3) $\|z - u\|^2 \le \|x - u\|^2 - (1 - a^2k^2)\|x - y\|^2 \le \|x - u\|^2$.

Proof. We prove (1). Let $u \in VI(C, A)$. Since A is monotone, we have

$$\langle y - u, Ay\rangle \ge \langle y - u, Au\rangle \ge 0.$$

From $a > 0$ we have that

$$\langle y - z, aAy\rangle - \langle u - z, aAy\rangle = a\langle y - u, Ay\rangle \ge a\langle y - u, Au\rangle \ge 0$$

and hence $\langle y - z, aAy\rangle \ge \langle u - z, aAy\rangle$. We prove (2). By $y = P_C(x - aAx)$ and $z \in C$, we have

$$\langle z - y, (x - aAx) - y\rangle \le 0.$$

Then the following inequality holds:

$$\begin{aligned}\langle z - y, x - y\rangle - \langle z - y, aAy\rangle &= \langle z - y, (x - aAx) - y\rangle + a\langle z - y, Ax - Ay\rangle \\ &\le a\langle z - y, Ax - Ay\rangle.\end{aligned}$$

Since A is k-Lipschitz continuous and $ak \le 1$, it follows that

$$\begin{aligned}&\|x - z\|^2 + 2\langle z - y, aAy\rangle \\ &\qquad = \big(\|x - y\|^2 + \|z - y\|^2 - 2\langle z - y, x - y\rangle\big) + 2\langle z - y, aAy\rangle \\ &\qquad \ge \|x - y\|^2 + \|z - y\|^2 - 2a\langle z - y, Ax - Ay\rangle \\ &\qquad \ge \|x - y\|^2 + \|y - z\|^2 - 2ak\|z - y\|\,\|x - y\| \\ &\qquad = (1 - a^2k^2)\|x - y\|^2 + (ak\|x - y\| - \|y - z\|)^2 \ge 0.\end{aligned}$$

We prove (3). Using $z = P_C(x - aAy)$, (1), (2) and properties of P_C, we have

$$\begin{aligned}
\|z-u\|^2 &\le \|(x-aAy)-u\|^2 - \|(x-aAy)-z\|^2 \\
&= (\|x-u\|^2 + \|aAy\|^2 - 2\langle x-u, aAy\rangle) \\
&\qquad - (\|x-z\|^2 + \|aAy\|^2 - 2\langle x-z, aAy\rangle) \\
&= \|x-u\|^2 - \|x-z\|^2 - 2\langle z-u, aAy\rangle \\
&\le \|x-u\|^2 - \|x-z\|^2 - 2\langle z-y, aAy\rangle \\
&\le \|x-u\|^2 - (1-a^2k^2)\|x-y\|^2 - (ak\|x-y\| - \|z-y\|)^2 \\
&\le \|x-u\|^2 - (1-a^2k^2)\|x-y\|^2 \le \|x-u\|^2.
\end{aligned}$$

This completes the proof. □

Lemma 3.6. *Let C be a non-empty, closed and convex subset of a Hilbert space H. Let $k > 0$ and let A be a monotone and k-Lipschitz continuous mapping of C into H such that $VI(C, A) \neq \emptyset$. Let $0 < d < 1/k$ and $\{a_n\}$ be a sequence in $(0, d]$. Let $\{V_{a_n}\}$ be a sequence of mappings on C defined by $V_{a_n}x = P_C(I - a_nA)x$ for $x \in C$ and let $\{U_{a_n}\}$ be a sequence of mappings on C defined by*

$$U_{a_n}x = P_C(I - a_nAV_{a_n})x$$

for $x \in C$. Then each U_{a_n} is a quasi-nonexpansive mapping such that $F(V_{a_n}) = F(U_{a_n}) = VI(C, A)$ and $\{U_{a_n}\}$ satisfies Condition (E) with $\{V_{a_n}\}$.

Proof. We show that $\{U_{a_n}\}$ satisfies Condition (E_1). Fix $n \in N$ arbitrarily. Since $0 < a_nk \le dk < 1$, P_C is nonexpansive and A is k-Lipschitz continuous, we have that for all $x \in C$

$$\begin{aligned}
\|U_{a_n}x - V_{a_n}x\| &= \|P_C(x - a_nAV_{a_n}x) - P_C(x - a_nAx)\| \\
&\le \|(x-x) - a_n(AV_{a_n}x - Ax)\| \le a_nk\|V_{a_n}x - x\|
\end{aligned}$$

and hence

$$\begin{aligned}
\|U_{a_n}x - x\| &\le \|U_{a_n}x - V_{a_n}x\| + \|V_{a_n}x - x\| \\
&\le a_nk\|V_{a_n}x - x\| + \|V_{a_n}x - x\| \\
&\le (1 + a_nk)\|V_{a_n}x - x\| \le 2\|V_{a_n}x - x\|.
\end{aligned}$$

This implies that $\{U_{a_n}\}$ satisfies Condition (E_1) as $M_1 = 2$. We show that $\{U_{a_n}\}$ satisfies Condition (E_2). Fix $n \in N$ arbitrarily. Let $x \in C$, $u \in VI(C, A)$ and set $y = V_{a_n}x$. By $U_{a_n}x = P_C(x - a_nAy)$ and Lemma 3.5 (3), we have

$$\|U_{a_n}x - u\|^2 \le \|x-u\|^2 - (1 - a_n^2k^2)\|x-y\|^2 \le \|x-u\|^2.$$

Thus we have that for $x \in C$ and $u \in VI(C, A)$

(*a*) $\|U_{a_n}x - u\| \le \|x - u\|$;

(*b*) $(1 - d^2k^2)\|x - V_{a_n}x\|^2 \le (1 - a_n^2k^2)\|x - V_{a_n}x\|^2 \le \|x-u\|^2 - \|U_{a_n}x - u\|^2$.

From (*b*), it follows that $\{U_{a_n}\}$ satisfies Condition (E_2) as $M_2 = 1/(1 - d^2k^2)$. We have from Lemma 3.4 that $F(V_{a_n}) = F(U_{a_n}) = VI(C, A)$ for each $n \in N$. By (*a*), each U_{a_n} is a quasi–nonexpansive mapping. This completes the proof. □

4. Main Results

We present our main results.

Theorem 4.1. *Let C be a closed and convex subset of a Hilbert space H and let $\alpha > 0$. Let A be an α-inverse strongly monotone mapping of C into H. Let $\{a_n\}$ be a sequence in $[c, d]$ as $0 < c \leq d < 2\alpha$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n}x = P_C(I - a_nA)x$ for all $x \in C$. Let S be a mapping of C into itself such that $F(S) \subset A(S)$ and $I - S$ is demiclosed at 0. Assume $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a, b]$ as $0 < a \leq b < 1$. Let $x_1 \in C$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined by*

$$y_n = V_{a_n}x_n, \qquad x_{n+1} = \alpha_n SV_{a_n}x_n + (1 - \alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $u \in F(S) \cap VI(C, A)$.

Proof. Under our assumptions, it follows that each V_{a_n} is a nonexpansive mapping such that $F(V_{a_n}) = VI(C, A) \neq \emptyset$. Since $F(S) \subset A(S)$ and $F(S) \neq \emptyset$, S is also quasi-nonexpansive. Let $w \in F(S) \cap VI(C, A)$. We have that

$$\begin{aligned}\|x_{n+1} - w\| &\leq \alpha_n\|SV_{a_n}x_n - w\| + (1 - \alpha_n)\|x_n - w\| \\ &\leq \alpha_n\|x_n - w\| + (1 - \alpha_n)\|x_n - w\| = \|x_n - w\|\end{aligned}$$

for all $n \in N$. Then $\{\|x_n - w\|\}$ is non-increasing and converges to some $s \in [0, \infty)$. It follows that $\{x_n\}$ are bounded. We also have that

$$\begin{aligned}&\alpha_n\|x_{n+1} - w\| + (1 - \alpha_n)(\|x_{n+1} - w\| - \|x_n - w\|) \\ &\qquad \leq \alpha_n\|SV_{a_n}x_n - w\| \leq \alpha_n\|V_{a_n}x_n - w\| \leq \alpha_n\|x_n - w\|.\end{aligned}$$

Since $\alpha_n \in [a, b]$ and $\|x_{n+1} - w\| - \|x_n - w\| \leq 0$, we have that

$$\|x_{n+1} - w\| + \tfrac{1}{a}(\|x_{n+1} - w\| - \|x_n - w\|) \leq \|V_{a_n}x_n - w\| \leq \|x_n - w\|$$

for all $n \in N$. This implies $\lim_n \|V_{a_n}x_n - w\| = \lim_n \|x_n - w\| = s$. We have from Lemma 3.3 that $\lim_n \|V_{a_n}x_n - x_n\| = 0$. On the other hand, we have from (2.1) that for any $x, y \in H$ and $\alpha \in R$

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

Setting $\alpha = \alpha_n$, $x = SV_{a_n}x_n - w$, $y = x_n - w$, we have that for any $n \in N$

$$\begin{aligned}&\alpha_n(1 - \alpha_n)\|SV_{a_n}x_n - x_n\|^2 \\ &\qquad = \alpha_n\|SV_{a_n}x_n - w\|^2 + (1 - \alpha_n)\|x_n - w\|^2 - \|x_{n+1} - w\|^2 \\ &\qquad \leq \alpha_n\|x_n - w\|^2 + (1 - \alpha_n)\|x_n - w\|^2 - \|x_{n+1} - w\|^2 \\ &\qquad = \|x_n - w\|^2 - \|x_{n+1} - w\|^2.\end{aligned}$$

Since $\{\|x_n - w\|\}$ is a convergent sequence and $\alpha_n \in [a, b]$ for all $n \in N$, we have that $\lim_n \|SV_{a_n}x_n - x_n\| = 0$. Moreover, since

$$\|SV_{a_n}x_n - V_{a_n}x_n\| \leq \|SV_{a_n}x_n - x_n\| + \|V_{a_n}x_n - x_n\|.$$

for all $n \in N$, we have that

$$\lim_n \|Sy_n - y_n\| = \lim_n \|SV_{a_n}x_n - V_{a_n}x_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a weakly convergent subsequence. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $u \in C$. From $\lim_n \|V_{a_n}x_n - x_n\| = 0$, $\{y_{n_j}\}$ also converges weakly to u. Since A is monotone and

$1/\alpha$-Lipschitz continuous, from $\lim_j \|V_{a_{n_j}} x_{n_j} - x_{n_j}\| = 0$ and Lemma 3.2, we have $u \in VI(C, A)$. Since $I - S$ is demi–closed at 0 and $\lim_n \|SV_{a_n} x_n - V_{a_n} x_n\| = 0$, we also have $u \in F(S)$. Thus $u \in VI(C, A) \cap F(S)$.

Finally, let us show that $\{x_n\}$ converges weakly to $u \in VI(C, A) \cap F(S)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ which converge weakly to $u, v \in VI(C, A) \cap F(S)$, respectively. To have the result, it is sufficient to show $u = v$. Assume $u \neq v$. By the Opial property, we have that

$$\begin{aligned} \lim_i \|x_{n_i} - u\| &< \lim_i \|x_{n_i} - v\| = \lim_j \|x_{n_j} - v\| \\ &< \lim_j \|x_{n_j} - u\| = \lim_i \|x_{n_i} - u\|. \end{aligned}$$

This is a contradiction. Then we have $u = v$. Therefore we have the desired result. $\square$

Theorem 4.2. *Let C be a closed and convex subset of a Hilbert space H and let $k > 0$. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{a_n\}$ be a sequence in $[c, \infty)$ as $c \in (0, \infty)$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n} x = P_C(I - a_n A)x$ for all $x \in C$. Let $\{W_n\}$ be a sequence of mappings on C with $F(W_n) \subset A(W_n)$ such that $\{W_n\}$ satisfies Condition (E) with $\{V_{a_n}\}$. Let S be a mapping of C into itself such that $F(S) \subset A(S)$ and $I - S$ is demiclosed at 0. Assume $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a, b]$ as $0 < a \leq b < 1$. Let $x_1 \in C$ and let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be sequences defined by*

$$y_n = V_{a_n} x_n, \; z_n = W_n x_n, \quad x_{n+1} = \alpha_n S W_n x_n + (1 - \alpha_n) x_n, \quad \forall n \in N.$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge weakly to a point $u \in F(S) \cap VI(C, A)$.

Proof. By Lemma 3.4, we know that W_n is quasi-nonexpansive and $F(W_n) = VI(C, A)$ for all $n \in N$. Since $F(S) \subset A(S)$ and $F(S) \cap VI(C, A) \neq \emptyset$, S is also quasi-nonexpansive. Let $w \in F(S) \cap VI(C, A)$. We have that

$$\begin{aligned} \|x_{n+1} - w\| &\leq \alpha_n \|SW_n x_n - w\| + (1 - \alpha_n)\|x_n - w\| \\ &\leq \alpha_n \|x_n - w\| + (1 - \alpha_n)\|x_n - w\| = \|x_n - w\| \end{aligned}$$

for all $n \in N$. Then $\{\|x_n - w\|\}$ is non–increasing and converges to some $s \in [0, \infty)$. Thus we have that $\{x_n\}$ are bounded. As in the proof of Theorem 4.1, we also have that

$$\begin{aligned} &\alpha_n \|x_{n+1} - w\| + (1 - \alpha_n)(\|x_{n+1} - w\| - \|x_n - w\|) \\ &\qquad \leq \alpha_n \|SW_n x_n - w\| \leq \alpha_n \|W_n x_n - w\| \leq \alpha_n \|x_n - w\|. \end{aligned}$$

Since $\alpha_n \in [a, b]$ and $\|x_{n+1} - w\| - \|x_n - w\| \leq 0$, we have

$$\|x_{n+1} - w\| + \tfrac{1}{a}(\|x_{n+1} - w\| - \|x_n - w\|) \leq \|W_n x_n - w\| \leq \|x_n - w\|$$

for all $n \in N$. This implies $\lim_n \|W_n x_n - w\| = s$. By (E_2) of Condition (E), there is $M_2 > 0$ such that

$$\|V_{a_n} x_n - x_n\|^2 \leq M_2(\|x_n - w\|^2 - \|W_n x_n - w\|^2)$$

for all $n \in N$. Since $\lim_n \|x_n - w\| = \lim_n \|W_n x_n - w\| = s$, we have that $\lim_n \|V_{a_n} x_n - x_n\| = 0$. By (E_1) of Condition (E), we also have that $\lim_n \|W_n x_n -$

$x_n\| = 0$. On the other hand, using (2.1), we have that for any $n \in N$

$$\begin{aligned}
&\alpha_n(1-\alpha_n)\|SW_nx_n - x_n\|^2\\
&\qquad = \alpha_n\|SW_nx_n - w\|^2 + (1-\alpha_n)\|x_n - w\|^2 - \|x_{n+1} - w\|^2\\
&\qquad \le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|x_n - w\|^2 - \|x_{n+1} - w\|^2\\
&\qquad = \|x_n - w\|^2 - \|x_{n+1} - w\|^2.
\end{aligned}$$

Since $\{\|x_n - w\|\}$ converges and $\alpha_n \in [a,b]$ for all $n \in \mathbb{N}$, we have $\lim_n \|SW_nx_n - x_n\| = 0$. Moreover, since

$$\|SW_nx_n - W_nx_n\| \le \|SW_nx_n - x_n\| + \|W_nx_n - x_n\|.$$

for all $n \in N$, we have that

$$\lim_n \|Sz_n - z_n\| = \lim_n \|SW_nx_n - W_nx_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a weakly convergent subsequence. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $u \in C$. By $\lim_n \|V_{a_n}x_n - x_n\| = 0$ and $\lim_n \|W_nx_n - x_n\| = 0$, we also have that $\{y_{n_j}\}$ and $\{z_{n_j}\}$ converge weakly to u. Since A is monotone and k-Lipschitz continuous, from $\lim_j \|V_{a_{n_j}}x_{n_j} - x_{n_j}\| = 0$ and Lemma 3.2, we have that $u \in VI(C,A)$. Since $I - S$ is demi–closed at 0 and $\lim_j \|SW_{n_j}x_{n_j} - W_{n_j}x_{n_j}\| = 0$, we also have $u \in F(S)$. Thus $u \in VI(C,A)\cap F(S)$. To show that $\{x_n\}$ converges weakly to a point of $VI(C,A)\cap F(S)$, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ which converge weakly to $u, v \in VI(C,A) \cap F(S)$, respectively. To have the result, it is sufficient to show $u = v$. Assume $u \ne v$. As in the proof of Theorem 4.1, we have that

$$\begin{aligned}
\lim_i \|x_{n_i} - u\| &< \lim_i \|x_{n_i} - v\| = \lim_j \|x_{n_j} - v\|\\
&< \lim_j \|x_{n_j} - u\| = \lim_i \|x_{n_i} - u\|.
\end{aligned}$$

This is a contradiction. Then we have the desired result. □

5. Applications

Using Theorems 4.1 and 4.2, we present some new results. The following are extensions of Theorem 1.1.

Theorem 5.1. *Let C be a closed and convex subset of a Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H. Let $\{a_n\}$ be a sequence in $[c,d]$ as $0 < c \le d < 2\alpha$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n}x = P_C(I - a_nA)x$ for all $x \in C$. Let S be a generalized hybrid mapping of C into itself. Assume that $F(S)\cap VI(C,A) \ne \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a,b]$ as $0 < a \le b < 1$. Let $x_1 \in C$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined by*

$$y_n = V_{a_n}x_n, \quad x_{n+1} = \alpha_nSV_{a_n}x_n + (1-\alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $u \in F(S) \cap VI(C,A)$.

Proof. Since $S : C \to C$ is generalized hybrid, S satisfies $F(S) \subset A(S)$. By Lemma 2.1 we have that $I - S$ is demiclosed at 0. Then, by Theorem 4.1, we have the desired result. □

Theorem 5.2. *Let C be a closed and convex subset of a Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H. Let $\{a_n\}$ be a sequence in $[c,d]$ as $0 < c \le d < 2\alpha$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n}x = P_C(I - a_nA)x$ for all $x \in C$. Let $S : C \to C$ be a mapping such that, for some $s \in [0,\infty)$,*

$$\|x - Ty\| \le s\|x - Tx\| + \|x - y\|, \quad \forall x, y \in C. \tag{5.1}$$

Assume that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a,b]$ as $0 < a \le b < 1$. Let $x_1 \in C$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined by

$$y_n = V_{a_n}x_n, \quad x_{n+1} = \alpha_n SV_{a_n}x_n + (1-\alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $u \in F(S) \cap VI(C,A)$.

Proof. Since S is a mapping satisfying (5.1), S satisfies $F(S) \subset A(S)$. By Lemma 2.2 we have that $I - S$ is demiclosed at 0. Then, by Theorem 4.1, we have the desired result. □

Using Theorem 5.2, we have the following result.

Theorem 5.3. *Let C be a closed and convex subset of a Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H. Let $\{a_n\}$ be a sequence in $[c,d]$ as $0 < c \le d < 2\alpha$. For each $n \in N$, let V_{a_n} be a mapping of C into itself defined by $V_{a_n}x = P_C(I - a_nA)x$ for $x \in C$. Let $S : C \to C$ be a mapping which satisfies Condition (C). Assume that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a,b]$ as $0 < a \le b < 1$. Let $x_1 \in C$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined by*

$$y_n = V_{a_n}x_n, \quad x_{n+1} = \alpha_n SV_{a_n}x_n + (1-\alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $u \in F(S) \cap VI(C,A)$.

Proof. If a mapping S satisfies Condition (C), then we know that S satisfies (5.1). Thus we obtain the desired result from Theorem 5.2. □

As in the proofs of Theorems 5.1 and 5.2 we have the following extensions of Theorem 1.2 from Lemma 3.6 and Theorem 4.2.

Theorem 5.4. *Let C be a closed and convex subset of a Hilbert space H and A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{a_n\}$ be a sequence in $[c,d]$ as $0 < c \le d < 1/k$. For each $n \in N$, let V_{a_n} and U_{a_n} be mappings of C into itself defined by*

$$V_{a_n}x = P_C(I - a_nA)x, \quad U_{a_n}x = P_C(I - a_nAV_{a_n})x, \quad \forall x \in C,$$

respectively. Let $S : C \to C$ be a generalized hybrid mapping. Assume that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[a,b]$ as $0 < a \le b < 1$. Let $x_1 \in C$ and let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be sequences defined by

$$y_n = V_{a_n}x_n, \quad z_n = U_{a_n}x_n, \quad x_{n+1} = \alpha_n SU_{a_n}x_n + (1-\alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge weakly to a point $u \in F(S) \cap VI(C,A)$.

Theorem 5.5. *Let C be a closed and convex subset of a Hilbert space H and A be a monotone and k-Lipschitz continuous mapping of C into H. Let $\{a_n\}$ be a* *sequence in* $[c, d]$ *as* $0 < c \leq d < 1/k$. *For each* $n \in N$, *let* V_{a_n} *and* U_{a_n} *be mappings of* C *into itself defined by*

$$V_{a_n}x = P_C(I - a_nA)x, \quad U_{a_n}x = P_C(I - a_nAV_{a_n})x, \quad \forall x \in C,$$

respectively. Let $S : C \to C$ *be a mapping such that, for some* $s \in [0, \infty)$,

$$\|x - Ty\| \leq s\|x - Tx\| + \|x - y\|, \quad \forall x, y \in C.$$

Assume that $F(S) \cap VI(C, A) \neq \emptyset$. *Let* $\{\alpha_n\}$ *be a sequence in* $[a, b]$ *as* $0 < a \leq b < 1$. *Let* $x_1 \in C$ *and let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *be sequences defined by*

$$y_n = V_{a_n}x_n, \quad z_n = U_{a_n}x_n, \quad x_{n+1} = \alpha_nSU_{a_n}x_n + (1 - \alpha_n)x_n, \quad \forall n \in N.$$

Then $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge weakly to a point* $u \in F(S) \cap VI(C, A)$.

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