SOME OPERATOR DIVERGENCES BASED ON PETZ-BREGMAN DIVERGENCE

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Abstract. Let A and B be strictly positive operators on a Hilbert space. For relative operator entropies $S(A|B) \equiv A^{\frac{1}{2}} \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$, $T_\alpha(A|B) \equiv \frac{1}{\alpha}(A \natural_\alpha B - A)$ and $S_\alpha(A|B) \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$, we showed

$$(*) \quad S_1(A|B) \geq -T_{1-\alpha}(B|A) \geq S_\alpha(A|B) \geq T_\alpha(A|B) \geq S(A|B) \quad \text{for } \alpha \in (0,1).$$

Petz gave an operator divergence $D_0(A|B) = B - A - S(A|B)$ which we call Petz-Bregman divergence. Petz also defined Bregman divergence $D_\Psi(X,Y)$ for an operator valued smooth function $\Psi : \mathbf{C} \to B(H)$ and $X, Y \in \mathbf{C}$, where $\mathbf{C}$ is a convex set in a Banach space.

In this paper, firstly, we define new operator divergences as the differences between two terms in $(*)$ and represent them by using $D_0(A|B)$. Secondly, we let $\mathbf{C} = \mathbb{R}$ and show $D_\Psi(x,y) = D_0(A \natural_y B|A \natural_x B)$ for $\Psi(t) = A \natural_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ and $x, y \in \mathbb{R}$. Then we have $D_\Psi(1,0) = D_0(A|B)$ in particular. Based on this interpretation, we discuss Bregman divergences $D_\Psi(1,0)$ for several functions Ψ which relate to the operator divergences defined above.

1 Introduction. Throughout this paper, a bounded linear operator T on a Hilbert space H is positive (denoted by $T \geq 0$) if $(T\xi, \xi) \geq 0$ for all $\xi \in H$, and T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

Fujii and Kamei [2] defined the following relative operator entropy for strictly positive operators A and B:

$$S(A|B) \equiv A^{\frac{1}{2}} \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

Furuta [7] defined generalized relative operator entropy as follows:

$$S_\alpha(A|B) \equiv A^{\frac{1}{2}} \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) A^{\frac{1}{2}}, \ \alpha \in \mathbb{R},$$

We know immediately $S_0(A|B) = S(A|B)$. Yanagi, Kuriyama and Furuichi [15] introduced Tsallis relative operator entropy as follows:

$$T_\alpha(A|B) \equiv \frac{A \natural_\alpha B - A}{\alpha}, \ \alpha \in (0,1],$$

where $A \natural_\alpha B \equiv A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}}$ for $\alpha \in [0,1]$ is the weighted geometric operator mean (cf. [12]). Since $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$ holds for $a > 0$, we have $T_0(A|B) \equiv \lim_{\alpha \to 0} T_\alpha(A|B) =$

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$S(A|B)$. Tsallis relative operator entropy can be extended as the notion for $\alpha \in \mathbb{R}$. In [8], we had given the following relations among these relative operator entropies:

$$(*) \qquad S_1(A|B) \geq -T_{1-\alpha}(B|A) \geq S_\alpha(A|B) \geq T_\alpha(A|B) \geq S(A|B),\ \alpha \in (0,1).$$

A path $A \natural_x B$ passing through A and B is given as follows ([3, 4, 11] etc.):

$$A \natural_x B \equiv A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^x A^{\frac{1}{2}},\ x \in \mathbb{R}.$$

We remark that $A \natural_x B = B \natural_{1-x} A$ holds for $x \in \mathbb{R}$. If $x \in [0,1]$, the path $A \natural_x B$ coincides with $A \sharp_x B$. We can regard $S_\alpha(A|B)$ as the slope of the tangent line of the path $A \natural_x B$ at $x = \alpha$ and $T_\alpha(A|B)$ as the slope of the line passing through A and $A \sharp_\alpha B$ on the path.

Fujii [1] defined an operator valued α-divergence $D_\alpha(A|B)$ for $\alpha \in (0,1)$ as follows:

$$D_\alpha(A|B) \equiv \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)},$$

where $A \nabla_\alpha B \equiv (1-\alpha)A + \alpha B$ is the weighted arithmetic operator mean. The operator valued α-divergence has the following relations at end points for interval $(0,1)$.

Theorem A ([5, 6])**.** *For strictly positive operators A and B, the following hold*:

$$\begin{aligned} D_0(A|B) &\equiv \lim_{\alpha \to +0} D_\alpha(A|B) = B - A - S(A|B), \\ D_1(A|B) &\equiv \lim_{\alpha \to 1-0} D_\alpha(A|B) = A - B - S(B|A). \end{aligned}$$

Petz [14] introduced the right hand side in the first equation in Theorem A as an operator divergence, so we call $D_0(A|B)$ *Petz-Bregman divergence*. We remark that $D_1(A|B) = D_0(B|A)$ holds. Figure 1 shows our interpretation of $D_0(A|B)$.

In [9], we represented $D_\alpha(A|B)$ as follows:

$$D_\alpha(A|B) = -T_{1-\alpha}(B|A) - T_\alpha(A|B),\ \alpha \in (0,1),$$

which is a difference between two of five terms in $(*)$. Moreover, $D_0(A|B)$ can be also represented as $D_0(A|B) = T_1(A|B) - S(A|B)$. From these facts, we regard the differences between the relative operator entropies in $(*)$ as operator divergences. In section 2, we represent these operator divergences by using Petz-Bregman divergence.

On the other hand, for an operator valued smooth function $\Psi : \mathbf{C} \to B(H)$ and $X, Y \in \mathbf{C}$, where $\mathbf{C}$ is a convex set in a Banach space, Petz [14] defined a divergence $D_\Psi(X,Y)$ as follows:

$$D_\Psi(X,Y) \equiv \Psi(X) - \Psi(Y) - \lim_{\alpha \to +0} \frac{\Psi(Y + \alpha(X-Y)) - \Psi(Y)}{\alpha}.$$

We call $D_\Psi(X,Y)$ Ψ-*Bregman divergence* of Y and X in this paper. Petz gave some examples for invertible density matrices X and Y. If $\Psi(X) = \eta(X) \equiv X \log X$ and X commutes with Y, then $D_\Psi(X,Y) = Y - X + X(\log X - \log Y)$, and if $\Psi(X) = tr\ \eta(X)$, then $D_\Psi(X,Y) = tr\ X(\log X - \log Y)$, which is the usual quantum relative entropy.

In section 3, we let $\mathbf{C} = \mathbb{R}$ and show $D_\Psi(x,y) = D_0(A \natural_y B | A \natural_x B)$ for $\Psi(t) = A \natural_t B$ and $x, y \in \mathbb{R}$. Then we have $D_\Psi(1,0) = D_0(A|B)$ in particular. Based on this interpretation, we discuss Ψ-Bregman divergences $D_\Psi(1,0)$ for several functions Ψ which relate to the operator divergences given in section 2.

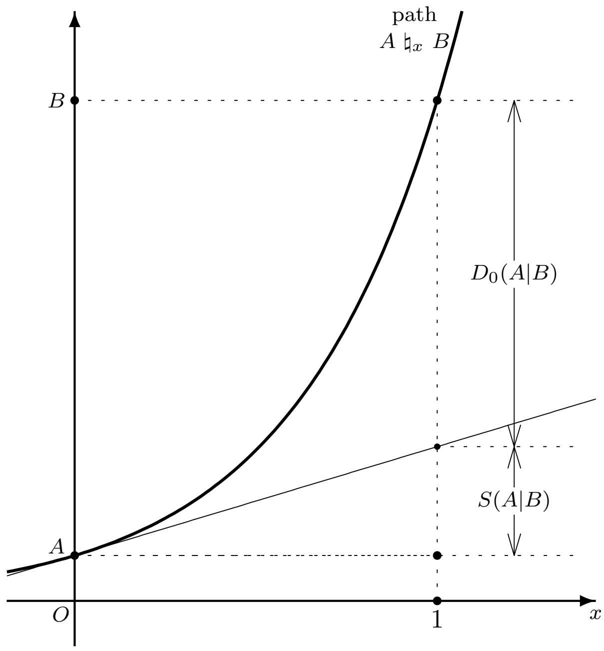

Figure 1: An interpretation of $D_0(A|B)$.

2 Divergences given by the differences among relative operator entropies. As we mentioned in section 1, we regard the differences between the relative operator entropies in $(*)$ as operator divergences. There are 10 such divergences. For convenience, we use symbols Δ_i for them as follows:

$$\begin{aligned} &\Delta_1 = T_\alpha(A|B) - S(A|B), && \Delta_2 = S_\alpha(A|B) - T_\alpha(A|B), \\ &\Delta_3 = -T_{1-\alpha}(B|A) - S_\alpha(A|B), && \Delta_4 = S_1(A|B) + T_{1-\alpha}(B|A), \\ &\Delta_5 = S_\alpha(A|B) - S(A|B), && \Delta_6 = -T_{1-\alpha}(B|A) - T_\alpha(A|B) = D_\alpha(A|B), \\ &\Delta_7 = S_1(A|B) - S_\alpha(A|B), && \Delta_8 = -T_{1-\alpha}(B|A) - S(A|B), \\ &\Delta_9 = S_1(A|B) - T_\alpha(A|B), && \Delta_{10} = S_1(A|B) - S(A|B). \end{aligned}$$

In this section, we consider a relation between each of $\Delta_1, \cdots, \Delta_{10}$ and the Petz-Bregman divergence $D_0(A|B)$. It is sufficient to consider Δ_1, Δ_2, Δ_3 and Δ_4 since the following relations hold:

$$\begin{aligned} &\Delta_5 = \Delta_1 + \Delta_2, && \Delta_6 = \Delta_2 + \Delta_3, && \Delta_7 = \Delta_3 + \Delta_4, \\ &\Delta_8 = \Delta_1 + \Delta_2 + \Delta_3, && \Delta_9 = \Delta_2 + \Delta_3 + \Delta_4, && \Delta_{10} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4. \end{aligned}$$

The order of the differences among $\Delta_1, \cdots, \Delta_{10}$ are given as in Table 1.

The next lemma is essential tools in our discussion.

Lemma 2.1 ([8, 9]). *For strictly positive operators A and B, the following hold for $s, t \in \mathbb{R}$:*

(1) $$S_t(A|A \natural_s B) = sS_{st}(A|B),$$

(2) $$S_t(A|B) = -S_{1-t}(B|A).$$

Table 1

$$
\begin{array}{ccccccccc}
S_1(A|B)-S(A|B) & \geq & S_1(A|B)-T_\alpha(A|B) & \geq & S_1(A|B)-S_\alpha(A|B) & \geq & S_1(A|B)+T_{1-\alpha}(B|A) & \geq & 0 \\
\rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & & & \\
-T_{1-\alpha}(B|A)-S(A|B) & \geq & -T_{1-\alpha}(B|A)-T_\alpha(A|B) & \geq & -T_{1-\alpha}(B|A)-S_\alpha(A|B) & \geq & 0 & & \\
\rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & & & & & \\
S_\alpha(A|B)-S(A|B) & \geq & S_\alpha(A|B)-T_\alpha(A|B) & & & & & & \\
\rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & & & & & \\
T_\alpha(A|B)-S(A|B) & & 0 & & & & & & \\
\rotatebox{-90}{$\geq$} & & & & & & & & \\
0 & & & & & & & &
\end{array}
$$

The following are results on Δ_1, Δ_2, Δ_3 and Δ_4.

Theorem 2.2. *For strictly positive operators A and B, the following hold*:

(1) $$\Delta_1 = T_\alpha(A|B)-S(A|B) = \frac{1}{\alpha}D_0(A|A \natural_\alpha B) \ \textit{for}\ \alpha\in(0,1],$$

(2) $$\Delta_2 = S_\alpha(A|B)-T_\alpha(A|B) = \frac{1}{\alpha}D_0(A \natural_\alpha B|A) \ \textit{for}\ \alpha\in(0,1],$$

(3) $$\Delta_3 = -T_{1-\alpha}(B|A)-S_\alpha(A|B) = \frac{1}{1-\alpha}D_0(A \natural_\alpha B|B) \ \textit{for}\ \alpha\in[0,1),$$

(4) $$\Delta_4 = S_1(A|B)+T_{1-\alpha}(B|A) = \frac{1}{1-\alpha}D_0(B|A \natural_\alpha B) \ \textit{for}\ \alpha\in[0,1).$$

Proof. (1) By (1) in Lemma 2.1, we have

$$
\begin{aligned}
T_\alpha(A|B)-S(A|B) &= \frac{A \natural_\alpha B - A}{\alpha} - S(A|B) = \frac{1}{\alpha}\big(A \natural_\alpha B - A - \alpha S(A|B)\big) \\
&= \frac{1}{\alpha}\big(A \natural_\alpha B - A - S(A|A \natural_\alpha B)\big) = \frac{1}{\alpha}D_0(A|A \natural_\alpha B).
\end{aligned}
$$

(2) By Lemma 2.1, we have

$$
\begin{aligned}
S_\alpha(A|B)-T_\alpha(A|B) &= \frac{A - A \natural_\alpha B}{\alpha} + S_\alpha(A|B) = \frac{1}{\alpha}\big(A - A \natural_\alpha B + \alpha S_\alpha(A|B)\big) \\
&= \frac{1}{\alpha}\big(A - A \natural_\alpha B + S_1(A|A \natural_\alpha B)\big) \\
&= \frac{1}{\alpha}\big(A - A \natural_\alpha B - S(A \natural_\alpha B|A)\big) = \frac{1}{\alpha}D_0(A \natural_\alpha B|A).
\end{aligned}
$$

(3) By Lemma 2.1 and (2) in this theorem, we have

$$
\begin{aligned}
-T_{1-\alpha}(B|A)-S_\alpha(A|B) &= -T_{1-\alpha}(B|A)+S_{1-\alpha}(B|A) = \frac{1}{1-\alpha}D_0(B \natural_{1-\alpha} A|B) \\
&= \frac{1}{1-\alpha}D_0(A \natural_\alpha B|B).
\end{aligned}
$$

(4) By (2) in Lemma 2.1 and (1) in this theorem, we have

$$
\begin{aligned}
T_{1-\alpha}(B|A) + S_1(A|B) &= T_{1-\alpha}(B|A) - S(B|A) \\
&= \frac{1}{1-\alpha} D_0(B|B \sharp_{1-\alpha} A) = \frac{1}{1-\alpha} D_0(B|A \sharp_\alpha B).
\end{aligned}
$$

□

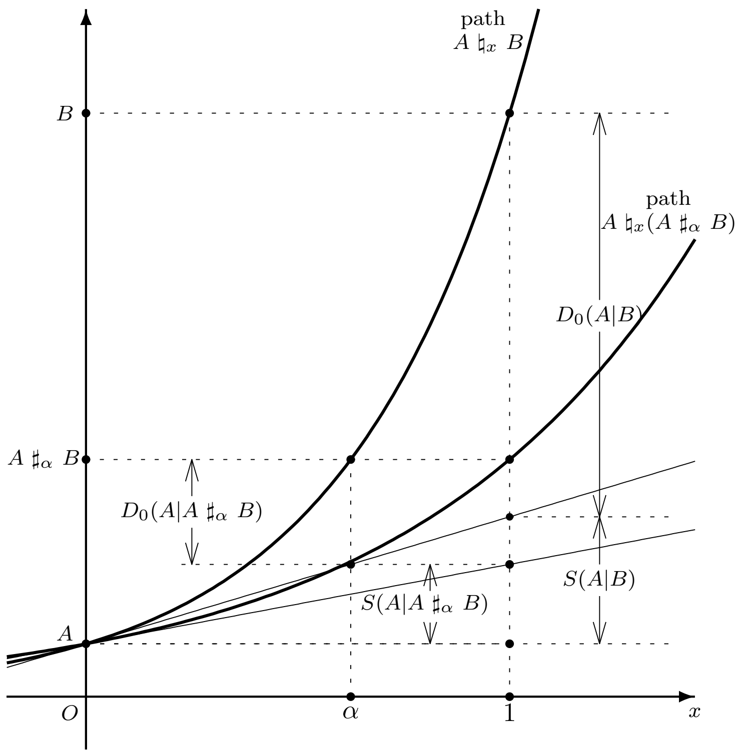

Figure 2: An interpretation of $D_0(A|A \sharp_\alpha B) = A \sharp_\alpha B - A - S(A|A \sharp_\alpha B)$.

Figure 2 shows an interpretation of $D_0(A|A \sharp_\alpha B)$ appeared in (1) in Theorem 2.2, and in Figure 3 we illustrate an interpretation of (1) and (2) in Theorem 2.2.

Theorem 2.2 leads the next theorem.

Theorem 2.3. *For strictly positive operators A and B, the following hold*:

$$
D_\alpha(A|B) = \frac{1}{1-\alpha} D_0(A \sharp_\alpha B|B) + \frac{1}{\alpha} D_0(A \sharp_\alpha B|A) \quad \textit{for } \alpha \in (0,1).
$$

Proof. By (2) and (3) in Theorem 2.2, we have

$$
\begin{aligned}
D_\alpha(A|B) &= -T_{1-\alpha}(B|A) - T_\alpha(A|B) \\
&= \big(-T_{1-\alpha}(B|A) - S_\alpha(A|B)\big) + \big(S_\alpha(A|B) - T_\alpha(A|B)\big) \\
&= \frac{1}{1-\alpha} D_0(A \sharp_\alpha B|B) + \frac{1}{\alpha} D_0(A \sharp_\alpha B|A).
\end{aligned}
$$

□

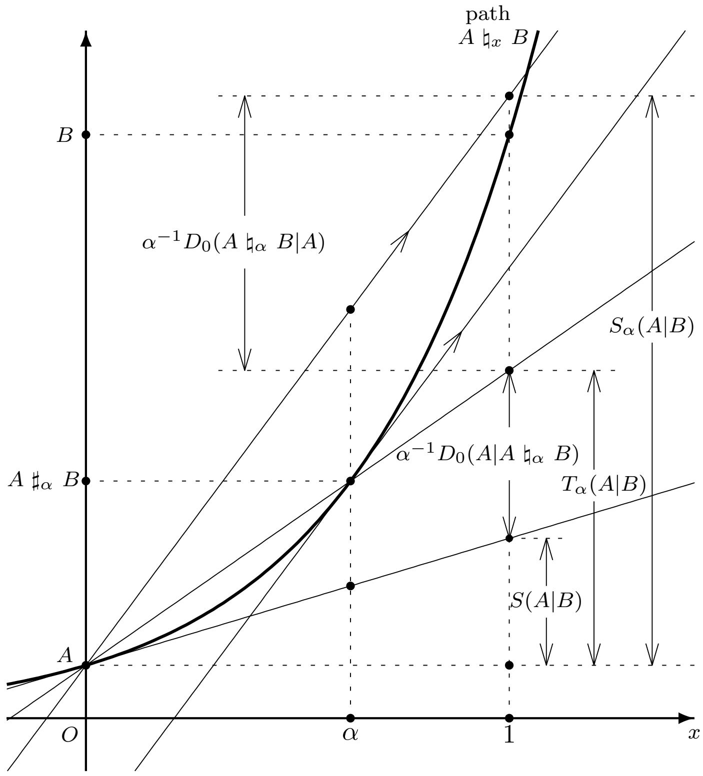

Figure 3: An interpretation of $S_\alpha(A|B) - T_\alpha(A|B) = \alpha^{-1}D_0(A \natural_\alpha B|A)$ and $T_\alpha(A|B) - S(A|B) = \alpha^{-1}D_0(A|A \natural_\alpha B)$.

By Theorem 2.3, we have

$$\begin{aligned}\alpha(1-\alpha)D_\alpha(A|B) &= \alpha D_0(A \sharp_\alpha B|B) + (1-\alpha)D_0(A \sharp_\alpha B|A)\\ &= \alpha\big(B - A \sharp_\alpha B - S(A \sharp_\alpha B|B)\big) + (1-\alpha)\big(A - A \sharp_\alpha B - S(A \sharp_\alpha B|A)\big)\\ &= A \nabla_\alpha B - A \sharp_\alpha B - \big((1-\alpha)S(A \sharp_\alpha B|A) + \alpha S(A \sharp_\alpha B|B)\big),\end{aligned}$$

and then

$$(1-\alpha)S(A \sharp_\alpha B|A) + \alpha S(A \sharp_\alpha B|B) = 0,$$

since $D_\alpha(A|B) = \dfrac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)}$. This means that $A \sharp_\alpha B$ is a solution of $(1-\alpha)S(X|A) + \alpha S(X|B) = 0$ which is called Karcher equation. For 2-variable cases, we can rewrite the result of Lawson-Lim [13] as follows:

Theorem 2.4 ([13]). *For strictly positive operators A, B and X, and for $\alpha \in [0,1]$,*

$$(1-\alpha)S(X|A) + \alpha S(X|B) = 0 \ \textit{ if and only if } X = A \sharp_\alpha B.$$

3 Ψ-Bregman divergences on the differences of relative operator entropies. In this section, we consider Ψ-Bregman divergence in the case $\mathbf{C} = \mathbb{R}$ as follows: For an operator valued smooth function $\Psi : \mathbb{R} \to B(H)$ and $x, y \in \mathbb{R}$,

$$D_\Psi(x,y) \equiv \Psi(x) - \Psi(y) - \lim_{\alpha \to +0} \frac{\Psi(y + \alpha(x-y)) - \Psi(y)}{\alpha}.$$

From the following proposition, it is natural that we consider $D_\Psi(1,0)$ as a divergence of operators A and B.

Proposition 3.1. *Let $\Psi(t) = A \natural_t B$ for strictly positive operators A and B. Then for $x, y \in \mathbb{R}$,*

$$D_\Psi(x,y) = D_0(A \natural_y B | A \natural_x B).$$

In particular, $D_\Psi(1,0) = D_0(A|B)$.

Proof.

$$\begin{aligned} D_\Psi(x,y) &= A \natural_x B - A \natural_y B - \lim_{\alpha\to+0} \frac{A \natural_{y+\alpha(x-y)} B - A \natural_y B}{\alpha} \\ &= A \natural_x B - A \natural_y B - \lim_{\alpha\to+0} \frac{(A \natural_y B) \natural_\alpha (A \natural_x B) - A \natural_y B}{\alpha} \quad \text{by [10, Lemma 2.2]} \\ &= A \natural_x B - A \natural_y B - S(A \natural_y B | A \natural_x B) = D_0(A \natural_y B | A \natural_x B). \end{aligned}$$

□

In the rest of this section, we obtain $D_\Psi(1,0)$ for functions Ψ which relate to the operator divergences Δ_1, Δ_2, Δ_5 and Δ_6 in section 2.

Theorem 3.2. *For strictly positive operators A and B, the following hold:*

(1) If $\Psi(t) = T_t(A|B) - S(A|B)$, then

$$D_\Psi(1,0) = D_0(A|B) - \frac{1}{2} S(A|B) A^{-1} S(A|B).$$

(2) If $\Psi(t) = S_t(A|B) - S(A|B)$, then

$$D_\Psi(1,0) = D_0(A|B) + D_0(B|A) - S(A|B) A^{-1} S(A|B).$$

(3) If $\Psi(t) = S_t(A|B) - T_t(A|B)$, then

$$D_\Psi(1,0) = D_0(B|A) - \frac{1}{2} S(A|B) A^{-1} S(A|B).$$

Proof. (1) For $a > 0$, we have

$$\lim_{\alpha\to+0} \frac{a^\alpha - 1 - \alpha \log a}{\alpha^2} = \frac{1}{2}(\log a)^2.$$

Replacing a by $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$, we have

$$\begin{aligned} \lim_{\alpha\to+0} \frac{T_\alpha(A|B) - S(A|B)}{\alpha} &= \lim_{\alpha\to+0} \frac{A^{\frac{1}{2}} \big((A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha - I - \alpha \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \big) A^{\frac{1}{2}}}{\alpha^2} \\ &= \frac{1}{2} A^{\frac{1}{2}} \big(\log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \big)^2 A^{\frac{1}{2}} = \frac{1}{2} S(A|B) A^{-1} S(A|B), \end{aligned}$$

then

$$\begin{aligned} D_\Psi(1,0) &= T_1(A|B) - S(A|B) - \big(T_0(A|B) - S(A|B) \big) \\ &\quad - \lim_{\alpha\to+0} \frac{T_\alpha(A|B) - S(A|B) - \big(T_0(A|B) - S(A|B) \big)}{\alpha} \\ &= T_1(A|B) - S(A|B) - \lim_{\alpha\to+0} \frac{T_\alpha(A|B) - S(A|B)}{\alpha} \\ &= D_0(A|B) - \frac{1}{2} S(A|B) A^{-1} S(A|B). \end{aligned}$$

(2) For $a > 0$, we have

$$\lim_{\alpha\to+0}\frac{a^\alpha\log a-\log a}{\alpha}=(\log a)^2.$$

Replacing a by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have

$$\begin{aligned}\lim_{\alpha\to+0}&\frac{S_\alpha(A|B)-S(A|B)}{\alpha}\\ &= \lim_{\alpha\to+0}\frac{A^{\frac{1}{2}}\big((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})-\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\big)A^{\frac{1}{2}}}{\alpha}\\ &= A^{\frac{1}{2}}\big(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\big)^2A^{\frac{1}{2}} = S(A|B)A^{-1}S(A|B),\end{aligned}$$

then by (2) in Lemma 2.1,

$$\begin{aligned}D_\Psi(1,0) &= S_1(A|B)-S(A|B)-\big(S_0(A|B)-S(A|B)\big)\\ &\quad -\lim_{\alpha\to+0}\frac{S_\alpha(A|B)-S(A|B)-\big(S_0(A|B)-S(A|B)\big)}{\alpha}\\ &= S_1(A|B)-S(A|B)-\lim_{\alpha\to+0}\frac{S_\alpha(A|B)-S(A|B)}{\alpha}\\ &= \big(B-A-S(A|B)\big)+\big(A-B-S(B|A)\big)-S(A|B)A^{-1}S(A|B)\\ &= D_0(A|B)+D_0(B|A)-S(A|B)A^{-1}S(A|B).\end{aligned}$$

(3) This relation is obtained from (1) and (2) immediately.

□

Theorem 3.3. *Let $\Psi(t) = D_t(A|B)$ for $t \in [0,1]$ and strictly positive operators A and B. Then*

$$D_\Psi(1,0) = D_0(B|A)-2D_0(A|B)+\frac{1}{2}S(A|B)A^{-1}S(A|B).$$

Proof. For $a > 0$, we have

$$\begin{aligned}\lim_{\alpha\to+0}&\frac{1-\alpha+\alpha a-a^\alpha-\alpha(1-\alpha)(a-1-\log a)}{\alpha^2(1-\alpha)}\\ &= \lim_{\alpha\to+0}\frac{-1+a-a^\alpha\log a-(1-2\alpha)(a-1-\log a)}{2\alpha-3\alpha^2}\\ &= \lim_{\alpha\to+0}\frac{-a^\alpha(\log a)^2+2(a-1-\log a)}{2-6\alpha}\\ &= -\frac{1}{2}(\log a)^2+a-1-\log a.\end{aligned}$$

Replacing a by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have

$$\begin{aligned}\lim_{\alpha\to+0}\frac{D_\alpha(A|B)-D_0(A|B)}{\alpha} &= \lim_{\alpha\to+0}\frac{\frac{A\ \nabla_\alpha\ B-A\ \sharp_\alpha\ B}{\alpha(1-\alpha)}-\big(B-A-S(A|B)\big)}{\alpha}\\ &= \lim_{\alpha\to+0}\frac{A\ \nabla_\alpha\ B-A\ \sharp_\alpha\ B-\alpha(1-\alpha)\big(B-A-S(A|B)\big)}{\alpha^2(1-\alpha)}\\ &= -\frac{1}{2}S(A|B)A^{-1}S(A|B)+D_0(A|B),\end{aligned}$$

then

$$\begin{aligned} D_\Psi(1,0) &= D_1(A|B) - D_0(A|B) - \lim_{\alpha\to+0}\frac{D_\alpha(A|B)-D_0(A|B)}{\alpha} \\ &= D_0(B|A) - 2D_0(A|B) + \frac{1}{2}S(A|B)A^{-1}S(A|B). \end{aligned}$$

□

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