DYNAMICAL SYSTEM FOR EPITAXIAL GROWTH MODEL UNDER DIRICHLET CONDITIONS

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Abstract. This paper treats the initial-boundary value problem for a semilinear parabolic equation of forth order which has been presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [8] to describe the large-scale features of a growing crystal surface under molecular beam epitaxy. In the preceding papers [4, 5, 6, 7], we have already treated the problem under the Neumann like boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\varDelta u = 0$. In this paper, we want to handle the same equation but under the Dirichlet boundary conditions $u = \frac{\partial u}{\partial n} = 0$, more natural boundary conditions than before. In the previous case, the leading linear operator $\varDelta^2$ was decomposed into the product $(-\varDelta)^2$, where $-\varDelta$ is a negative Laplace operator equipped with the usual Neumann boundary conditions and is a positive definite self-adjoint operator of L_2 space. Such a favorable decomposition is now no longer available. We have to handle a very fourth order operator $\varDelta^2$ equipped with the homogeneous Dirichlet boundary conditions.

Our goal of this paper is to construct a dynamical system generated by the initial-boundary value problem as done in [4] for the Neumann like boundary conditions.

1 Introduction We study the initial-boundary value problem for a nonlinear parabolic equation of fourth order

$$
(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = -a\Delta^2 u - \mu\nabla\cdot\left(\dfrac{\nabla u}{1+|\nabla u|^2}\right) & \text{in} \quad \Omega\times(0,\infty), \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}
$$

in a two-dimensional bounded domain Ω. Here, Ω denotes a substrate domain and the unknown function $u = u(x,t)$ denotes a displacement of surface height from the standard level at position $x \in \Omega$ and time t. And $n(x)$ denotes the outer normal vector of the boundary at boundary point $x \in \partial\Omega$.

Such a nonlinear parabolic equation has been presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [8] in order to describe the large-scale features of a growing crystal surface under molecular beam epitaxy. They pay attentions on the two main effects. One is diffusion of adatoms on the surface caused by the difference of the chemical potential proportional to the curvature of the surface. The adatoms have tendency to migrate from the positions of large curvature to those of small one. Such a current is called the surface diffusion. According to Mullins [10], a linearized surface diffusion is described by the fourth order equation $\frac{\partial u}{\partial t} \approx -a\varDelta^2 u$. The other is a uphill current of adatoms caused by step edge barriers [3, 11, 14]. The step edge barriers prevent adatoms from hopping down from the upper terraces to lower ones. As a consequence, diffusing adatoms preferably

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attach to steps from the terrace below rather than from above and non-equilibrium uphill currents are induced. Such a current is called the roughening. In the mentioned paper [8], the authors introduced as a macroscopic representation of the roughening the negative diffusion equation $\frac{\partial u}{\partial t} \approx -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$. Combining these positive and negative diffusion equations, we obtain the fourth order equation of (1.1).

In the preceding papers [4, 5, 6, 7], we used the Neumann like boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = 0$. The fourth order operator Δ^2 equipped with these homogeneous boundary conditions is then decomposed into $\Delta^2 = (-\Delta)^2$, where $-\Delta$ is the negative Laplace operator equipped with the usual Neumann boundary conditions. Although mathematical treatments are easier, the boundary conditions $\frac{\partial}{\partial n}\Delta u = 0$ seem to be somewhat artificial. In this paper, we want to impose on u the Dirichlet boundary conditions $u = \frac{\partial u}{\partial n} = 0$. Physically, this means that the surface level is always controlled to $u = 0$ on the boundary $\partial\Omega$ together with the conditions $\frac{\partial u}{\partial n} = 0$ on the normal derivatives.

We first construct a global solution for any $u_0 \in H^{-2}(\Omega)$. For this purpose, we will appeal to the general theory of abstract parabolic equations in infinite-dimensional spaces, see [9, 12, 15]. The theory is available to the higher order semilinear parabolic equations, too. We secondly construct a dynamical system generated by (1.1) in the underlying space $H^{-2}(\Omega)$. Furthermore, it is shown that the dynamical system has an exponential attractor, see [1, 13, 15]. In particular, for any initial function $u_0 \in H^{-2}(\Omega)$, the trajectory starting from u_0 admits a nonempty ω-limit set.

Throughout the paper, Ω denotes a convex or $\mathcal{C}^2$, bounded domain in $\mathbb{R}^2$. For $s \geq 0$, $H^s(\Omega)$ is the complex Sobolev space with exponent s. As usual, $H^0(\Omega) = L_2(\Omega)$. For $s > 0$, $H_0^s(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ (space of infinitely differentiable functions in Ω with compact support) in the topology $H^2(\Omega)$. We shall also use the Sobolev space $H^{-s}(\Omega) = [H_0^s(\Omega)]'$ with negative exponent $-s$. The coefficients $a > 0$ and $\mu > 0$ are fixed constants.

2 Abstract formulation In order to employ the theory of abstract parabolic equations, let us formulate (1.1) as the Cauchy problem for an abstract evolution equation. We first define a realization of the operator $a\Delta^2$ under the conditions $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. For this purpose, we consider a symmetric sesquilinear form

$$a(u,v) = a\int_\Omega \Delta u\cdot\Delta\overline{v}\,dx, \qquad u,\ v \in H_0^2(\Omega),$$

defined on $H_0^2(\Omega)$. Since $\nabla u \in H_0^1(\Omega)$ if $u \in H_0^2(\Omega)$, $u \in H_0^2(\Omega)$ satisfies $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Of course, $u \in H_0^2(\Omega)$ satisfies $u = 0$ on $\partial\Omega$. Therefore, $u \in H_0^2(\Omega)$ satisfies the homogeneous Dirichlet boundary conditions. Furthermore, as Ω is convex or of class $\mathcal{C}^2$, in either case, the elliptic estimates yield that

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L_2}, \qquad u \in H^2(\Omega)\cap H_0^1(\Omega). \tag{2.1}$$

This then implies the coercive estimate

$$a(u,u) \geq \delta\|u\|_{H^2}^2 \qquad \text{for all } u \in H_0^2(\Omega),$$

with some constant $\delta > 0$. As a consequence, we see that $a(u,v)$ determines a linear operator A from $H_0^2(\Omega)$ into $H^{-2}(\Omega)$ by the formula $a(u,v) = \langle Au, v\rangle_{H^{-2}\times H_0^2}$, see [2]. Here, $H^{-2}(\Omega)$ is the dual space of $H_0^2(\Omega)$ and these spaces compose a triplet

$$H_0^2(\Omega) \subset L_2(\Omega) \subset H^{-2}(\Omega). \tag{2.2}$$

The operator A thus defined is considered as a realization of $a\Delta^2$ under the homogeneous Dirichlet boundary conditions which is a densely defined, closed operator in $H^{-2}(\Omega)$ whose spectrum is contained in the positive real line $(0,\infty)$. (Note that the part of A in $L_2(\Omega)$ is a positive definite self-adjoint operator of $L_2(\Omega)$.)

For $0 \le \theta \le 1$, A^θ denotes the fractional power of A of exponent θ. Of course, $A^0 = I$ (identity operator on $H^{-2}(\Omega)$) and $A^1 = A$. As a general result (cf. [15, Theorem 2.35]), it follows from (2.2) that $\mathcal{D}(A^{\frac{1}{2}}) = L_2(\Omega)$ with norm equivalence. From this fact it is further deduced that, for $\frac{1}{2} \le \theta \le 1$.

$$\mathcal{D}(A^\theta) = [\mathcal{D}(A^{\frac{1}{2}}), \mathcal{D}(A)]_{2\theta-1} = [L_2(\Omega), H_0^2(\Omega)]_{2\theta-1} \subset H^{4\theta-2}(\Omega). \tag{2.3}$$

As well, (2.1) can be extended for $\frac{1}{2} \le \theta \le 1$ by

$$\|u\|_{H^{4\theta-2}} \le C\|A^{\theta-\frac{1}{2}}u\|_{L^2}, \qquad u \in \mathcal{D}(A^\theta).$$

We next define a realization of the nonlinear operator $-\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ in the framework of (2.2). Since ∇ is a bounded operator from $L_2(\Omega)$ into $H^{-1}(\Omega)$, if $\frac{\nabla u}{1+|\nabla u|^2}$ is in $L_2(\Omega)$, then we see that $\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) \in H^{-1}(\Omega) \subset H^{-2}(\Omega)$. So, it is natural to set

$$f(u) = -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right), \qquad u \in H^1(\Omega). \tag{2.4}$$

In view of (2.3), $\mathcal{D}(A^{\frac{3}{4}}) \subset H^1(\Omega)$. This shows that f is defined on the domain $\mathcal{D}(A^{\frac{3}{4}})$ and can be regarded as a subordinate operator to A.

We thus arrive at an abstract formulation of (1.1) which is written as

$$\begin{cases} \dfrac{du}{dt} + Au = f(u), & 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{2.5}$$

in the underlying space $H^{-2}(\Omega)$. It is now possible to apply the various results of the theory of semilinear abstract parabolic equations.

3 Construction of solutions We begin with constructing local solutions to (2.5) by using [15, Theorem 4.4]. To this end, it suffices to verify a suitable Lipschitz condition for $f(u)$. In fact, for u, $v \in H^1(\Omega)$,

$$\begin{aligned} \frac{\nabla u}{1+|\nabla u|^2} - \frac{\nabla v}{1+|\nabla v|^2} &= \frac{(1+|\nabla v|^2)\nabla(u-v) - (|\nabla u|^2 - |\nabla v|^2)\nabla v}{(1+|\nabla u|^2)(1+|\nabla v|^2)} \\ &= \frac{\nabla(u-v)}{1+|\nabla u|^2} - \frac{(|\nabla u| - |\nabla v|)(|\nabla u| + |\nabla v|)\nabla v}{(1+|\nabla u|^2)(1+|\nabla v|^2)}. \end{aligned}$$

Therefore,

$$\left\|\frac{\nabla u}{1+|\nabla u|^2} - \frac{\nabla v}{1+|\nabla v|^2}\right\|_{L_2} \le C\|u-v\|_{H^1}.$$

This then yields that

$$\|f(u) - f(v)\|_{H^{-1}} \le C\|A^{\frac{3}{4}}(u-v)\|_{H^{-2}}, \qquad u,\, v \in \mathcal{D}(A^{\frac{3}{4}}),$$

i.e., f fulfills [15, (4.21)] with $\eta = \frac{3}{4}$.

As a direct consequence of [15, Theorem 4.4], for any $u_0 \in H^{-2}(\Omega)$, there exists a unique local solution to (2.5) in the function space:

$$u \in \mathcal{C}([0, T_{u_0}]; H^{-2}(\Omega)) \cap \mathcal{C}^1((0, T_{u_0}]; H^{-2}(\Omega)) \cap \mathcal{C}((0, T_{u_0}]; H_0^2(\Omega)). \tag{3.1}$$

The local solution u satisfies the estimate

$$t\|u(t)\|_{H^2} + t^{\frac{3}{4}}\|u(t)\|_{H^1} + \|u(t)\|_{H^{-2}} \le C_{u_0}, \qquad 0 < t \le T_{u_0}. \tag{3.2}$$

The time $T_{u_0} > 0$ and constant C_{u_0} are determined by the norm $\|u_0\|_{H^{-2}}$ alone.

For constructing global solutions, the essential thing is to establish the *a priori* estimates for local solutions, cf. [15, Corollary 4.3]. By the smoothing effect of solutions seen by (3.1) we have $u(t) \in H^2(\Omega)$ for any $t > 0$. So, in proving the *a priori* estimates (and hence constructing a global solution to (2.5)), there is no loss of generality to assume that $u_0 \in L_2(\Omega) = \mathcal{D}(A^{\frac{1}{2}})$. Under this assumption, let u denote any local solution to (2.5) in the space:

$$u \in \mathcal{C}([0, T_u]; L_2(\Omega)) \cap \mathcal{C}^1((0, T_u]; H^{-2}(\Omega)) \cap \mathcal{C}((0, T_u]; H_0^2(\Omega)). \tag{3.3}$$

Proposition 3.1. *There exists a constant $C > 0$ such that the estimate*

$$\|u(t)\|_{L_2} \le C(\|u_0\|_{L_2} + 1), \quad 0 \le t \le T_u,$$

holds true for any local solution u lying in (3.3)*, C being independent of the interval* $[0, T_u]$.

Proof. Take a scaler product of the equation of (2.5) and $\overline{u}$. Noting that $\|u(t)\|_{L_2}^2$ is differentiable for $t > 0$ with derivative $\frac{d}{dt}\|u(t)\|_{L_2}^2 = 2\mathrm{Re}\,\langle \frac{du}{dt}(t), u(t)\rangle_{H^{-2}\times H_0^2}$ and that $\langle Au(t), u(t)\rangle_{H^{-2}\times H_0^2} = a(u(t), u(t))$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + a\int_\Omega |\Delta u|^2 dx &= \mu\int_\Omega \frac{|\nabla u|^2}{1 + |\nabla u|^2}\, dx \\ &\le \mu|\Omega|.\end{aligned}$$

By (2.1) there exists a constant $\delta > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \delta\int_\Omega |u|^2 dx \le \mu|\Omega|.$$

Solving this integral inequality, we obtain that

$$\|u(t)\|_{L^2}^2 \le e^{-2\delta t}\|u_0\|_{L^2}^2 + \mu\delta^{-1}|\Omega|, \qquad 0 \le t \le T_u.$$

□

Proposition 3.1 shows that the norm $\|u(t)\|_{L_2}$ remains uniformly bounded for any interval $[0, T_u]$. This then means that one can always extend any local solution with a uniform time interval to obtain a global solution in the space:

$$u \in \mathcal{C}([0, \infty); L_2(\Omega)) \cap \mathcal{C}^1((0, \infty); H^{-2}(\Omega)) \cap \mathcal{C}((0, \infty); H_0^2(\Omega)). \tag{3.4}$$

Of course, the global solution satisfies the similar estimate

$$\|u(t)\|_{L_2}^2 \le e^{-2\delta t}\|u_0\|_{L_2}^2 + \mu\delta^{-1}|\Omega|, \qquad 0 \le t < \infty. \tag{3.5}$$

Finally, let us remark that, if the initial function u_0 is real, then the solution $u(t)$ with $u(0) = u_0$ is also real for every time $t > 0$. In fact, we notice that the complex conjugate $\overline{u}$ of the solution u to (2.5) satisfies the same evolution equation for every t. So, $\overline{u}$ is a solution satisfying an initial condition $\overline{u(0)} = \overline{u_0}$. If u_0 is real, i.e., $u_0 = \overline{u_0}$, then uniqueness of solution implies $u(t) = \overline{u(t)}$ and $u(t)$ must be real for every t.

4 Dynamical systems The next step is to observe that the problem (2.5) generates a dynamical system. For this purpose, we can again follow the general procedure for semilinear abstract parabolic equations, see [15, Section 6.5].

For $u_0 \in H^{-2}(\Omega)$, let $u(t; u_0)$ denote the global solution of (2.5), and set

$$S(t)u_0 = u(t; u_0), \qquad 0 \le t < \infty.$$

Then, $S(t)$ is a nonlinear semigroup acting on $H^{-2}(\Omega)$, i.e., $S(0) = I$ and $S(t+s) = S(t)S(s)$ for $0 \le s,\, t < \infty$. Furthermore, $S(t)$ is seen to be continuous in the sense that $(t, u_0) \mapsto S(t)u_0$ is continuous from $[0, \infty) \times H^{-2}(\Omega)$ into $H^{-2}(\Omega)$. Whence, $S(t)$ defines a dynamical system in $H^{-2}(\Omega)$ which is denoted by $(S(t), H^{-2}(\Omega))$.

We can see from the dissipative estimate (3.5) that $(S(t), H^{-2}(\Omega))$ has an exponential attractor. Remember that a set $\mathcal{M}$ satisfying the following conditions is called the exponential attractor:

1. $\mathcal{M}$ is a compact subset of $H^{-2}(\Omega)$ with finite fractal dimension.
2. $\mathcal{M}$ is a positively invariant set of $S(t)$, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for any $0 < t < \infty$.
3. There exists an exponent $k > 0$ such that, for any bounded subset B of $H^{-2}(\Omega)$, it holds true that
$$h(S(t)B, \mathcal{M}) \le C_B e^{-kt}, \qquad 0 < t < \infty,$$
with a constant $C_B > 0$.

Here, $h(B_1, B_2) = \sup_{f \in B_1} \inf_{g \in B_2} \|f - g\|_{H^{-2}}$ is a semi-distance of two bounded subsets B_1 and B_2.

As explained in [15, Section 6.4], the compact smoothing property

$$\|S(t^*)u_0 - S(t^*)v_0\|_{L_2} \le C\|u_0 - v_0\|_{H^{-2}}, \qquad u_0,\, v_0 \in \mathcal{B}, \tag{4.1}$$

of $S(t)$ provides existence of exponential attractors, where $\mathcal{B}$ is an attractive, positively invariant, compact subset of $H^{-2}(\Omega)$ and where $t^* > 0$ is any fixed time. But, this property is also easily verified from the known estimates (3.2) and (3.5). In fact, let B be any bounded subset of $H^{-2}(\Omega)$. Then, it follows from (3.2) that there exist a bounded ball $B_{2,B}$ of $L_2(\Omega)$ and time $t_B > 0$ both depending on B such that $S(T_B)B \subset B_{2,B}$. In addition, (3.5) yields that, for any $u_0 \in B$,

$$\|S(t)u_0\|_{L_2}^2 = \|S(t - T_B)S(T_B)u_0\|_{L_2}^2 \le e^{-2\delta(t - T_B)}R_{2,B} + \mu\delta^{-1}|\Omega|, \qquad \forall t \ge T_B,$$

where $R_{2,B}$ is the radius of $B_{2,B}$. This shows that the ball $B(0; \sqrt{1 + \mu\delta^{-1}|\Omega|})$ of $L_2(\Omega)$ is an absorbing set. Let $\mathcal{B}$ be the collection of all trajectories starting from this ball. Obviously, $\mathcal{B}$ is an absorbing and invariant set; moreover, since $\mathcal{B}$ is a bounded subset of $L_2(\Omega)$, it is a compact set of $H^{-2}(\Omega)$. Finally, the desired Lipschitz condition (4.1) can be verified by using the standard techniques described in [15, Subsection 6.5.3]. In this way, we verify that our dynamical system admits an exponential attractor.

Finally, let us notice that $S(t)$ defines a dynamical system even in the space $L_2(\Omega)$ and the restricted dynamical system denoted by $(S(t), L_2(\Omega))$ also admits an exponential attractor. In fact, as seen in (3.4), $S(t)$ maps $L_2(\Omega)$ into itself. In addition, it is proved that $S(t)$ is continuous from $L_2(\Omega)$ into itself. Therefore, (2.5) generates a dynamical system in $L_2(\Omega)$, too. Furthermore, the exponential attractor $\mathcal{M}$ in $H^{-2}(\Omega)$ constructed above is obviously a bounded subset of $\mathcal{D}(A)$ $(= H_0^2(\Omega))$, and remains to be an exponential attractor of $(S(t), L_2(\Omega))$.

5 Lyapunov function Multiply the equation of (1.1) by $-\frac{\partial \overline{u}}{\partial t}$ and integrate the product in Ω. By somewhat formal computations, its real part is given by

$$
\begin{aligned}
-\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx &= a\mathrm{Re}\int_\Omega \Delta u \frac{\partial}{\partial t}\Delta \overline{u}\,dx - \mu \mathrm{Re}\int_\Omega \left[\frac{\nabla u}{1+|\nabla u|^2}\right]\cdot \frac{\partial}{\partial t}\nabla \overline{u}\,dx \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega \big[a|\Delta u|^2 - \mu\log(1+|\nabla u|^2)\big]dx.
\end{aligned}
$$

These computations then suggest that the functional

$$
\Phi(u) = \frac{1}{2}\int_\Omega \big[a|\Delta u|^2 - \mu\log(1+|\nabla u|^2)\big]dx, \qquad u\in H_0^2(\Omega), \tag{5.1}
$$

becomes a Lyapunov function of the dynamical system.

In order to justify this, however, we need a higher regularity of solution u than the known (3.4). Let $u_0 \in H_0^2(\Omega)$ and assume that the solution to (2.5) belongs to

$$
u \in \mathcal{C}^1((0,\infty); L_2(\Omega)) \quad \text{and} \quad \Delta^2 u \in \mathcal{C}((0,\infty); L_2(\Omega)). \tag{5.2}
$$

It is clear that

$$
\begin{aligned}
&\|\Delta u(t+h)\|_{L_2}^2 - \|\Delta u(t)\|_{L_2}^2 \\
&\qquad = (\Delta[u(t+h)-u(t)], \Delta u(t+h)) + (\Delta u(t), \Delta[u(t+h)-u(t)]) \\
&\qquad = (u(t+h)-u(t), \Delta^2 u(t+h)) + (\Delta^2 u(t), u(t+h)-u(t)).
\end{aligned}
$$

In view of (5.2), it is observed that

$$
\begin{aligned}
\frac{d}{dt}\|\Delta u(t)\|_{L_2}^2 &= \left(\frac{du}{dt}(t), \Delta^2 u(t)\right) + \left(\Delta^2 u(t), \frac{du}{dt}(t)\right) \\
&= 2\mathrm{Re}\left(\Delta^2 u(t), \frac{du}{dt}(t)\right).
\end{aligned}
$$

In the meantime, for u, $v\in H_0^2(\Omega)$, consider

$$
\int_\Omega [\log(1+|\nabla v|^2) - \log(1+|\nabla u|^2)]dx.
$$

For a.e. $x\in\Omega$, we have

$$
\begin{aligned}
\log[1+|\nabla v(x)|^2] - \log[1+|\nabla u(x)|^2] &= \int_0^1 \frac{d}{d\theta}\log\{1+|\nabla[\theta v(x)+(1-\theta)u(x)]|^2\}d\theta \\
&= \int_0^1 \frac{2\mathrm{Re}\nabla[v(x)-u(x)]\cdot\nabla\overline{u}(x) + 2\theta|\nabla[v(x)-u(x)]|^2}{1+|\nabla[\theta v(x)+(1-\theta)u(x)]|^2}d\theta.
\end{aligned}
$$

Moreover, since

$$
\begin{aligned}
\frac{1}{1+|\nabla[\theta v(x)+(1-\theta)u(x)]|^2} &= \frac{1}{1+|\nabla u(x)|^2} \\
&\quad - \frac{2\theta\mathrm{Re}\nabla[v(x)-u(x)]\cdot\nabla\overline{u}(x) + \theta^2|\nabla[v(x)-u(x)]|^2}{\{1+|\nabla[\theta v(x)+(1-\theta)u(x)]|^2\}(1+|\nabla u(x)|^2)},
\end{aligned}
$$

we have

$$\left|\log[1+|\nabla v(x)|^2] - \log[1+|\nabla u(x)|^2] - \frac{2\mathrm{Re}\nabla[v(x)-u(x)]\cdot\nabla\overline{u}(x)}{1+|\nabla u(x)|^2}\right| \\ \le C\{|\nabla[v(x)-u(x)]|^2 + |\nabla[v(x)-u(x)]|^4\}.$$

Therefore, integration in Ω yields that

$$\left|\int_\Omega \left[\log(1+|\nabla v|^2] - \log(1+|\nabla u|^2) - \frac{2\mathrm{Re}\nabla[v-u]\cdot\nabla\overline{u}}{1+|\nabla u|^2}\right] dx\right| \\ \le C\{\|\nabla(v-u)\|_{L_2}^2 + \|\nabla(v-u)\|_{L_4}^4\}.$$

We here use Galiardo-Nireberg's inequality ([15, Theorem 1.37]) to obtain that

$$\|\nabla(v-u)\|_{L_4} \le C\|\nabla(v-u)\|_{L_2}^{\frac{1}{2}}\|\nabla(v-u)\|_{H^1}^{\frac{1}{2}} \le C\|v-u\|_{H^1}^{\frac{1}{2}}\|v-u\|_{H^2}^{\frac{1}{2}} \\ \le C\|v-u\|_{L_2}^{\frac{1}{4}}\|v-u\|_{H^2}^{\frac{3}{4}}.$$

Then,

$$\left|\int_\Omega \left\{\log(1+|\nabla v|^2] - \log(1+|\nabla u|^2) + 2\mathrm{Re}\left[\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)(\overline{v}-\overline{u})\right]\right\} dx\right| \\ \le C\|v-u\|_{L_2}(\|v-u\|_{H^2} + \|v-u\|_{H^2}^3).$$

Let us apply this estimate with $v = u(t+h)$ and $u = u(t)$, where u is the solution mentioned above. Then, since $\|u(t+h)-u(t)\|_{H^2} \to 0$ as $h \to 0$, it is easily verified that

$$\frac{d}{dt}\int_\Omega \log[1+|\nabla u(t)|^2]dx = -2\mathrm{Re}\int_\Omega \nabla\cdot\left(\frac{\nabla u(t)}{1+|\nabla u(t)|^2}\right)\frac{d\overline{u}}{dt}(t)\,dx.$$

We have thus proved that, for any solution lying in (5.2), the function $\Phi(u(t))$ is differentiable with derivative

$$\frac{d}{dt}\Phi(u(t)) = -\left\|\frac{du}{dt}(t)\right\|_{L_2}^2, \qquad 0 < t < \infty. \tag{5.3}$$

6 Numerical Results We shall conclude this paper with illustrating some numerical examples. Let us consider (1.1) in the square domain $\Omega = (0,1)\times(0,1)$. The coefficient a is fixed as $a = 1$ but $\mu > 0$ is treated as a control parameter. The initial function is taken as

$$u_0(x_1,x_2) = 0.1[\sin(2\cdot 3.14x_1)\times\sin(2\cdot 3.14x_2)], \qquad (x_1,x_2)\in\Omega,$$

which is a perturbation of the null solution $u \equiv 0$. Clearly, the null solution is a unique homogeneous stationary solution.

Set first $\mu = 12$. As seen by Figure 1, the solution tends to the null solution as $t \to \infty$. The graph of Lyapunov function along this trajectory is given by Figure 2.

Take next $\mu = 13$. As seen by Figure 3, the solution no longer tends to the null solution. Instead, the small perturbation grows into two columns of ridges. One can count in each column 12 ridges. The graph of Lyapunov function along the trajectory is given by Figure 4.

Finally, take a sufficiently large μ, say $\mu = 40$. As seen by Figure 5, the perturbation again grows into two columns of ridges. The number of ridges in a column increases more than in the case of $\mu = 13$. As before, the Lyapunov function is monotone decreasing along the trajectory, see Figure 6.

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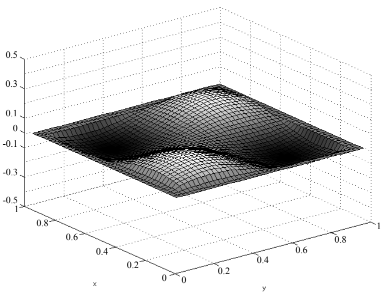

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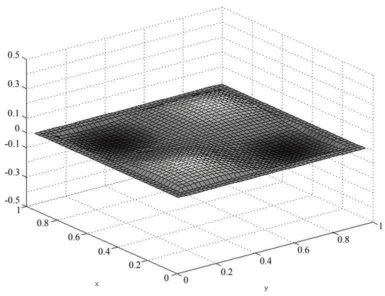

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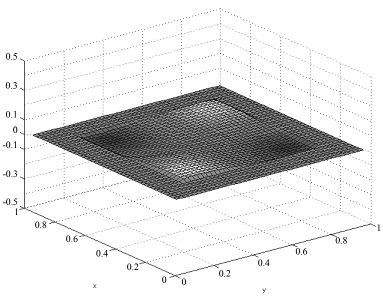
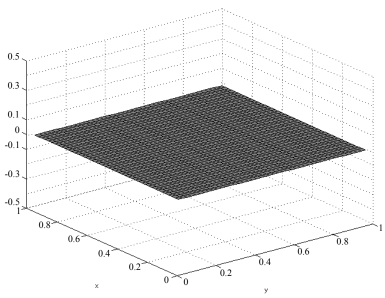


(c) t=40



(d) t=60

Fig. 1: Dynamics for $\mu = 12$

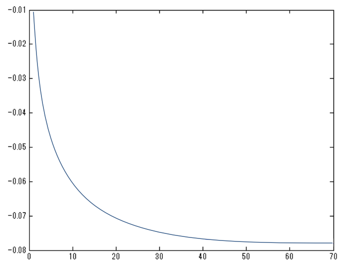
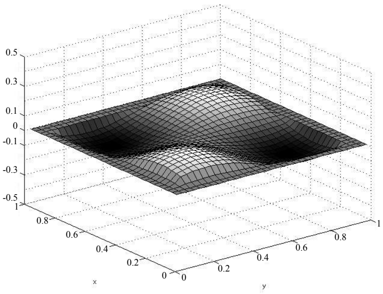
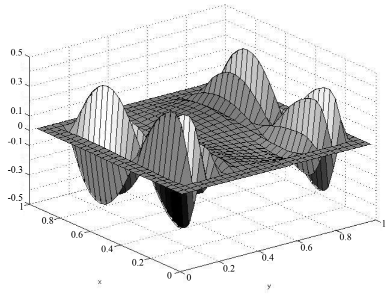


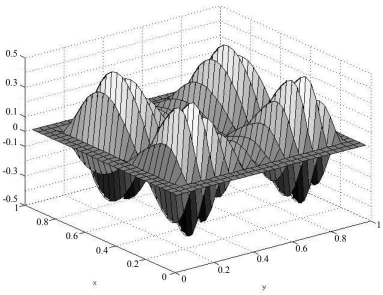
Fig. 2: Lyapunov function for $\mu = 12$



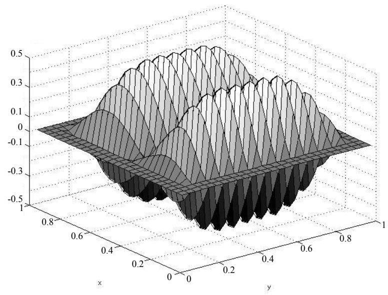
(a) t=0



(b) t=30



(c) t=60



(d) t=90

Fig. 3: Dynamics for $\mu = 13$

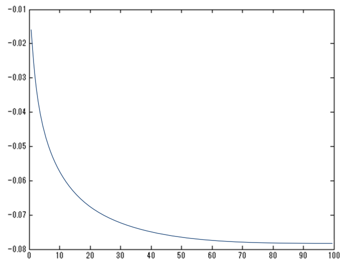
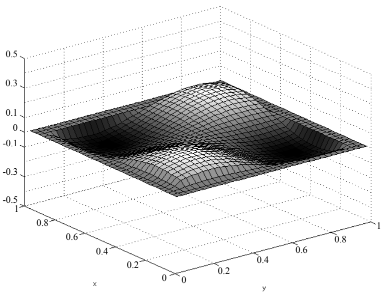
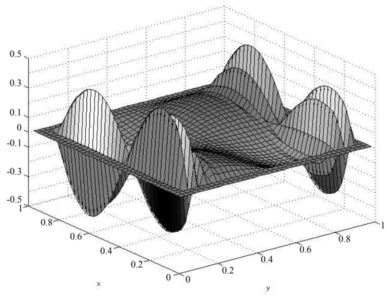


Fig. 4: Lyapunov function for $\mu = 13$



(a) t=0



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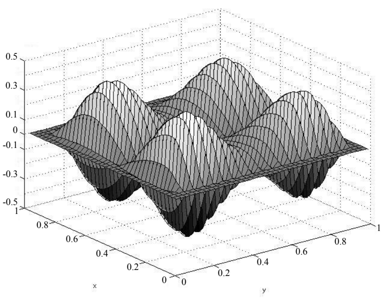
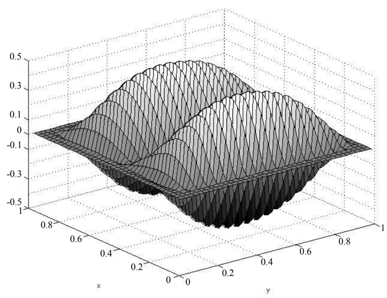


(c) t=120



(d) t=180

Fig. 5: Dynamics for $\mu = 40$

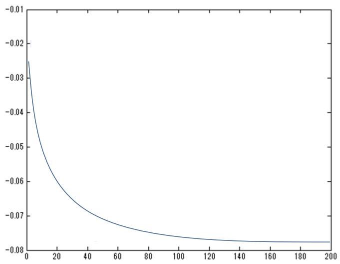


Fig. 6: Lyapunov function for $\mu = 40$