HOMOGENEOUS STATIONARY SOLUTION TO EPITAXIAL GROWTH MODEL UNDER DIRICHLET CONDITIONS

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Received October 2, 2015 ; revised December 9, 2015

Abstract. This paper continues a study on the initial-boundary value problem for a nonlinear parabolic equation of fourth order under the homogeneous Dirichlet boundary conditions. The parabolic equation has been presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [10] in order to describe the large-scale features of a growing crystal surface under molecular beam epitaxy. In the previous papers [1, 2], we constructed a dynamical system generated by the problem and showed that every trajectory converges to some stationary solution as $t \to \infty$. This paper is then devoted to investigating stability or instability of the null solution which is a unique homogeneous stationary solution. We shall also illustrate some numerical results to observe how changes the structure of stationary solutions as the roughening coefficient increases.

1 Introduction We are concerned with the initial-boundary value problem for a nonlinear parabolic equation of fourth order

$$
(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = -a\Delta^2 u - \mu\nabla\cdot\left(\dfrac{\nabla u}{1+|\nabla u|^2}\right) & \text{in} \quad \Omega\times(0,\infty),\\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}
$$

in a two-dimensional bounded domain Ω. Such a nonlinear parabolic equation has been presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [10] in order to describe the large-scale features of a growing crystal surface under molecular beam epitaxy. Here, Ω denotes a substrate domain and the unknown function $u = u(x,t)$ denotes a displacement of surface height from the standard level at position $x \in \Omega$ and time t. For detailed physical background, see [5, 12, 13, 16].

As in the preceding papers [1, 2], we will formulate (1.1) as the Cauchy problem for an abstract parabolic equation of the form (2.1) with underlying space $L_2(\Omega)$. In [1], we constructed a dynamical system $(S(t), L_2(\Omega))$ generated by (2.1), where $S(t)$ is a continuous nonlinear semigroup acting on $L_2(\Omega)$ determined by global solutions of (2.1). In addition, the dynamical system was shown to have a finite-dimensional attractor and to admit a Lyapunov function given by (2.8). In the subsequent paper [2], we succeeded in proving longtime convergence. For any $u_0 \in L_2(\Omega)$, $S(t)u_0$ was shown to converge as $t \to \infty$ to a stationary solution $\overline{u}$ of (2.1).

This paper is then concerned with stationary solutions of (2.1). Among others, we are concerned with stability and instability of the null solution $\overline{u} \equiv 0$. Clearly, the null solution is a unique homogeneous stationary solution. For this purpose, we will appeal

[1]This work is supported by Grant-in-Aid for Scientific Research (No. 26400166) of the Japan Society for the Promotion of Science.

2000 *Mathematics Subject Classification.* (2010) 35K55, 37L15, 74E15.

Key words and phrases. Epitaxial Growth, Stationary Solutions, Stability and Instability.

to the linearized principle in infinite-dimensional spaces invented by Babin-Vishik [3] and Temam [15], see also [17, Section 6.6]. Indeed, we shall prove that, when $\mu < ad^{-2}$, where $d > 0$ is a constant determined by (3.5), the null solution is globally stable and that, when $\mu > ad^{-2}$, the null solution is unstable. The constant d can be estimated by an optimal coefficient of the Poincare inequality. In the latter case, there must exist non-null stationary solutions (remember that every trajectory converges to some stationary solution).

In the papers [6, 7, 8, 9], we handled the same fourth order parabolic equation but under the Neumann like boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = 0$. Among others in [8] we studied stability and instability of the homogeneous stationary solution using the fact that, under these Neumann like boundary conditions, the fourth order operator Δ^2 can be reduced into the product $(-\Delta)^2$ of the negative Laplace operator $-\Delta$ equipped with the usual Neumann boundary conditions which is a positive definite self-adjoint operator of $L_2(\Omega)$. In the present case, however, such a favorable reduction is not available and we have to handle a very fourth order elliptic operator.

Throughout the paper, Ω is a rectangular or $\mathcal{C}^4$, bounded domain in $\mathbb{R}^2$. And $n(x)$ denotes the outer normal vector of the boundary at boundary point $x \in \partial\Omega$. As noticed by [2, Proposition 2.1], for $f \in L_2(\Omega)$, the elliptic problem $-\Delta^2 u = f$ in Ω under the conditions $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ admits a unique solution u such that $u \in H^4(\Omega)$. For $1 \le p \le \infty$, $L_p(\Omega)$ is the space of complex valued L_p functions in Ω. For $s \ge 0$, $H^s(\Omega)$ is the complex Sobolev space in Ω with exponent s. For $s \ge 0$, $H_0^s(\Omega)$ denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ (the space of all infinitely differentiable functions with compact support) in the topology of $H^s(\Omega)$. The coefficients $a > 0$ and $\mu > 0$ are given constants.

2 Reviews of known results

In this section, let us review known results obtained in the previous papers [1, 2].

Abstract Formulation. As in [1, 2], we formulate (1.1) as the Cauchy problem for a semilinear abstract evolution equation

$$\begin{cases} \dfrac{du}{dt} + Au = f(u), & 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

in the underlying space $X = L_2(\Omega)$. Here, A is an associated linear operator in the framework of a triplet $H_0^2(\Omega) \subset L_2(\Omega) \subset H^{-2}(\Omega)$ $(= H_0^2(\Omega)')$ with a symmetric sesquilinear form defined by

$$a(u, v) = a \int_\Omega \Delta u \cdot \Delta \overline{v}\, dx, \qquad u,\, v \in H_0^2(\Omega),$$

(cf. [4]). Then, A is a positive definite self-adjoint operator of X with domain $\mathcal{D}(A) \subset H_0^2(\Omega)$. The operator A is considered as a realization of the fourth order operator $a\Delta^2$ in X under the conditions $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

As seen by [2, Proposition 2.1], our assumption on Ω yields a characterization of $\mathcal{D}(A)$ in such a way that $\mathcal{D}(A) = H^4(\Omega) \cap H_0^2(\Omega)$ with norm equivalence. As the sesquilinear form is symmetric, $\mathcal{D}(A^{\frac{1}{2}})$ coincides with the from domain, i.e., $\mathcal{D}(A^{\frac{1}{2}}) = H_0^2(\Omega)$ with norm equivalence. By interpolation, we can then verify that, for $\frac{1}{2} \le \theta \le 1$,

$$\mathcal{D}(A^\theta) \subset H^{4\theta}(\Omega) \cap H_0^2(\Omega),$$

and for $0 \le \theta < \frac{1}{2}$,

$$\mathcal{D}(A^\theta) \subset H^{4\theta}(\Omega).$$

In addition, for any $0 \le \theta \le 1$, the inequality

$$\|u\|_{H^{4\theta}} \le C\|A^\theta u\|_X, \qquad u \in \mathcal{D}(A^\theta), \tag{2.2}$$

is satisfied, namely, the embedding described above is continuous.

Meanwhile, f is a nonlinear operator defined by

$$\begin{aligned} f(u) &= -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) \\ &= -\mu\left[\frac{\Delta u}{1+|\nabla u|^2} - \frac{\nabla|\nabla u|^2\cdot\nabla u}{(1+|\nabla u|^2)^2}\right], \qquad u \in \mathcal{D}(A^{\frac{7}{8}}). \end{aligned} \tag{2.3}$$

Note that, since $\mathcal{D}(A^{\frac{7}{8}}) \subset H^{\frac{7}{2}}(\Omega)$ due to (2.2) and $H^{\frac{7}{2}}(\Omega) \subset \mathcal{C}^2(\overline{\Omega})$, $u \in \mathcal{D}(A^{\frac{7}{8}})$ certainly implies $f(u) \in L_2(\Omega)$. Furthermore, according to [2, (2.8)], it holds true that

$$\begin{aligned} \|f(u)-f(v)\|_X \le C\big[&\|A^{\frac{1}{2}}(u-v)\|_X \\ &+ (\|A^{\frac{7}{8}}u\|_X + \|A^{\frac{7}{8}}v\|_X)\|A^{\frac{1}{4}}(u-v)\|_X\big], \qquad u,\, v \in \mathcal{D}(A^{\frac{7}{8}}). \end{aligned} \tag{2.4}$$

The general result on abstract semilinear evolution equations (cf. [17, Theorem 4.1]) readily provides local existence of solutions. For any $u_0 \in \mathcal{D}(A^{\frac{1}{4}})$, (2.1) possesses a unique local solution. As a matter of fact, we can formulate (1.1) even in a larger underlying space $H^{-2}(\Omega)$ of the form (2.1). As shown in [1], for any $u_0 \in H^{-2}(\Omega)$, there exists a unique local solution. Combining these two existence results, we can claim that, for any $u_0 \in L_2(\Omega) = X$, (2.1) possesses a unique local solution in the function space:

$$u \in \mathcal{C}((0, T_{u_0}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{u_0}]; X) \cap \mathcal{C}^1((0, T_{u_0}]; X), \tag{2.5}$$

$T_{u_0} > 0$ being determined by the norm $\|u_0\|_X$ alone.

In the subsequent sections, we need to use differentiability of $f(u)$.

Proposition 2.1. *$f : \mathcal{D}(A^{\frac{7}{8}}) \to X$ is Fréchet differentiable with derivative*

$$f'(u)h = -\mu\nabla\cdot\left(\frac{\nabla h}{1+|\nabla u|^2} - \frac{2(\nabla u\cdot\nabla h)\nabla u}{(1+|\nabla u|^2)^2}\right), \qquad u,\, h \in \mathcal{D}(A^{\frac{7}{8}}).$$

Proof. Let $u,\, h \in \mathcal{D}(A^{\frac{7}{8}})$. From (2.3) it follows that

$$\begin{aligned} f(u+h) - f(u) &= -\mu\nabla\cdot\left[\left(\frac{1}{1+|\nabla(u+h)|^2} - \frac{1}{1+|\nabla u|^2}\right)\nabla(u+h)\right] \\ &\qquad - \mu\nabla\cdot\left(\frac{\nabla(u+h)-\nabla u}{1+|\nabla u|^2}\right) \\ &= -\mu\nabla\cdot\left[\frac{(-2\nabla u\cdot\nabla h - |\nabla h|^2)\nabla(u+h)}{(1+|\nabla(u+h)|^2)(1+|\nabla u|^2)}\right] - \mu\nabla\cdot\left(\frac{\nabla h}{1+|\nabla u|^2}\right). \end{aligned}$$

By the similar calculations as for (2.4),

$$\|f(u+h) - f(u) - f'(u)h\|_X \le C\|A^{\frac{7}{8}}h\|_X^2(\|A^{\frac{7}{8}}u\|_X + \|A^{\frac{7}{8}}h\|_X).$$

This means that $f : \mathcal{D}(A^{\frac{7}{8}}) \to X$ is Fréchet differentiable at u. □

Proposition 2.2. *Let $u \in \mathcal{D}(A^{\frac{7}{8}})$ varies in a ball $B^{\mathcal{D}(A^{\frac{1}{2}})}(0;1)$. Then, $f'(u)$ satisfies the Lipschitz condition*

$$\|[f'(u) - f'(v)]h\|_X \le C\|A^{\frac{1}{2}}(u-v)\|_X\|A^{\frac{7}{8}}h\|_X,$$
$$u,\, v \in \mathcal{D}(A^{\frac{7}{8}}) \cap B^{\mathcal{D}(A^{\frac{1}{2}})}(0;1);\ h \in \mathcal{D}(A^{\frac{7}{8}}).$$

Proof. From the formula giving $f'(u)$, we can estimate directly the difference $f'(u) - f'(v)$. □

Dynamical System. The [2, Proposition 3.1] provides *a priori* estimates for local solutions obtained above in the space (2.5). Indeed, any local solution to (2.1) on interval $[0, T_u]$ satisfies the estimate

$$\|u(t)\|_X^2 \le e^{-2\delta t}\|u_0\|_X^2 + \mu\delta^{-1}, \qquad 0 \le t \le T_u,$$

with some fixed exponent $\delta > 0$. Then, by the standard argument, we conclude that, for any $u_0 \in X$, (2.1) possesses a unique global solution u in the function space:

$$u \in \mathcal{C}((0,\infty); \mathcal{D}(A)) \cap \mathcal{C}([0,\infty); X) \cap \mathcal{C}^1((0,\infty); X). \tag{2.6}$$

Furthermore, u also satisfies the same estimate

$$\|u(t)\|_X^2 \le e^{-2\delta t}\|u_0\|_X^2 + \mu\delta^{-1}, \qquad 0 \le t < \infty, \tag{2.7}$$

which shows dissipation of u. Set a nonlinear semigroup $S(t)$, $0 \le t < \infty$, on X by $S(t)u_0 = u(t; u_0)$, using the global solution $u(t; u_0)$ to (2.1) with initial data $u_0 \in X$. Then, we obtain a dynamical system $(S(t), X)$ generated by (2.1). The dissipate estimates yield existence of a finite-dimensional attractor $\mathcal{M}$ which attracts every trajectory $S(t)u_0$ at an exponential rate. Such an attractor is called the exponential attractor. In particular, we know that every trajectory has a nonempty ω-limit set $\omega(u_0)$.

As shown by [1, Section 5], our system $(S(t), X)$ admits a Lyapunov function of the from

$$\Phi(u) = \frac{1}{2}\int_\Omega [a|\Delta u|^2 - \mu \log(1 + |\nabla u|^2)]dx, \qquad u \in H_0^2(\Omega). \tag{2.8}$$

That is, the value $\Phi(S(t)u_0)$ is monotone decreasing as $t \to \infty$ along any trajectory. Furthermore, it is seen that, for $\overline{u} \in \mathcal{D}(A)$, $\Phi'(\overline{u}) = 0$ and $A\overline{u} = f(\overline{u})$ (i.e., $\overline{u}$ is a stationary solution) are equivalent. From this equivalence, we see that, if $\overline{u} \in \omega(u_0)$, then $\overline{u}$ must be a stationary solution of (2.1). The set $\omega(u_0)$ consists only of stationary solutions.

Convergence of Solutions. The objective of [2] was then to show that $\omega(u_0)$ is a singleton for every u_0. We proved that $\Phi(u)$ satisfies the Łojasiewicz-Simon inequality

$$\|\Phi'(u)\|_{H^{-2}} \ge D|\Phi(u) - \Phi(\overline{u})|^{1-\theta}$$

in a neighborhood of $\overline{u}$, where $\overline{u} \in \omega(u_0)$, with some exponent $0 < \theta \le \frac{1}{2}$. This inequality readily implies that

$$\|S(t)u_0 - \overline{u}\|_X \le C[\Phi(S(t)u_0) - \Phi(\overline{u})]^\theta. \tag{2.9}$$

As $\Phi(S(t)u_0)$ converges to $\Phi(\overline{u})$ as $t \to \infty$, we observe that $S(t)u_0$ converges to $\overline{u}$ in X with some rate of convergence.

3 Linearized Stability Let us now investigate stability and instability of the stationary solutions of (2.1). For this purpose, we will employ the general methods for abstract evolution equations, see [3, 15].

Let $\overline{u} \in \mathcal{D}(A)$ be any stationary solution to (2.1), i.e., $A\overline{u} = f(\overline{u})$. By Propositions 2.1 and 2.2, $f : \mathcal{D}(A^{\frac{7}{8}}) \to X$ is of class $\mathcal{C}^{1,1}$ in a neighborhood of $\overline{u}$, and the derivative satisfies a Lipschitz condition

$$\|[f'(u) - f'(v)]h\|_X \le C\|A^{\frac{1}{2}}(u-v)\|_X\|A^{\frac{7}{8}}h\|_X, \quad u,\, v \in \mathcal{D}(A^{\frac{7}{8}}) \cap \mathcal{O}(\overline{u});\ h \in \mathcal{D}(A^{\eta}),$$

$\mathcal{O}(\overline{u})$ being a neighborhood of $\overline{u}$ in $\mathcal{D}(A^{\frac{1}{2}})$. It is known that this condition in turn implies Fréchet differentiability of the semigroup. Indeed, for $0 \le t \le t^*$ where $t^* > 0$ is arbitrarily fixed time, $S(t) : \mathcal{D}(A^{\frac{1}{2}}) \to \mathcal{D}(A^{\frac{1}{2}})$ is of class $\mathcal{C}^{1,1}$ in a neighborhood $\mathcal{O}'(\overline{u})$ of $\overline{u}$ in $\mathcal{D}(A^{\frac{1}{2}})$ together with the estimate

$$\|S(t)'u - S(t)'v\|_{\mathcal{L}(\mathcal{D}(A^{\frac{1}{2}}))} \le C\|A^{\frac{1}{2}}(u-v)\|_X, \qquad u,\, v \in \mathcal{O}'(\overline{u});\ 0 \le t \le t^*. \tag{3.1}$$

For the detailed proof, see the proof of [17, Subsection 6.6.3].

We here assume a spectral separation condition for $\sigma(A - f'(\overline{u}))$ of the form

$$\sigma(A - f'(\overline{u})) \cap \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda = 0\} = \emptyset.$$

Then, since $S(t)'\overline{u} = e^{-t\overline{A}}$, where $\overline{A} = A - F'(\overline{u})$, we have in turn a spectral separation for $S(t)'\overline{u}$ of the from

$$\sigma(S(t)'\overline{u}) \cap \{\lambda \in \mathbb{C};\ |\lambda| = 1\} = \emptyset. \tag{3.2}$$

According to [17, Theorem 6.9], under (3.1) and (3.2), a smooth local unstable manifold $\mathcal{M}_+(\overline{u}; \mathcal{O})$ can be constructed in a neighborhood $\mathcal{O}$ of $\overline{u}$ in $\mathcal{D}(A^{\frac{1}{2}})$. When

$$\sigma(A - F'(\overline{u})) \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda > 0\}, \tag{3.3}$$

we have $\sigma(S(t)'\overline{u}) \subset \{\lambda \in \mathbb{C};\ |\lambda| < 1\}$ and $\mathcal{M}_+(\overline{u}; \mathcal{O})$ reduces to a singleton $\{\overline{u}\}$. Whence, if (3.3) takes place, $\overline{u}$ is stable. In the meantime, when

$$\sigma(A - f'(\overline{u})) \cap \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda < 0\} \ne \emptyset, \tag{3.4}$$

we have $\sigma(S(t)'\overline{u}) \cap \{\lambda \in \mathbb{C};\ |\lambda| > 1\} \ne \emptyset$ and $\mathcal{M}_+(\overline{u}; \mathcal{O})$ is not trivial. Whence, if (3.4) takes place, $\overline{u}$ is unstable.

Let us now apply these discussions to the null solution $\overline{u} \equiv 0$. We see from Proposition 2.1 that $A - f'(0) = a\varDelta^2 + \mu\varDelta$. So, it is necessary to investigate the spectrum of the operator $a\varDelta^2 + \mu\varDelta$. To this end, we will introduce a normalization of A; indeed, when $a = 1$, we denote $A = A_1$; and, regarding a as a positive parameter, we denote in general $A = aA_1$. Of course, A_1 is a realization of the operator $\varDelta^2$ in $L_2(\Omega)$ under the homogeneous Dirichlet conditions on $\partial\Omega$, and is a positive definite self-adjoint operator of X. As verified above, we have $\mathcal{D}(A_1) = H^4(\Omega) \cap H_0^2(\Omega)$ with norm equivalence and $\mathcal{D}(A_1^{\frac{1}{2}}) = H_0^2(\Omega)$ with norm equivalence.

We here notice a fact that a mapping $u \mapsto \frac{\|\nabla u\|_X}{\|\varDelta u\|_X}$ is continuous from $H_0^2(\Omega) - \{0\}$ into $\mathbb{R}$ and has a maximum on the sphere $\|A_1 u\|_X = 1$ because of compact embedding $\mathcal{D}(A_1) \subset \mathcal{D}(A_1^{\frac{1}{2}})$. Put

$$d \equiv \max_{\|A_1 u\|_X = 1} \frac{\|\nabla u\|_X}{\|\varDelta u\|_X}. \tag{3.5}$$

In other words, the d is an optimal coefficient in the inequality

$$\|\nabla u\|_X \le d\|\Delta u\|_X \qquad u \in \mathcal{D}(A_1).$$

Stability of the null solution is then determined by dominance in magnitude of the two coefficients a and μ to the other but with weight d^{-2} for a.

Theorem 3.1. *If $ad^{-2} > \mu$, then the null solution is stable. If $ad^{-2} < \mu$, then the null solution is unstable.*

Proof. We notice that $a\Delta^2+\mu\Delta$ is a self-adjoint operator of X whose domain $H^4(\Omega)\cap H_0^2(\Omega)$ is compactly embedded in $L_2(\Omega)$. Therefore, the spectrum set $\sigma(a\Delta^2+\mu\Delta)$ is contained in the real axis and consists of point spectrum alone.

For any $u \in \mathcal{D}(A_1)-\{0\}$, we observe that

$$(a\Delta^2 u + \mu\Delta u, u) = a\|\Delta u\|_X^2 - \mu\|\nabla u\|_X^2 \ge (ad^{-2}-\mu)\|\nabla u\|_X^2 > 0,$$

provided $ad^{-2} > \mu$. Therefore, if μ is dominated as $\mu < ad^{-2}$, then $\sigma(a\Delta^2+\mu\Delta) \subset (0,\infty)$ and the null solution is stable. To the contrary, if μ is large enough so that $\mu > ad^{-2}$, i.e., $d > \sqrt{\frac{a}{\mu}}$, then there exists an element $u_0 \in \mathcal{D}(A_1)-\{0\}$ such that $\|\nabla u_0\|_X > \sqrt{\frac{a}{\mu}}\|\Delta u_0\|_X$. Therefore,

$$(a\Delta^2 u_0 + \mu\Delta u_0, u_0) = a\|\Delta u_0\|_X^2 - \mu\|\nabla u_0\|_X^2 < 0.$$

This means that $\sigma(a\Delta^2+\mu\Delta)\cap(-\infty,0) \ne \emptyset$. Hence, the null solution is unstable. □

As a matter of fact, when $ad^{-2} > \mu$, every trajectory converges to 0, that is, the null solution is globally stable.

Theorem 3.2. *Let $ad^{-2} > \mu$. For any $u_0 \in X$, $S(t)u_0$ converges to 0 as $t \to \infty$ at an exponential rate.*

Proof. Multiply the equation of (1.1) by $\overline{u}$ and integrate the product in Ω. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + a\int_\Omega |\Delta u|^2 dx &= \mu\int_\Omega \frac{|\nabla u|^2}{1+|\nabla u|^2}dx \\ &\le \mu\int_\Omega |\nabla u|^2 dx.\end{aligned}$$

It then follows from (3.5) that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_X^2 dx \le -(ad^{-2}-\mu)\|\nabla u(t)\|_X^2 \le -(ad^{-2}-\mu)D^{-1}\|u(t)\|_X^2,$$

where $D > 0$ is a coefficient for the Poincare inequality given by (4.1) below. Hence, $\|u(t)\|_X \le e^{-(ad^{-2}-\mu)D^{-1}t}\|u_0\|_X$ for $t \ge 0$. □

4 Estimation of d from above.

The weight constant d can be easily estimated from above from the Poincare inequality

$$\|u\|_X \le D\|\nabla u\|_X \qquad u \in H_0^1(\Omega). \tag{4.1}$$

Theorem 4.1. *Let d be the constant determined by* (3.5) *and let D be an optimal coefficient for the Poincare inequality* (4.1)*. Then, it always holds true that $d \le D$.*

Proof. Indeed,

$$\|\nabla u\|_X^2 = (-\Delta u, u) \le \|\Delta u\|_X \|u\|_X \le D\|\Delta u\|_X \|\nabla u\|_X, \qquad u \in H_0^2(\Omega).$$

Therefore, $\|\nabla u\|_X \le D\|\Delta u\|_X$ for $u \in H_0^2(\Omega)$. Of course, it holds that $\|\nabla u\|_X \le D\|\Delta u\|_X$ for $u \in \mathcal{D}(A_1)$. □

The coefficient D is usually estimated by the band width of Ω, see [4, Section 4.7].

The rest of this section is devoted to obtaining an optimal estimate of D in the specific case where

$$\Omega = \{(x_1, x_2);\ 0 < x_1 < \ell_1,\ 0 < x_2 < \ell_2\}.$$

Let Λ denote a realization of $-\Delta$ equipped with the boundary condition $u = 0$ in $L_2(\Omega)$. Then, Λ is a positive definite self-adjoint operator of $L_2(\Omega)$ with domain $\mathcal{D}(\Lambda) = H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, since its minimal eigenvalue is $\frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}$ with eigenfunction $\sin \frac{\pi}{\ell_1} x_1 \cdot \sin \frac{\pi}{\ell_2} x_2$, we have $(\Lambda u, u) \ge \left(\frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}\right) \|u\|_X^2$ for any $u \in \mathcal{D}(\Lambda)$. It then follows that

$$\|\nabla u\|_X^2 = (-\Delta u, u) \ge \left(\frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}\right) \|u\|_X^2, \qquad u \in \mathcal{D}(\Lambda).$$

Since $\mathcal{D}(\Lambda)$ is dense in $\mathcal{D}(\Lambda^{\frac{1}{2}})$ and since $\mathcal{D}(\Lambda^{\frac{1}{2}})$ coincides with $H_0^1(\Omega)$, this inequality holds true for every $u \in H_0^1(\Omega)$. Hence, (4.1) takes place with $D = \left(\frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}\right)^{-\frac{1}{2}}$ and, in fact, this is optimal.

Theorem 4.2. *Let $\Omega = (0, \ell_1) \times (0, \ell_2)$. Then, an optimal coefficient D for the Poincare inequality* (4.1) *is given by $D = \frac{\ell_1 \ell_2}{\pi\sqrt{\ell_1^2 + \ell_2^2}}$. Consequently, the weight constant d is estimated by $d \le \frac{\ell_1 \ell_2}{\pi\sqrt{\ell_1^2 + \ell_2^2}}$.*

Corollary 1. *Let $\Omega = (0, \ell_1) \times (0, \ell_2)$. If $\mu < \frac{\pi^2(\ell_1^2 + \ell_2^2)a}{\ell_1^2 \ell_2^2}$, then the null solution is globally stable.*

5 Numerical Results Let us here illustrate some numerical examples which shows some agreements to Corollary 1. We consider (1.1) in one of the following rectangular domains

$$\Omega = \left(0, \tfrac{1}{\ell}\right) \times (0, \ell), \quad \text{where } \ell \text{ is } 1, 2 \text{ or } 4.$$

When $\ell = 1$, Ω is square. Otherwise, Ω is strictly rectangular. The area of Ω is constantly equal to 1. The coefficients a and μ are fixed as $a = 1$ and $\mu = 40$.

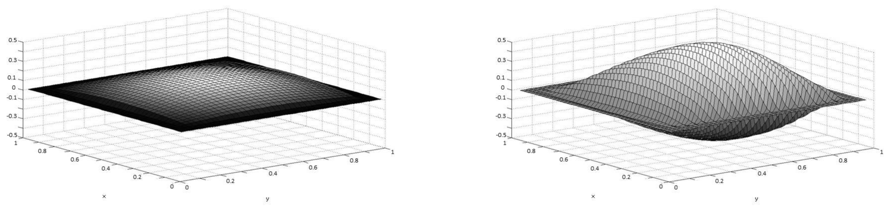
Set first $\Omega = (0, 1) \times (0, 1)$. We also set the initial function as

$$u_0(x_1, x_2) = 0.1[\sin(3.14 x_1) \times \sin(3.14 x_2)], \qquad (x_1, x_2) \in \Omega,$$

see Figure 1 (a). This is a small perturbation of the null solution. The solution then converges to some non-null stationary solution as $t \to \infty$. Its profile is given by Figure 1 (b). This means that the null stationary solution is unstable.

Set secondly $\Omega = \left(0, \frac{1}{2}\right) \times (0, 2)$. We accordingly replace the initial function with

$$u_0(x_1, x_2) = 0.1[\sin(2 \cdot 3.14 x_1) \times \sin(3.14 x_2)], \qquad (x_1, x_2) \in \Omega,$$



(a) t=0 (b) t=120

Fig. 1: Case where $\Omega = (0,1) \times (0,1)$

see Figure 2 (a). The solution again converges to some non-null stationary solution as $t \to \infty$ whose profile is given by Figure 2 (b). This means that the null stationary solution is still unstable.

Finally, set $\Omega = \left(0, \frac{1}{4}\right) \times (0,4)$, and replace the initial function with

$$u_0(x_1, x_2) = 0.1[\sin(4 \cdot 3.14 x_1) \times \sin(3.14 x_2)], \qquad (x_1, x_2) \in \Omega,$$

see Figure 3 (a). As seen by Figure 3 (b), the solution now converges to the null solution. The domain Ω is slender enough to reduce the weight constant d in such a way that $d \le \frac{\ell_1 \ell_2}{\pi\sqrt{\ell_1^2 + \ell_2^2}}$ (by Theorem 4.2) and to globally stabilize the null solution as ensured by Corollary 1.

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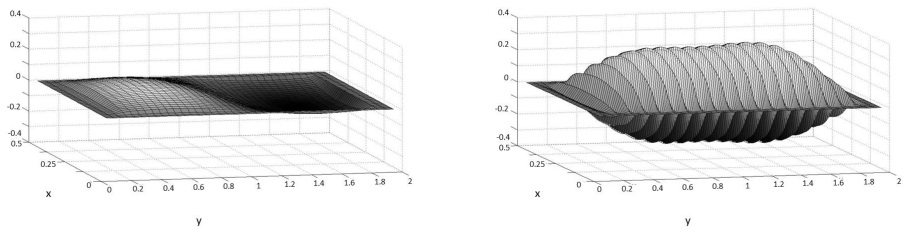

(a) t=0 (b) t=180

Fig. 2: Case where $\Omega = (0, \frac{1}{2}) \times (0, 2)$

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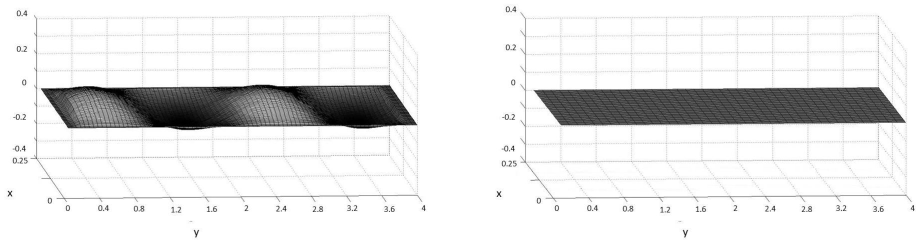

(a) t=0 (b) t=240

Fig. 3: Case where $\Omega = (0, \frac{1}{4}) \times (0, 4)$