ON THE DECOMPOSITION OF CONTRACTIONS AND ISOMETRIES

G. A. BAGHERI-BARDI

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ABSTRACT. It is proved (with given different proofs) that the von Neumann-Wold and the Nagy-Foias-Langer decompositions are valid in more general classes than the classical W*-algebras.

INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and let $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. The aim of the structure theory analysis is the structure of operators in $B(\mathcal{H})$. The structure of some operators are well-understood. As for unitaries a spectral theory and effective functional calculus are available. Another part is unilateral shifts. An operator a on $\mathcal{H}$ is called a unilateral shift if there is a sequence of pairwise orthogonal subspaces $\mathcal{H}_0, \mathcal{H}_1, \cdots$ such that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots$ and a maps $\mathcal{H}_n$ isometrically onto $\mathcal{H}_{n+1}$. For a comprehensive discussion about unilateral shifts, we refer to section 23 of [1].

Two fundamental theorems make the cornerstone of the structure theory. The first one provides the largest reducing subspace for a given contraction $a \in B(\mathcal{H})$ on which a will be unitary [5][8] and the second one gives much more details when a is an isometry[10].

Theorem 0.1. ***The Nagy-Foias-Langer Decomposition*** *To every contraction a on the Hilbert space $\mathcal{H}$ there corresponds a decomposition of $\mathcal{H}$ into an orthogonal sum of two subspaces reducing a, say $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, such that the restriction of a to $\mathcal{H}_0$ is unitary, and the restriction of a to $\mathcal{H}_1$ is completely non-unitary. This decomposition is uniquely determined.*

Theorem 0.2. ***The von Neumann-Wold Decomposition.*** *If x is an isometry on the Hilbert space $\mathcal{H}$ and $\mathcal{H}_0 = \bigcap_n x^n\mathcal{H}$, then $\mathcal{H}_0$ reduces x, $x_{|_{\mathcal{H}_0}}$ is unitary and $x_{|_{\mathcal{H}_0^\perp}}$ is a unilateral shift.*

The strategy of the original proofs of these decompositions are completely based on the geometry of the underlying Hilbert space. In this discussion, we give different proofs of these results which are more algebraic in nature than the the well-known proofs. These proof therefore offer valuable insight as to how one can extend the results to non-normed topological algebras. This is demonstrated in Section 2, where it is shown that the results are valid for locally W*-algebras[2], a class of (generally non-normed) topological *-algebras more general than W*-algebras.

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1. Wold decomposition

Throughout this section $\mathcal{M}$ stands for a W*-algebra with the unit 1. At first, we deal with the decomposition of contractions. To begin we need a convention. Let x be in $\mathcal{M}$. We denote by $[x]$ the relative unit of the w^*-closed algebra generated by xx^* in $\mathcal{M}$ and call it the range projection of x.

Remark 1.1. To make an illustration what the projection $[x]$ is in the concrete case, assume that x is a bounded linear operator on the Hilbert space $\mathcal{H}$. In this case $[x]$ will be the relative unit of the von Neumann algebra generated by xx^* in $B(\mathcal{H})$. One may check that $[x]$ is just the orthogonal projection onto $\overline{x\mathcal{H}}$.

Let a be a contraction in $\mathcal{M}$. A sequence of projections $\{e_n\}_{n\in\mathbb{Z}}$ in $\mathcal{M}$ is called a $U(a)$-solution if

$$\begin{cases} {a^*}^n a^n e_n = e_n & n \geq 0 \\ a^{-n} {a^*}^{-n} e_n = e_n & n \leq 0 \end{cases}$$

For two given solutions $\{e_n^j\}_{n\in\mathbb{Z}}$ $(j = 1, 2)$, we write $\{e_n^1\}_{n\in\mathbb{Z}} \preceq \{e_n^2\}_{n\in\mathbb{Z}}$ if $e_n^1 \leq e_n^2$ for all $n \in \mathbb{Z}$. Clearly $\preceq$ defines a partial order relation on $U(a)$-solutions.

Lemma 1.2. *Let a be a contraction. The set of $U(a)$-solutions has a maximal element.*

Proof. Let us consider

$$\begin{cases} p_n = 1 - [1 - {a^*}^n a^n] & n \geq 0 \\ p_n = 1 - [1 - a^{-n} {a^*}^{-n}] & n \leq 0. \end{cases}$$

Since $[1-{a^*}^n a^n]$ is the relative unit of the w^*-closed algebra generated by $1-{a^*}^n a^n$, then

$$\underbrace{(1 - (1 - {a^*}^n a^n))}_{{a^*}^n a^n} \underbrace{(1 - [1 - {a^*}^n a^n])}_{p_n} = p_n.$$

Similarly one may see that $a^{-n}{a^*}^{-n} p_n = p_n$ when $n \leq 0$. It means that $\{p_n\}_{n\in\mathbb{Z}}$ is a $U(a)$-solution. Assume $\{q_n\}_{n\in\mathbb{Z}}$ is another $U(a)$-solution. We have then

$$\begin{aligned} (1 - {a^*}^n a^n) q_n = 0 &\Longrightarrow [1 - {a^*}^n a^n] q_n = 0 \\ &\Longrightarrow q_n \leq 1 - [1 - {a^*}^n a^n] = p_n \end{aligned}$$

for all $n \geq 0$. Similarly $q_n \leq p_n$ for negative integers n. □

We put $v := \inf_{n\in\mathbb{N}} p_n$ where $\{p_n\}$ is the maximum of $U(a)$-solutions and call v the unitary projection of a. The unitary projection of a is zero if and only if either $\{{a^*}^n a^n\}$ or $\{a^{-n}{a^*}^{-n}\}$ converges to zero in the w^*-topology. Such a contraction is called completely non-unitary.

Lemma 1.3. *The unitary projection v of a commutes with a. Moreover*

$$va^*av = vaa^*v = v$$

Proof. We combine some points to obtain the assertion.

- Since v is majorized by p_1 then $a^*av = v$. Therefore ava^* is a projection, and hence the unit element of the von Neumann algebra generated by $(ava^*)(ava^*)^*$, being $[av]$, is ava^*.

- We now show that $[av]$ (as a constant sequence) is a $U(a)$-solution too: when n is negative:

$$\begin{aligned} a^{-n}{a^*}^{-n}\underbrace{(ava^*)}_{[av]} &= a^{-n}a^{*-n-1}\underbrace{a^*(ava^*)}_{va^*} \\ &= a^{-n}a^{*-n-1}(va^*) && (v \le p_1 \to a^*av = v) \\ &= ava^* && (v \le p_{-n-1} \to a^{-n-1}{a^*}^{-n-1}v = v) \end{aligned}$$

Let n be a positive integer. Since ${a^*}^{n+1}a^{n+1}v = v$ we have then

$$(ava^*)(1 - {a^*}^n a^n)(ava^*) = 0$$

On the other hand

$$\begin{aligned} (ava^*)(1 - {a^*}^n a^n)(ava^*) = 0 &\Longleftrightarrow \sqrt{(1 - a^{*n}a^n)}(ava^*) = 0 \\ &\Longrightarrow (1 - {a^*}^n a^n)(ava^*) = 0 \end{aligned}$$

Similarly one may see that $[a^*v]$ is a $U(a)$-solution too. Therefore v majorizes both $[av]$ and $[av^*]$.

- Finally we have

$$\begin{aligned} [av] \le v &\Longleftrightarrow (1 - v)[av] = 0 \\ &\Longleftrightarrow (1 - v)av = 0, \end{aligned}$$

which implies that $av = vav$. We apply $[a^*v] \le v$ to conclude that $a^*v = va^*v$. These two earlier identities finish the proof.

□

Combination of these two lemmas implies the following result:

Theorem 1.4. *Let a be a contraction in $\mathcal{M}$. Then v, the unitary projection of a is uniquely determined with the following properties:*

(1) *v commutes with a.*
(2) *$(1 - v)a(1 - v)$ is completely non unitary in W^*-algebra $(1 - v)\mathcal{M}(1 - v)$.*
(3) *vav is unitary in W^*-algebra $v\mathcal{M}v$ provided that v is non zero.*

Proof. Assume that w is another projection which satisfies the above axioms. Based on Lemma 1.2, v majorizes w. Therefore $v - w$ is a projection which commutes with a, majorized by $1 - w$ and $(v - w)a(v - w)$ is unitary in $(v - w)\mathcal{M}(v - w)$. It is contradiction with the axiom (2). □

Remark 1.5. Let $\mathcal{A}$ be a von Neumann subalgebra in $B(\mathcal{H})$. Let a be a contraction in $\mathcal{A}$ and consider v, the unitary projection of a, obtained in Lemma 1.2. The identity $va = av$ is equivalent to the point that $\mathcal{H}_0 = v\mathcal{H}$ is reduced by a. A glance at the proof of the Nagy-Foias-Langer decomposition theorem (see [9] page 9) shows that $\mathcal{H}_0$ is just the largest reducing subspace such that $a_{|\mathcal{H}_0}$ is unitary.

We now commence with the process of decomposition of an isometry x. To begin, the analogue of unilateral shifts in any arbitrary W^*-algebra is introduced.

Lemma 1.6. *Let x be an isometry in $\mathcal{M}$ and p be a projection in $\mathcal{M}$. We have*

$$[xp] = xpx^*$$

Proof. Since xpx^* is itself a projection, then similar to the argument in Lemma 1.3, $x = xpx^*$. □

Let x be an isometry. We shift forward the projection p to the projection $[xp]$ by x and continue this process to obtain the following sequence of projections

$$p_0 = p, p_1 = [xp] = xpx^*, \cdots, p_n = [xp_{n-1}] = x^n p {x^*}^n, \cdots$$

Such a sequence is called the p-shift spectrum of x with the initial projection p. This sequence is called an orthogonal p-shift spectrum if the projections p_n are pairwise mutually orthogonal.

Definition 1.7. Assume x is an isometry in $\mathcal{M}$. A projection p in $\mathcal{M}$ is called wandering for x if the corresponding shift spectrum (with the initial projection p) is orthogonal. If the total summation $\sum_0^\infty p_n$ (in the sense of w^*-topology) is just the unit of $\mathcal{M}$ then x is called an abstract unilateral shift.

Proposition 1.8. *Let x be an isometry in $\mathcal{M}$.*

(1) $1 - [x]$ *is a wandering projection of* x.

(2) *If x is an abstract unilateral shift, then there is unique orthogonal shift spectrum of x with total summation* 1. *Moreover the initial projection is* $1 - [x]$.

Proof. We apply Lemma 1.6 to obtain the following points:

i) $[x^{n+1}] = x[x^n]x^*$

ii) $[x([x^{n-1}] - [x^n])] = [x^n] - [x^{n+1}]$

Since $[x^{n+1}]$ is majorized by $[x^n]$, then ii) shows that $1-[x]$ is a wandering projection of x. As for the second item (2), assume x is an abstract unilateral shift. Let p be a wandering projection of x whose orthogonal shift spectrum has total summation 1. It means in the sense of w^*-topology that

$$\begin{aligned} 1 &= p + xpx^* + x^2 p {x^*}^2 + \cdots \\ &= p + x(p + xpx^* + x^2 p {x^*}^2 + \cdots)x^* \\ &= p + xx^* = p + [x] \end{aligned}$$

Therefore the initial projection p should be $1 - [x]$. □

Remark 1.9. Assume that x is a unilateral shift. We have then for every positive integer n that

i′) $[x^n] = x^*[x^{n+1}]x$

ii′) $[x^*([x^{n+1}] - [x^{n+2}])] = [x^n] - [x^{n+1}]$: To prove this, note that item i′) shows that the absolute value $|x^*([x^{n+1}] - [x^{n+2}])|$ is the just the projection $[x^n] - [x^{n+1}]$.

iii′) $[x^*(1 - [x])] = 0$

Based on these relations one may say that x^* acts as a backward shift.

Let x be an isometry and consider the following projections

$$\begin{aligned} s &= \sum [x^n] - [x^{n+1}] = 1 - \lim\ [x^n] \\ u &= \lim\ [x^n] = \inf [x^n] \end{aligned}$$

where the limits are taken in the w^*-topology. The pair (s, u) is called the Wold pair of x. We have the following main result when both s, u are non-trivial.

Theorem 1.10. *Let x be an isometry in a W^*-algebra $\mathcal{M}$. The Wold pair (s,u) of x is uniquely determined with the following properties*

1. *s and u are mutually orthogonal and $s+u=1$.*
2. *Both projections s and u commute with x.*
3. *sxs is an abstract unilateral shift in the W^*-algebra $s\mathcal{M}s$ and uxu is a unitary in the W^*-algebra $u\mathcal{M}u$.*

Proof. The first item is clear and the second one is directly obtained by the definition of the Wold pair (s,u). As for (3), since s commutes with x, then sxs is an isometry in the W*-algebra $s\mathcal{M}s$. The projection $1-[x]$ is majorized by s and so is a projection in $s\mathcal{M}s$. We apply the second item of the Proposition 1.8 to obtain $1-[x]$ is a wandering projection for sxs. Moreover

$$\sum(sx^n s)([x^n]-[x^{n+1}])(sx^{*^n}s)=s(\sum x^n([x^n]-[x^{n+1}])x^{*^n})s=s$$

which shows that sxs is an abstract unilateral shift in $s\mathcal{M}s$. The definition of u shows that $u\leq[x]$. Since u commutes with x

$$(uxu)(ux^*u)=u[x]u=u$$

Therefore uxu is unitary in $u\mathcal{M}u$.
Assume that (s_1,u_1) is a pair satisfying in conditions $(1),(2)$ and (3). We have then

$$\begin{aligned}1-[x]&=(s_1\oplus u_1)-[x(s_1\oplus u_1)]\\&=(s_1\oplus u_1)-(s_1[x]s_1\oplus u_1[x]u_1)\\&=s_1-s_1[x]s_1.\end{aligned}$$

We conclude x and s_1xs_1 have the same orthogonal shift spectrum with intial projection $1-[x]$ which implies that $s_1=s$ and so $u_1=u$. □

This theorem says that any isometry x is decomposed into an abstract unilateral shift and a unitary, which is exactly the von-Neumann-Wold decomposition:

$$x=sxs\oplus uxu.$$

Remark 1.11. Assume x is an isometry in $B(\mathcal{H})$ and p is a wandering projection of x. Let us denote $\mathcal{H}_n$ to be the range of the projection $p_n=x^npx^{*^n}$. Then $\mathcal{H}_0,\mathcal{H}_1,\cdots$ form pairwise orthogonal closed subspaces and x maps $\mathcal{H}_n$ isometrically onto $\mathcal{H}_{n+1}$. Therefore abstract unilateral shifts coincide with the unilateral shifts in $B(\mathcal{H})$. Moreover the corresponding Wold pair of x induces reducing subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ on which x is decomposed into a unitary and a unilateral shift.

The Wold decomposition is concerned with the structure of an isometry. It is extended for a particular finite sequence of isometries. We examine the current method for two such items.

The structure of an isometric tuple of operators in $B(\mathcal{H})$ is given in [6] as an extension of the Wold decomposition of an isometry. An n-tuple of operators $(x_1,\cdots,x_n)$ acting on $\mathcal{H}$ is said to be isometric if the row operator $[x_1,\cdots,x_n]:\mathcal{H}^n\to\mathcal{H}$ is an isometry. In fact, an isometric n-tuple is a sequence of isometries $x_1,\cdots,x_n$ such that the x_i's have pairwise orthogonal ranges. It is equivalent to say that the sequence of isometries $x_1,\cdots,x_n$ satisfies the Cuntz relations:

$$x_i^*x_j=\begin{cases}1 & i=j\\0 & i\neq j.\end{cases}$$

We now follow the structure of an isometric n-tuple in any arbitrary W^*-algebra.

Let $x_1, \cdots, x_n$ be a sequence of isometries in $\mathcal{M}$. The following are equivalent:

(1) The Cuntz relations hold for the sequence $x_1, \cdots, x_n$.
(2) If i and j are distinct then $[x_i][x_j] = 0$.
(3) $\sum_{i=1}^{n}[x_i] \leq 1$.

Assume that $x_1, \cdots, x_n$ is a sequence of isometries satisfying the Cuntz relations. We take $F_{m,n}$ to be the set of all functions from $\{1, \cdots, m\}$ to $\{1, \cdots, n\}$. For given $f \in F$, we set

$$x_f := x_{f(1)} \cdots x_{f(m)} \ , \ x_f^* := x_{f(m)}^* \cdots x_{f(1)}^*.$$

We also put $x_0 = x_{f(0)} = 1$ and $F = \bigcup_{m\geq 0} F_m$. Let us consider $p = 1 - \sum[x_i]$. A direct calculation shows that $[x_f p][x_g p] = 0$ (see Lemma 1.6) for all distinct functions $f, g \in F$. It allows us to say that $p = 1 - \sum[x_i]$ plays the role of a wandering projection for the isometric n-tuple $(x_1, \cdots, x_n)$. Let us consider the total summation $s := \sum_{f\in F}[x_f p]$. If $s = 1$ then $x_1, \cdots, x_n$ is called an n-orthogonal shift.

Just like the case $n = 1$, an n-orthogonal shift has a unique wandering projection with the total summation 1. To see this, assume q is a wandering projection for an isometric n-tuple $x_1, \cdots, x_n$ with $\sum_{f\in F}[x_f q] = 1$. Then in the sense of w^*-topology

$$\begin{aligned} 1 &= \sum_{f\in F}[x_f q] \\ &= q + (\sum_{f(1)=1}[x_f q]) + \cdots + (\sum_{f(1)=n}[x_f q]) \\ &= q + x_1(\sum_{f\in F}[x_f q])x_1^* + \cdots + x_n(\sum_{f\in F}[x_f q])x_n^* \\ &= q + \sum_{i=1}^{n}[x_i]. \end{aligned}$$

We put $u := 1 - s$ and call (s, u) the Wold pair of $x_1, \cdots, x_n$. One may apply Lemma 1.6 to conclude that both projections u, s commute with all x_i's. To sum up:

Theorem 1.12. *Let $x_1, \cdots, x_n$ be an isometric n-tuple in $\mathcal{M}$. Then the Wold pair (s, u) is uniquely determined with the following properties*

(1) *s and u are mutually orthogonal and $s + u = 1$.*
(2) *Both projections s and u commute with all x_i's.*
(3) *$sx_1s, \cdots, sx_ns$ is an n-orthogonal unilateral shift in the W^*-algebra $s\mathcal{M}s$ and*

$$\sum ux_ix_i^*u = u$$

Finally we examine the method for doubly commuting isometries in W*-algebras. A pair of commuting isometries (x_1, x_2) is called double commuting if $x_ix_j^* = x_j^*x_i$. In [7] Slocinski obtained an analogous result of the Wold decomposition for a pair of doubly commuting isometries.

Theorem 1.13. *Let $x = (x_1, x_2)$ be a pair of doubly commuting isometries on the Hilbert space $\mathcal{H}$. Then there exists a unique decomposition*

$$\mathcal{H} = \mathcal{H}_{ss} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{uu}$$

where $\mathcal{H}_{ij}$ are joint x-reducing subspaces of $\mathcal{H}$. Moreover x_1 on $\mathcal{H}_{ij}$ is a shift if $i = 1$ and unitary if $i = u$ and x_2 is a shift if $j = s$ and unitary if $j = u$.

Let $x = (x_1, x_2)$ be a pair of doubly commuting isometries in W*-algebra $\mathcal{M}$. Let (s_1, u_1) be the Wold pair of x_1 (see Theorem 1.10). Both projections s_1 and u_1 commute with x_2, since $x_i x_j^* = x_j^* x_i$. We again apply Theorem 1.10 for isometries $s_1 x_2 s_1$ and $u_1 x_2 u_1$ in W*-algebras $s_1 \mathcal{M} s_1$ and $u_1 \mathcal{M} u_1$ respectively. We then obtain two Wold pairs as follow

$$\begin{cases} u_1 = w_{uu} \oplus w_{us} \\ s_1 = w_{su} \oplus w_{ss} \end{cases}$$

One may check all these projections $w_{\alpha\beta}$'s commute with both x_1 and x_2. Moreover

$$\begin{cases} w_{\alpha\beta} x_1 w_{\alpha\beta} \text{ is a unitary in } w_{\alpha\beta}\mathcal{M}w_{\alpha\beta} \text{ if } \alpha = u \\ w_{\alpha\beta} x_1 w_{\alpha\beta} \text{ is a unilateral shift in } w_{\alpha\beta}\mathcal{M}w_{\alpha\beta} \text{ if } \alpha = s \\ w_{\alpha\beta} x_2 w_{\alpha\beta} \text{ is a unitary in } w_{\alpha\beta}\mathcal{M}w_{\alpha\beta} \text{ if } \beta = u \\ w_{\alpha\beta} x_2 w_{\alpha\beta} \text{ is a unilateral shift in } w_{\alpha\beta}\mathcal{M}w_{\alpha\beta} \text{ if } \beta = s \end{cases}$$

2. Application

Let us have a look at the proof of lemmas 1.2 and 1.3 again. We observe that the following points are used.

1. Every W*-algebra is unital.
2. The lattice of projections in any W*-algebra is complete.
3. There is a partial ordered relation on the hermitian part of $\mathcal{A}$ and any positive element has unique square root.

In the current decomposition of an isometry in any W*-algebra, in addition to the above points, the following are also applied

4. Any monotone sequence of projections is w^*-convergent to a projection.
5. Assume $a \leq b$. For given $x \in \mathcal{M}$ we have then $x^* a x \leq x^* b x$.

Hence one may conclude fundamental decompositions theorems 0.1 and 0.2 in any dual topological *-algebra satisfying these properties. A well behaved of these structures are locally W*-algebras. We recall these structures.

In [3] Inoue introduced the notion of locally Hilbert space and the analogue of $B(\mathcal{H})$ as well. Let Λ be a directed index set and $\{\mathcal{H}_\alpha\}_{\alpha\in\Lambda}$ a family of Hilbert spaces such that $\mathcal{H}_\alpha$ is embedded in $\mathcal{H}_\beta$ where $\alpha \leq \beta$.

Let $\mathbf{H}$ be the direct limit of $\{\mathcal{H}_\alpha\}_{\alpha\in\Lambda}$

$$\mathbf{H} := \lim_{\rightarrow} \mathcal{H}_\alpha = \bigcup_{\alpha\in\Lambda} \mathcal{H}_\alpha.$$

Endow $\mathbf{H}$ with the inductive limit topology, that is the finest locally convex topology making the injections $\mathcal{H}_\alpha \hookrightarrow \mathbf{H}$ continuous. Then $\mathbf{H}$ is called a locally Hilbert space which is not a Hilbert space in general. Let $\iota_{\alpha\beta} : \mathcal{H}_\alpha \hookrightarrow \mathcal{H}_\beta$ be the canonical injection, and define $\mathbf{L}(\mathbf{H})$ to be the set of all continuous linear maps $T : \mathbf{H} \to \mathbf{H}$ for which $T_\beta \circ \iota_{\alpha\beta} = \iota_{\alpha\beta} \circ T_\alpha$, where $T_\alpha \in B(\mathcal{H}_\alpha)$ is the restriction of T to $\mathcal{H}_\alpha$. We have then that $\mathbf{L}(\mathbf{H})$ is the inverse limit of $\{\mathbf{B}(\mathcal{H}_\alpha)\}_{\alpha\in\Lambda}$ that is,

$$\mathbf{L}(\mathbf{H}) = \lim_{\leftarrow} B(\mathcal{H}_\alpha),$$

where $\mathbf{L}(\mathbf{H})$ is endowed with the inverse limit topology

$$\sigma := \lim_{\leftarrow} \sigma_\alpha \text{ with } \sigma_\alpha = \sigma(B(\mathcal{H}_\alpha), B(\mathcal{H}_\alpha)_*).$$

The topology σ on $\mathbf{L}(\mathbf{H})$ is called σ-weak topology. The σ-weakly closed *-subalgebras of $\mathbf{L}(\mathbf{H})$ are concrete locally W*-algebras [2]:

Theorem 2.1. *Every locally W*-algebra $\mathcal{M}$ endowed with the inverse limit topology σ coincides, within an isomorphism of topological *-algebras, with a σ-weakly closed *-subalgebra of $\boldsymbol{L}(\boldsymbol{H})$ for some locally Hilbert space $\boldsymbol{H}$.*

A continuous linear map $x : \mathbf{H} \to \mathbf{H}$ in $\mathbf{L}(\mathbf{H})$ is called an isometry (contraction) if x_α (the restriction of x on $\mathcal{H}_\alpha$) is an isometry (contraction) in $B(\mathcal{H}_\alpha)$. Equivalently, x is an isometry (contraction) if $x^*x = 1$ ($x^*x \leq 1$).

Inoue proved any locally C*-algebra satisfies in (3) and (5). To conclude, for Theorems 0.1 and 0.2, it is enough to show that items (1), (2) and (4) are also valid in locally W*-algebras. They are routine based on Theorem 2.1.

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Communicated by *Maria Fragoulopoulou*

Department of Mathematics, Persian Gulf University, Boushehr 75168, Iran
E-mail address: bagheri@mail.pgu.ac.ir , bagheri@gmail.com