ORDER PRESERVING PROPERTY FOR FUZZY VECTORS

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Abstract. In the present paper, the order preserving property for fuzzy vectors is investigated, and some classes of fuzzy vectors, which have the order preserving property and seem to be useful for applications, are constructed and proposed.

1 Introduction and preliminaries The concept of fuzzy vectors is an extension of the concept of fuzzy numbers, and it is useful for representing uncertain multidimensional quantities. Some properties of fuzzy vectors are investigated in [8]. Fuzzy linear programming problems involving oblique fuzzy vectors and fuzzy mathematical programming problems involving fuzzy vectors are considered in [2] and [7], respectively. When an ordering between any two fuzzy vectors is defined, the order preserving property for fuzzy vectors make fuzzy mathematical programming problems involving fuzzy vectors easy to solve. The order preserving property for fuzzy vectors is considered in the present paper. In the following, some basic notations and definitions are given.

For $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a, b[= \{x \in \mathbb{R} : a \leq x < b\}$, $]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$. In addition, we set $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}\}$ and $\mathbb{R}^n_- = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \leq \boldsymbol{0}\}$. Let $\mathbb{N}$ be the set of all natural numbers. For $S \subset \mathbb{R}^n$, we denote the closure, interior, and complement of S by $\mathrm{cl}(S)$, $\mathrm{int}(S)$, and S^c, respectively.

A fuzzy set $\widetilde{s}$ on $\mathbb{R}^n$ is identified with its membership function $\widetilde{s} : \mathbb{R}^n \to [0, 1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all fuzzy sets on $\mathbb{R}^n$. Let $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$. For $\alpha \in]0, 1]$, the set $[\widetilde{s}]_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) \geq \alpha\}$ is called the α-level set of $\widetilde{s}$. The 0-level set of $\widetilde{s}$ is defined as $[\widetilde{s}]_0 = \mathrm{cl}(\{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\})$, and $[\widetilde{s}]_0$ is called the support of $\widetilde{s}$. The fuzzy set $\widetilde{s}$ is said to be closed if $\widetilde{s}$ is upper semicontinuous on $\mathbb{R}^n$. The fuzzy set $\widetilde{s}$ is closed if and only if $[\widetilde{s}]_\alpha$ is closed for any $\alpha \in]0, 1]$. The fuzzy set $\widetilde{s}$ is said to be convex if $\widetilde{s}(\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \geq \min\{\widetilde{s}(\boldsymbol{x}), \widetilde{s}(\boldsymbol{y})\}$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$, that is, $\widetilde{s}$ is quasiconcave on $\mathbb{R}^n$. The fuzzy set $\widetilde{s}$ is convex if and only if $[\widetilde{s}]_\alpha$ is convex for any $\alpha \in]0, 1]$.

We define fuzzy vectors.

Definition 1 (See [7]). A fuzzy set $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ is called a fuzzy vector on $\mathbb{R}^n$ if $\widetilde{s}$ satisfies the following conditions:

(i) there exists a unique vector $\boldsymbol{c} \in \mathbb{R}^n$, called the center of $\widetilde{s}$, such that $\widetilde{s}(\boldsymbol{c}) = 1$,

(ii) $\widetilde{s}$ is a closed fuzzy set, that is, $\widetilde{s}$ is upper semicontinuous on $\mathbb{R}^n$,

(iii) $\widetilde{s}$ is a convex fuzzy set, that is, $\widetilde{s}$ is quasiconcave on $\mathbb{R}^n$,

(iv) $[\widetilde{s}]_0$ is bounded.

Let $\mathcal{FV}(\mathbb{R}^n)$ be the set of all fuzzy vectors on $\mathbb{R}^n$. In [7], a fuzzy mathematical programming problem with a fuzzy vector-valued objective function is considered. Assume

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that an ordering between any two fuzzy vectors is defined based on an ordering between two α-level sets of the fuzzy vectors for any $\alpha \in [0,1]$. Then, the fuzzy mathematical programming problem is equivalent to a mathematical programming problem with infinite many set-valued objective functions. If the fuzzy vector-valued objective function has the order preserving property, then the fuzzy mathematical programming problem is equivalent to a mathematical programming problem with finite many set-valued objective functions. Therefore, the order preserving property of the fuzzy vector-valued objective function make the fuzzy mathematical programmig problem easy to solve. The order preserving property of a fuzzy vector-valued function is equivalent to the order preserving property of a class of fuzzy vectors.

In the present paper, the order preserving property for fuzzy vectors is investigated, and some classes of fuzzy vectors, which have the order preserving property and seem to be useful for applications, are constructed and proposed.

For a crisp set $S \subset \mathbb{R}^n$, the function $c_S : \mathbb{R}^n \to \{0,1\}$ defined as

$$c_S(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in S, \\ 0 & \text{if } \boldsymbol{x} \notin S \end{cases}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$ is called the indicator function of S. A fuzzy set $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ can be represented as

$$\widetilde{s} = \sup_{\alpha \in]0,1]} \alpha c_{[\widetilde{s}]_\alpha}, \tag{1}$$

which is known as the decomposition theoren; see, for example, [1]. In order to construct fuzzy sets from classes of crisp sets, we set

$$\mathcal{S}(\mathbb{R}^n) = \{\{S_\alpha\}_{\alpha \in]0,1]} : S_\alpha \subset \mathbb{R}^n, \alpha \in]0,1], \text{ and } S_\beta \supset S_\gamma \text{ for } \beta, \gamma \in]0,1] \text{ with } \beta < \gamma\},$$

and define a mapping $M : \mathcal{S}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ as

$$M(\{S_\alpha\}_{\alpha \in]0,1]}) = \sup_{\alpha \in]0,1]} \alpha c_{S_\alpha} \tag{2}$$

for each $\{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$. When $\widetilde{s} = M(\{S_\alpha\}_{\alpha \in]0,1]})$ for $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$ and $\{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, $\widetilde{s}$ is called the fuzzy set generated by $\{S_\alpha\}_{\alpha \in]0,1]}$, and $\{S_\alpha\}_{\alpha \in]0,1]}$ is called the generator of $\widetilde{s}$. For $\{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$ and $\boldsymbol{x} \in \mathbb{R}^n$, it follows that

$$M(\{S_\alpha\}_{\alpha \in]0,1]})(\boldsymbol{x}) = \sup_{\alpha \in]0,1]} \alpha c_{S_\alpha}(\boldsymbol{x}) = \sup\{\alpha \in]0,1] : \boldsymbol{x} \in S_\alpha\},$$

where $\sup \emptyset = 0$. Based on the mapping M defined by (2), the decomposition theorem (1) can be represented as $\widetilde{s} = M(\{[\widetilde{s}]_\alpha\}_{\alpha \in]0,1]})$ for $\widetilde{s} \in \mathcal{F}(\mathbb{R}^n)$.

The following proposition shows a relationship between level sets of a fuzzy set and the generator of the fuzzy set.

Proposition 1 (See [3]). Let $\{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, and let $\widetilde{s} = M(\{S_\alpha\}_{\alpha \in]0,1]})$. Then, $[\widetilde{s}]_\alpha = \bigcap_{\beta \in]0,\alpha[} S_\beta$ for any $\alpha \in]0,1]$.

The remainder of the present paper is organized as follows. In Section 2, orderings of fuzzy sets are defined, and their properties are investigated. In Section 3, the concept of the order preserving property for fuzzy vectors is introduced. Then, in order to construct some classes of fuzzy vectors which have the order preserving property, properties of orderings of crisp sets are investigated when the crisp sets vary parametrically. In Section 4, some classes of fuzzy vectors which have the order preserving property are constructed and proposed. Finally, conclusions are presented in Section 5.

2 Ordering of fuzzy sets In this section, orderings of fuzzy sets are defined, and their properties are investigated.

In order to define orderings of fuzzy sets based on level sets of the fuzzy sets, orderings of crisp sets are defined as follows.

Definition 2 (See [5, 6, 7]). Let $A, B \subset \mathbb{R}^n$.

(i) We write $A \leq_S B$ or $B \geq_S A$ if $B \subset A + \mathbb{R}^n_+$ and $A \subset B + \mathbb{R}^n_-$.

(ii) We write $A <_S B$ or $B >_S A$ if $B \subset A + \mathrm{int}(\mathbb{R}^n_+)$ and $A \subset B + \mathrm{int}(\mathbb{R}^n_-)$.

The binary relation $\leq_S$ in Definition 2 is a pseudo order on the set of all subsets of $\mathbb{R}^n$.
The following proposition shows fundamental properties of $\leq_S$ and $<_S$ in Definition 2.

Proposition 2 (See [4]). Let $A, B \subset \mathbb{R}^n$.

(i) The relation $A \leq_S B$ holds if and only if the following two conditions (i-1) and (i-2) are satisfied: (i-1) for any $\boldsymbol{y} \in B$, there exists $\boldsymbol{x} \in A$ such that $\boldsymbol{x} \leq \boldsymbol{y}$; (i-2) for any $\boldsymbol{x} \in A$, there exists $\boldsymbol{y} \in B$ such that $\boldsymbol{x} \leq \boldsymbol{y}$.

(ii) The relation $A <_S B$ holds if and only if the following two conditions (ii-1) and (ii-2) are satisfied: (ii-1) for any $\boldsymbol{y} \in B$, there exists $\boldsymbol{x} \in A$ such that $\boldsymbol{x} < \boldsymbol{y}$; (ii-2) for any $\boldsymbol{x} \in A$, there exists $\boldsymbol{y} \in B$ such that $\boldsymbol{x} < \boldsymbol{y}$.

(iii) $A \leq_S A$.

(iv) If $A <_S B$, then $A \leq_S B$.

(v) It does not always hold that $A <_S B$ even if $A \leq_S B$.

(vi) If $A = \emptyset$ and $B \neq \emptyset$, then $A \not\leq_S B$, $B \not\leq_S A$, $A \not<_S B$, and $B \not<_S A$.

(vii) $A <_S A$ and $A \not<_S A$ are both possible.

(viii) $\emptyset \leq_S \emptyset$, $\emptyset <_S \emptyset$, $\mathbb{R}^n \leq_S \mathbb{R}^n$, $\mathbb{R}^n <_S \mathbb{R}^n$.

Based on the orderings of crisp sets given in Definition 2 and level sets of fuzzy sets, orderings of fuzzy sets are defined as follows.

Definition 3 (See [4]). Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$.

(i) We write $\widetilde{a} \preceq \widetilde{b}$ or $\widetilde{b} \succeq \widetilde{a}$ if $[\widetilde{a}]_\alpha \leq_S [\widetilde{b}]_\alpha$ for any $\alpha \in [0, 1]$.

(ii) We write $\widetilde{a} \prec \widetilde{b}$ or $\widetilde{b} \succ \widetilde{a}$ if $[\widetilde{a}]_\alpha <_S [\widetilde{b}]_\alpha$ for any $\alpha \in [0, 1]$.

The binary relation $\preceq$ in Definition 3 is a pseudo order on $\mathcal{F}(\mathbb{R}^n)$, and $\preceq$ is called the fuzzy max order. In [7], for $\widetilde{a}, \widetilde{b} \in \mathcal{FV}(\mathbb{R}^n)$, $\preceq_M$ and $\prec_M$ are defined as follows:

- we write $\widetilde{a} \preceq_M \widetilde{b}$ or $\widetilde{b} \succeq_M \widetilde{a}$ if $\inf([\widetilde{b}]_\alpha) \subset \inf([\widetilde{a}]_\alpha) + \mathbb{R}^n_+$ and $\sup([\widetilde{a}]_\alpha) \subset \sup([\widetilde{b}]_\alpha) + \mathbb{R}^n_-$ for any $\alpha \in [0, 1]$,

- we write $\widetilde{a} \prec_M \widetilde{b}$ or $\widetilde{b} \succ_M \widetilde{a}$ if $\inf([\widetilde{b}]_\alpha) \subset \inf([\widetilde{a}]_\alpha) + \mathrm{int}(\mathbb{R}^n_+)$ and $\sup([\widetilde{a}]_\alpha) \subset \sup([\widetilde{b}]_\alpha) + \mathrm{int}(\mathbb{R}^n_-)$ for any $\alpha \in [0, 1]$, where $\inf(S) = \{\boldsymbol{x} \in S : \text{there does not exist } \boldsymbol{y} \in S \text{ such that } \boldsymbol{y} \leq \boldsymbol{x} \text{ and } \boldsymbol{y} \neq \boldsymbol{x}\}$ and $\sup(S) = \{\boldsymbol{x} \in S : \text{there does not exist } \boldsymbol{y} \in S \text{ such that } \boldsymbol{y} \geq \boldsymbol{x} \text{ and } \boldsymbol{y} \neq \boldsymbol{x}\}$ for $S \subset \mathbb{R}^n$. The binary relation $\preceq_M$ is an extension of the fuzzy max order for fuzzy numbers given in [9].

The following proposition shows that $\preceq$ and $\prec$ in Definition 3 coincide with $\preceq_M$ and $\prec_M$ on $\mathcal{FV}(\mathbb{R}^n)$, respectively. Therefore, $\preceq$ and $\prec$ are extensions of $\preceq_M$ and $\prec_M$, respectively.

Proposition 3. Let $\widetilde{a}, \widetilde{b} \in \mathcal{FV}(\mathbb{R}^n)$.

(i) $\widetilde{a} \preceq \widetilde{b}$ if and only if $\widetilde{a} \preceq_M \widetilde{b}$.

(ii) $\widetilde{a} \prec \widetilde{b}$ if and only if $\widetilde{a} \prec_M \widetilde{b}$.

Proof. Let $\alpha \in [0,1]$. We set $A = [\widetilde{a}]_\alpha$ and $B = [\widetilde{b}]_\alpha$. Since A and B are nonempty compact convex sets, it follows that $\inf(A) \neq \emptyset$, $\sup(A) \neq \emptyset$, $\inf(B) \neq \emptyset$, and $\sup(B) \neq \emptyset$. In order to show (i) and (ii), it is sufficient to show that (i-1) $B \subset A + \mathbb{R}^n_+$ if and only if $\inf(B) \subset \inf(A) + \mathbb{R}^n_+$, (i-2) $A \subset B + \mathbb{R}^n_-$ if and only if $\sup(A) \subset \sup(B) + \mathbb{R}^n_-$, (ii-1) $B \subset A + \mathrm{int}(\mathbb{R}^n_+)$ if and only if $\inf(B) \subset \inf(A) + \mathrm{int}(\mathbb{R}^n_+)$, and (ii-2) $A \subset B + \mathrm{int}(\mathbb{R}^n_-)$ if and only if $\sup(A) \subset \sup(B) + \mathrm{int}(\mathbb{R}^n_-)$. We show only (i-1). (i-2), (ii-1), and (ii-2) can be shown in the similar way to (i-1). If $B \subset A + \mathbb{R}^n_+$, then $\inf(B) \subset B \subset A + \mathbb{R}^n_+ \subset \inf(A) + \mathbb{R}^n_+ + \mathbb{R}^n_+ = \inf(A) + \mathbb{R}^n_+$. If $\inf(B) \subset \inf(A) + \mathbb{R}^n_+$, then $B \subset \inf(B) + \mathbb{R}^n_+ \subset \inf(A) + \mathbb{R}^n_+ + \mathbb{R}^n_+ = \inf(A) + \mathbb{R}^n_+ \subset A + \mathbb{R}^n_+$. $\square$

3 Order preserving property In this section, the concept of the order preserving property for fuzzy vectors is introduced. Then, in order to construct some classes of fuzzy vectors which have the order preserving property, properties of the orderings of crisp sets are investigated when the crisp sets vary parametrically.

The orderings of two fuzzy sets in Definition 3 are defined by infinite many orderings of level sets of the fuzzy sets. If finite many orderings of level sets of two fuzzy sets imply the orderings of the fuzzy sets, then it makes the orderings of fuzzy sets easy to deal with for applications. Such property is called the order preserving property, and defined for fuzzy vectors as follows.

Definition 4. (i) Fuzzy vectors $\widetilde{a}, \widetilde{b} \in \mathcal{FV}(\mathbb{R}^n)$ are said to be order preserving on $\mathbb{R}^n$ if $[\widetilde{a}]_0 \leq_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 \leq_S [\widetilde{b}]_1$ imply $\widetilde{a} \preceq \widetilde{b}$, or if $[\widetilde{a}]_0 \geq_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 \geq_S [\widetilde{b}]_1$ imply $\widetilde{a} \succeq \widetilde{b}$.

(ii) A class of fuzzy vectors, $\mathcal{G} \subset \mathcal{FV}(\mathbb{R}^n)$, is said to be order preserving on $\mathbb{R}^n$ if any $\widetilde{a}, \widetilde{b} \in \mathcal{G}$ are order preserving on $\mathbb{R}^n$.

Definition 5. (i) Fuzzy vectors $\widetilde{a}, \widetilde{b} \in \mathcal{FV}(\mathbb{R}^n)$ are said to be strictly order preserving on $\mathbb{R}^n$ if $[\widetilde{a}]_0 <_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 <_S [\widetilde{b}]_1$ imply $\widetilde{a} \prec \widetilde{b}$, or if $[\widetilde{a}]_0 >_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 >_S [\widetilde{b}]_1$ imply $\widetilde{a} \succ \widetilde{b}$.

(ii) A class of fuzzy vectors, $\mathcal{G} \subset \mathcal{FV}(\mathbb{R}^n)$, is said to be strictly order preserving on $\mathbb{R}^n$ if any $\widetilde{a}, \widetilde{b} \in \mathcal{G}$ are strictly order preserving on $\mathbb{R}^n$.

In the following, in order to construct some classes of fuzzy vectors which have the order preserving property, properties of the orderings of crisp sets are investigated when the crisp sets vary parametrically.

The following proposition shows properties of the orderings of crisp sets when the crisp sets vary parametrically.

Proposition 4. Let $A, B \subset \mathbb{R}^n$, and let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. In addition, let $r : [0,1] \to [0,1]$ be a monotone decreasing function. Assume that $r(0) = 1$ and $r(1) = 0$. We set $F(\alpha) = r(\alpha)A + \boldsymbol{a}$ and $G(\alpha) = r(\alpha)B + \boldsymbol{b}$ for each $\alpha \in [0, 1]$.

(i) If $F(0) \leq_S G(0)$ and $F(1) \leq_S G(1)$, then $F(\alpha) \leq_S G(\alpha)$ for any $\alpha \in [0, 1]$.

(ii) If $F(0) <_S G(0)$ and $F(1) <_S G(1)$, then $F(\alpha) <_S G(\alpha)$ for any $\alpha \in [0, 1]$.

Proof. We show only (i). (ii) can be shown in the similar way to (i). ¿From Proposition 2, if $A = \emptyset$ or $B = \emptyset$, then the conclusion is obtained. Suppose that $A \neq \emptyset$ and $B \neq \emptyset$. Let $\alpha \in [0, 1]$. Since $F(0) \leq_S G(0)$ and $F(1) \leq_S G(1)$, it follows that $B + \boldsymbol{b} \subset A + \boldsymbol{a} + \mathbb{R}^n_+$, $A + \boldsymbol{a} \subset B + \boldsymbol{b} + \mathbb{R}^n_-$, and $\boldsymbol{a} \leq \boldsymbol{b}$. Though it needs to show that (i-1) $r(\alpha)B + \boldsymbol{b} \subset r(\alpha)A + \boldsymbol{a} + \mathbb{R}^n_+$ and (i-2) $r(\alpha)A + \boldsymbol{a} \subset r(\alpha)B + \boldsymbol{b} + \mathbb{R}^n_-$, we show only (i-1). (i-2) can be shown in the similar way to (i-1). Let $\boldsymbol{x} \in r(\alpha)B + \boldsymbol{b}$. Then, there exists $\boldsymbol{y} \in B$ such that $\boldsymbol{x} = r(\alpha)\boldsymbol{y} + \boldsymbol{b}$. Since $\boldsymbol{a} \leq \boldsymbol{b}$, there exists $\boldsymbol{d}_1 \in \mathbb{R}^n_+$ such that $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{d}_1$. Since $B + \boldsymbol{b} \subset A + \boldsymbol{a} + \mathbb{R}^n_+$, there exist $\boldsymbol{z} \in A$ and $\boldsymbol{d}_2 \in \mathbb{R}^n_+$ such that $\boldsymbol{y} + \boldsymbol{b} = \boldsymbol{z} + \boldsymbol{a} + \boldsymbol{d}_2$. Therefore, we have $\boldsymbol{x} = r(\alpha)\boldsymbol{y} + \boldsymbol{b} = r(\alpha)(\boldsymbol{y} + \boldsymbol{b}) + (1 - r(\alpha))\boldsymbol{b} = r(\alpha)(\boldsymbol{z} + \boldsymbol{a} + \boldsymbol{d}_2) + (1 - r(\alpha))(\boldsymbol{a} + \boldsymbol{d}_1) = r(\alpha)\boldsymbol{z} + \boldsymbol{a} + (r(\alpha)\boldsymbol{d}_2 + (1 - r(\alpha))\boldsymbol{d}_1) \in r(\alpha)A + \boldsymbol{a} + \mathbb{R}^n_+$. □

The following proposition shows sufficient conditions for generated fuzzy sets by the mapping M defined by (2) to be fuzzy vectors.

Proposition 5. Let $A \subset \mathbb{R}^n$ be a convex set containing the origin, and let $\boldsymbol{a} \in \mathbb{R}^n$. In addition, let $r : [0, 1] \to [0, 1]$ be a monotone decreasing function. We set $F(\alpha) = r(\alpha)A + \boldsymbol{a}$ for each $\alpha \in [0, 1]$, and $\widetilde{s} = M(\{F(\alpha)\}_{\alpha \in]0,1]})$.

(i) $\widetilde{s}$ is a convex fuzzy set.

(ii) If A is a closed set, then $\widetilde{s}$ is a closed fuzzy set.

(iii) If A is a compact set, $r(1) = 0$, and r is left-continuous at 1, then $\widetilde{s} \in \mathcal{FV}(\mathbb{R}^n)$.

Proof. (i) and (ii) follow from Proposition 1. We show (iii). Since A is a closed convex set, $\widetilde{s}$ is a closed convex fuzzy set from (i) and (ii).

We show that $\{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\}$ is bounded. For $\boldsymbol{x} \in \mathbb{R}^n \setminus (A + \boldsymbol{a})$, since $F(\alpha) \subset F(0) = r(0)A + \boldsymbol{a} \subset A + \boldsymbol{a}$ for any $\alpha \in [0, 1]$, it follows that $\widetilde{s}(\boldsymbol{x}) = \sup_{\alpha \in]0,1]} \alpha c_{F(\alpha)}(\boldsymbol{x}) = 0$. Since $(A + \boldsymbol{a})^c \subset \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) = 0\}$, it follows that $\{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\} \subset A + \boldsymbol{a}$. Therefore, $\{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\}$ is bounded.

For $\boldsymbol{x} \in \mathbb{R}^n$, we show that $\widetilde{s}(\boldsymbol{x}) = 1$ if and only if $\boldsymbol{x} = \boldsymbol{a}$. Since $\boldsymbol{a} = r(\alpha)\boldsymbol{0} + \boldsymbol{a} \in r(\alpha)A + \boldsymbol{a} = F(\alpha)$ for any $\alpha \in [0, 1]$, it follows that $\widetilde{s}(\boldsymbol{a}) = \sup_{\alpha \in]0,1]} \alpha c_{F(\alpha)}(\boldsymbol{a}) = 1$. For $\boldsymbol{b} \in \mathbb{R}^n$, we show that $\boldsymbol{b} \neq \boldsymbol{a}$ implies $\widetilde{s}(\boldsymbol{b}) < 1$. Since A is bounded, there exists $L > 0$ such that $A \subset B_L = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq L\}$, where $\| \cdot \|$ is the Euclidean norm. Since $r(\alpha)A \subset r(\alpha)B_L$ for any $\alpha \in [0, 1]$, it follows that $F(\alpha) = r(\alpha)A + \boldsymbol{a} \subset r(\alpha)B_L + \boldsymbol{a}$ for any $\alpha \in [0, 1]$. We set $\beta = \|\boldsymbol{b} - \boldsymbol{a}\|$. Since $r(1) = 0$ and r is left-continuous at 1, there exists $\delta > 0$ such that $|\alpha - 1| < \delta$ and $\alpha \in [0, 1]$ imply $r(\alpha) < \frac{\beta}{L}$. Thus, there exists $\alpha_0 \in]0, 1[$ such that $\alpha \in [\alpha_0, 1]$ implies $\boldsymbol{b} \notin r(\alpha)B_L + \boldsymbol{a}$, and then $\boldsymbol{b} \notin F(\alpha) = r(\alpha)A + \boldsymbol{a}$ for any $\alpha \in [\alpha_0, 1]$. Therefore, we have $\widetilde{s}(\boldsymbol{b}) = \sup_{\alpha \in]0,1]} \alpha c_{F(\alpha)}(\boldsymbol{b}) \leq \alpha_0 < 1$. □

The following proposition shows a property of the ordering of crisp sets decreasing parametrically.

Proposition 6. Let $F(\beta)$, $G(\beta) \subset \mathbb{R}^n$, $\beta \in]0, 1]$ be closed sets. Assume that $F(\gamma) \supset F(\delta)$ and $G(\gamma) \supset G(\delta)$ for $\gamma, \delta \in]0, 1]$ with $\gamma \leq \delta$, and that $\cup_{\beta \in]0,1]} F(\beta)$ and $\cup_{\beta \in]0,1]} G(\beta)$ are bounded. Let $\alpha \in]0, 1]$. Assume that $\cap_{\beta \in]0,\alpha[} F(\beta) \neq \emptyset$ and $\cap_{\beta \in]0,\alpha[} G(\beta) \neq \emptyset$. If $F(\beta) \leq_S G(\beta)$ for any $\beta \in]0, \alpha[$, then $\cap_{\beta \in]0,\alpha[} F(\beta) \leq_S \cap_{\beta \in]0,\alpha[} G(\beta)$.

Proof. For any $\beta \in \,]0,\alpha[$, since $F(\beta) \leq_S G(\beta)$, it follows that $G(\beta) \subset F(\beta) + \mathbb{R}^n_+$ and $F(\beta) \subset G(\beta) + \mathbb{R}^n_-$. Though it needs to show that (i) $\cap_{\beta\in]0,\alpha[}G(\beta) \subset \cap_{\beta\in]0,\alpha[}F(\beta) + \mathbb{R}^n_+$ and (ii) $\cap_{\beta\in]0,\alpha[}F(\beta) \subset \cap_{\beta\in]0,\alpha[}G(\beta) + \mathbb{R}^n_-$, we show only (i). (ii) can be shown in the similar way to (i). Since $G(\beta) \subset F(\beta) + \mathbb{R}^n_+$ for any $\beta \in \,]0,\alpha[$, it follows that $\cap_{\beta\in]0,\alpha[}G(\beta) \subset \cap_{\beta\in]0,\alpha[} \left(F(\beta) + \mathbb{R}^n_+\right)$. Thus, it is sufficient to show that $\cap_{\beta\in]0,\alpha[} \left(F(\beta) + \mathbb{R}^n_+\right) \subset \cap_{\beta\in]0,\alpha[} F(\beta) + \mathbb{R}^n_+$. Let $\boldsymbol{x} \in \cap_{\beta\in]0,\alpha[} \left(F(\beta) + \mathbb{R}^n_+\right)$. For each $\beta \in \,]0,\alpha[$, there exist $\boldsymbol{y}_\beta \in F(\beta)$ and $\boldsymbol{d}_\beta \in \mathbb{R}^n_+$ such that $\boldsymbol{x} = \boldsymbol{y}_\beta + \boldsymbol{d}_\beta$. Fix any $\{\beta_k\} \subset \,]0,\alpha[$ with $\beta_k \to \alpha$. Since $\{\boldsymbol{y}_{\beta_k}\} \subset \cup_{\beta\in]0,1]}F(\beta)$ is bounded, without loss of generality, suppose that $\boldsymbol{y}_{\beta_k} \to \boldsymbol{y}_0$ for some $\boldsymbol{y}_0 \in \mathbb{R}^n$. Then, it follows that $\boldsymbol{d}_{\beta_k} = \boldsymbol{x} - \boldsymbol{y}_{\beta_k} \to \boldsymbol{x} - \boldsymbol{y}_0 \in \mathbb{R}^n_+$. For any $\beta \in \,]0,\alpha[$, there exists $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies $\beta_k \in \,]\beta,\alpha[$, and it follows that $\{\boldsymbol{y}_{\beta_k}\}_{k\geq k_0} \subset F(\beta)$, and that $\boldsymbol{y}_{\beta_k} \to \boldsymbol{y}_0 \in F(\beta)$ since $F(\beta)$ is a closed set. Since $\boldsymbol{y}_0 \in F(\beta)$ for any $\beta \in \,]0,\alpha[$, we have $\boldsymbol{x} = \boldsymbol{y}_0 + (\boldsymbol{x} - \boldsymbol{y}_0) \in \cap_{\beta\in]0,\alpha[}F(\beta) + \mathbb{R}^n_+$. □

The following proposition shows a property of 0-level sets of generated fuzzy sets by the mapping M defined by (2).

Proposition 7. Let $A \subset \mathbb{R}^n$ be a compact convex set containing the origin, and let $\boldsymbol{a} \in \mathbb{R}^n$. In addition, let $r : [0,1] \to [0,1]$ be a monotone decreasing function. Assume that $r(0) = 1$, and that r is right-continuous at 0. We set $F(\alpha) = r(\alpha)A + \boldsymbol{a}$ for each $\alpha \in [0,1]$, and $\widetilde{s} = M\left(\{F(\alpha)\}_{\alpha\in]0,1]}\right)$. Then, $A + \boldsymbol{a} = [\widetilde{s}]_0$.

Proof. Since $F(\alpha) = r(\alpha)A + \boldsymbol{a} \subset F(0) = A + \boldsymbol{a}$ for any $\alpha \in [0,1]$, it follows that $\widetilde{s}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \mathbb{R}^n \setminus (A + \boldsymbol{a})$. Since $(A + \boldsymbol{a})^c \subset \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) = 0\}$, it follows that $A + \boldsymbol{a} \supset \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\}$. Therefore, we have $A + \boldsymbol{a} \supset [\widetilde{s}]_0$ since $A + \boldsymbol{a}$ is a closed set.

Let $\boldsymbol{x}_0 \in A + \boldsymbol{a}$. Then, there exists $\boldsymbol{y}_0 \in A$ such that $\boldsymbol{x}_0 = \boldsymbol{y}_0 + \boldsymbol{a}$. If $\boldsymbol{y}_0 = \boldsymbol{0}$, then $\boldsymbol{x}_0 = \boldsymbol{a} \in [\widetilde{s}]_0$. Thus, suppose that $\boldsymbol{y}_0 \neq \boldsymbol{0}$. We set $\lambda_0 = \max\{\lambda \geq 0 : \lambda\boldsymbol{y}_0 \in A\} \geq 1$.

Suppose that $\lambda_0 > 1$, and fix any sufficiently small $\delta > 0$. Since r is right-continuous at 0, $\alpha \in [0,\delta[$ implies $1 - r(\alpha) < 1 - \frac{1}{\lambda_0}$. For any $\alpha \in [0,\delta[$, it follows that $0 < \frac{1}{r(\alpha)\lambda_0} < 1$ and $r(\alpha)\lambda_0\boldsymbol{y}_0 \in r(\alpha)A$, and that $\boldsymbol{y}_0 = \frac{1}{r(\alpha)\lambda_0} \cdot r(\alpha)\lambda_0\boldsymbol{y}_0 \in r(\alpha)A$, and that $\boldsymbol{x}_0 = \boldsymbol{y}_0 + \boldsymbol{a} \in r(\alpha)A + \boldsymbol{a} = F(\alpha)$, and that $c_{F(\alpha)}(\boldsymbol{x}_0) = 1$. Therefore, since $\widetilde{s}(\boldsymbol{x}_0) = \sup_{\alpha\in]0,1]} \alpha c_{F(\alpha)}(\boldsymbol{x}_0) \geq \delta > 0$, we have $\boldsymbol{x}_0 \in [\widetilde{s}]_0$.

Suppose that $\lambda_0 = 1$. By the same arguments as in the case $\lambda_0 > 1$, it can be seen that $\widetilde{s}(\lambda\boldsymbol{y}_0 + \boldsymbol{a}) > 0$ for any $\lambda \in \,]0,1[$. Choose any $\{\lambda_k\} \subset \,]0,1[$ with $\lambda_k \to 1$. Since $\{\lambda_k\boldsymbol{y}_0 + \boldsymbol{a}\} \subset \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{s}(\boldsymbol{x}) > 0\}$, we have $\lambda_k\boldsymbol{y}_0 + \boldsymbol{a} \to \boldsymbol{y}_0 + \boldsymbol{a} = \boldsymbol{x}_0 \in [\widetilde{s}]_0$. □

The following proposition shows a property of crisp sets decreasing parametrically.

Proposition 8. Let $A \subset \mathbb{R}^n$ be a compact convex set containing the origin, and let $\boldsymbol{a} \in \mathbb{R}^n$. In addition, let $r : [0,1] \to [0,1]$ be a monotone decreasing function. We set $F(\beta) = r(\beta)A + \boldsymbol{a}$ for each $\beta \in [0,1]$. If r is left-continuous at $\alpha \in \,]0,1]$, then $F(\alpha) = \cap_{\beta\in]0,\alpha[}F(\beta)$.

Proof. It follows that $F(\alpha) = r(\alpha)A + \boldsymbol{a} \subset \cap_{\beta\in]0,\alpha[}(r(\beta)A + \boldsymbol{a}) = \cap_{\beta\in]0,\alpha[}F(\beta)$. In order to show that $r(\alpha)A + \boldsymbol{a} \supset \cap_{\beta\in]0,\alpha[}(r(\beta)A + \boldsymbol{a})$, suppose that $\boldsymbol{x}_0 \in \cap_{\beta\in]0,\alpha[}(r(\beta)A + \boldsymbol{a})$ and $\boldsymbol{x}_0 \notin r(\alpha)A + \boldsymbol{a}$. Since $\boldsymbol{x}_0 \in \cap_{\beta\in]0,\alpha[}(r(\beta)A + \boldsymbol{a}) \subset A + \boldsymbol{a}$, it follows that $\boldsymbol{x}_0 - \boldsymbol{a} \in A$. Since $\boldsymbol{x}_0 \notin r(\alpha)A + \boldsymbol{a}$, it follows that $\boldsymbol{x}_0 - \boldsymbol{a} \notin r(\alpha)A$. Thus, it follows that $r(\alpha) < 1$ and $\boldsymbol{x}_0 - \boldsymbol{a} \neq \boldsymbol{0}$. We set $\lambda_0 = \max\{\lambda \geq 0 : \lambda(\boldsymbol{x}_0 - \boldsymbol{a}) \in A\} \geq 1$. Then, since $\boldsymbol{x}_0 - \boldsymbol{a} \in \frac{1}{\lambda_0}A$ and $\boldsymbol{x}_0 - \boldsymbol{a} \notin r(\alpha)A$, it follows that $\frac{1}{\lambda_0} > r(\alpha)$. Fix any sufficiently small $\delta > 0$. Since r is left-continuous at α, $\beta \in \,]\alpha - \delta, \alpha]$ implies $r(\beta) - r(\alpha) < \frac{1}{\lambda_0} - r(\alpha)$. Fix any $\beta_0 \in \,]\alpha - \delta, \alpha[$. Then, it follows that $r(\beta_0) < \frac{1}{\lambda_0}$. If $\boldsymbol{x}_0 - \boldsymbol{a} \notin r(\beta_0)A$, then it follows that $\boldsymbol{x}_0 \notin r(\beta_0)A + \boldsymbol{a}$, and that $\boldsymbol{x}_0 \notin \cap_{\beta \in]0,\alpha[}(r(\beta)A + \boldsymbol{a})$, wihch is a contradiction. Thus, in order to show that $\boldsymbol{x}_0 - \boldsymbol{a} \notin r(\beta_0)A$, suppose that $\boldsymbol{x}_0 - \boldsymbol{a} \in r(\beta_0)A$. Then, since $\lambda_0(\boldsymbol{x}_0 - \boldsymbol{a}) \in \lambda_0 r(\beta_0)A$ and $\lambda_0 r(\beta_0) < 1$, for sufficiently small $\varepsilon > 0$, it follows that $(1+\varepsilon)\lambda_0(\boldsymbol{x}_0 - \boldsymbol{a}) \in (1+\varepsilon)\lambda_0 r(\beta_0)A \subset A$ and $\lambda_0 < (1+\varepsilon)\lambda_0$, which contradict the definition of λ_0. □

4 Main results In this section, based on the mapping M defined by (2), some classes of fuzzy vectors which have the order preserving property are constructed and proposed.

The following proposition shows sufficient conditions for generated fuzzy vectors by the mapping M defined by (2) to have the order preserving property.

Proposition 9. Let $A, B \subset \mathbb{R}^n$ be compact convex sets containing the origin, and let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. In addition, let $r : [0,1] \to [0,1]$ be a monotone decreasing function. Assume that $r(0) = 1$ and $r(1) = 0$, and that r is right-continuous at 0 and left-continuous at 1. We set $F(\alpha) = r(\alpha)A + \boldsymbol{a}$ and $G(\alpha) = r(\alpha)B + \boldsymbol{b}$ for each $\alpha \in [0,1]$, and $\widetilde{a} = M\left(\{F(\alpha)\}_{\alpha \in]0,1]}\right)$ and $\widetilde{b} = M\left(\{G(\alpha)\}_{\alpha \in]0,1]}\right)$.

(i) If $[\widetilde{a}]_0 \leq_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 \leq_S [\widetilde{b}]_1$, then $\widetilde{a} \preceq \widetilde{b}$.

(ii) Assume that r is left-continuous. If $[\widetilde{a}]_0 <_S [\widetilde{b}]_0$ and $[\widetilde{a}]_1 <_S [\widetilde{b}]_1$, then $\widetilde{a} \prec \widetilde{b}$.

Proof. (i) It follows that $A + \boldsymbol{a} \leq_S B + \boldsymbol{b}$ from Proposition 7, and that $\boldsymbol{a} \leq \boldsymbol{b}$ from Proposition 5. ¿From Proposition 4, it follows that $r(\alpha)A + \boldsymbol{a} \leq_S r(\alpha)B + \boldsymbol{b}$ for any $\alpha \in [0,1]$. Since $[\widetilde{a}]_\alpha = \cap_{\beta \in]0,\alpha[}(r(\beta)A + \boldsymbol{a})$ and $[\widetilde{b}]_\alpha = \cap_{\beta \in]0,\alpha[}(r(\beta)B + \boldsymbol{b})$ for any $\alpha \in]0,1]$ from Proposition 1, it follows that $[\widetilde{a}]_\alpha \leq_S [\widetilde{b}]_\alpha$ for any $\alpha \in [0,1]$ from Proposition 6. Therefore, we have $\widetilde{a} \preceq \widetilde{b}$.

(ii) It follows that $A + \boldsymbol{a} <_S B + \boldsymbol{b}$ from Proposition 7, and that $\boldsymbol{a} < \boldsymbol{b}$ from Proposition 5. ¿From Proposition 4, it follows that $r(\alpha)A + \boldsymbol{a} <_S r(\alpha)B + \boldsymbol{b}$ for any $\alpha \in [0,1]$. Since $[\widetilde{a}]_\alpha = \bigcap_{\beta \in]0,\alpha[}(r(\beta)A + \boldsymbol{a}) = r(\alpha)A + \boldsymbol{a}$ and $[\widetilde{b}]_\alpha = \bigcap_{\beta \in]0,\alpha[}(r(\beta)B + \boldsymbol{b}) = r(\alpha)B + \boldsymbol{b}$ for any $\alpha \in]0,1]$ from Propositions 1 and 8, it follows that $[\widetilde{a}]_\alpha <_S [\widetilde{b}]_\alpha$ for any $\alpha \in [0,1]$. Therefore, we have $\widetilde{a} \prec \widetilde{b}$. □

In the following, some classes of fuzzy vectors which have the order preserving property are constructed based on the obtained results. Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all compact convex subsets of $\mathbb{R}^n$ containing the origin, and let $\mathcal{R}$ be the set of all monotone decreasing functions from $[0,1]$ to $[0,1]$. We set

$$\begin{aligned}
\mathcal{R}_1 &= \{r \in \mathcal{R} : r(0) = 1, r(1) = 0, \\
&\qquad\qquad \text{and } r \text{ is right-continuous at 0 and left-continuous at 1 }\}, \\
\mathcal{R}_2 &= \{r \in \mathcal{R}_1 : \ r \text{ is left-continuous }\}.
\end{aligned}$$

In addition, we set

$$\begin{aligned}
\mathcal{S}^r(\mathbb{R}^n) &= \{\{r(\alpha)A + \boldsymbol{a}\}_{\alpha \in]0,1]} : A \in \mathcal{C}(\mathbb{R}^n), \boldsymbol{a} \in \mathbb{R}^n\}, \\
\mathcal{FV}_1^r(\mathbb{R}^n) &= \{M(\{S_\alpha\}_{\alpha \in]0,1]}) : \{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}^r(\mathbb{R}^n)\} = M(\mathcal{S}^r(\mathbb{R}^n))
\end{aligned}$$

for each $r \in \mathcal{R}_1$, and

$$\mathcal{FV}_2^r(\mathbb{R}^n) = \{M(\{S_\alpha\}_{\alpha \in]0,1]}) : \{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}^r(\mathbb{R}^n)\} = M(\mathcal{S}^r(\mathbb{R}^n))$$

for each $r \in \mathcal{R}_2$.

The following proposition shows that the classes of fuzzy sets constructed in the above are classes of fuzzy vectors which have the order preserving property.

Proposition 10. (i) $\mathcal{FV}_1^r(\mathbb{R}^n) \subset \mathcal{FV}(\mathbb{R}^n)$ for any $r \in \mathcal{R}_1$, and $\mathcal{FV}_2^r(\mathbb{R}^n) \subset \mathcal{FV}(\mathbb{R}^n)$ for any $r \in \mathcal{R}_2$.

(ii) $\mathcal{FV}_1^r(\mathbb{R}^n)$ is order preserving for any $r \in \mathcal{R}_1$.

(iii) $\mathcal{FV}_2^r(\mathbb{R}^n)$ is strictly order preserving for any $r \in \mathcal{R}_1$.

Proof. (i) follows from Proposition 5. (ii) and (iii) follow from Proposition 9. □

5 Conclusions In the present paper, we dealt with orderings of fuzzy vectors. When orderings of two fuzzy vectors were defined based on orderings of level sets of the fuzzy vectors, it needed to consider infinite many orderings of level sets of the fuzzy vectors. If finite many orderings of level sets of two fuzzy vectors imply the orderings of the fuzzy vectors, then it makes the orderings of fuzzy vectors easy to deal with for applications. Such property was defined as the order preserving property, and the order preserving property for fuzzy vectors was investigated. Based on classes of crisp sets decreasing parametrically, some classes of fuzzy vectors, which had the order preserving property and seemed to be useful for applications, were constructed and proposed.

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