HOPF HYPERSURFACES ADMITTING ϕ-INVARIANT RICCI TENSORS IN A NONFLAT COMPLEX SPACE FORM

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Abstract. We investigate real hypersurfaces with ϕ-invariant Ricci tensors in a non-flat complex space form $\widetilde{M}_n(c)$. In particular, we classify Hopf hypersurfaces having weakly ϕ-invariant Ricci tensor in $\widetilde{M}_n(c)$. In addition, we verify the non-existence of Hopf hypersurfaces with strongly ϕ-invariant Ricci tensor in $\widetilde{M}_n(c)$ and the non-existence of ruled real hypersurfaces with weakly ϕ-invariant Ricci tensor in $\widetilde{M}_n(c)$.

1 Introduction We denote by $\widetilde{M}_n(c)\,(n \geqq 2)$ an n-dimensional non-flat complex space form. Namely, $\widetilde{M}_n(c)$ is congruent to either a complex projective space of constant holomorphic sectional curvature $c(>0)$ or a complex hyperbolic space of constant holomorphic sectional curvature $c(<0)$. Let M^{2n-1} be a real hypersurface in $\widetilde{M}_n(c)$. It is well-known that real hypersurfaces in $\widetilde{M}_n(c)$ admitting almost contact metric structure (ϕ, ξ, η, g) induced from Kähler structure J of $\widetilde{M}_n(c)$ (see Section 2). From the viewpoint of contact geometry, real hypersurfaces are interesting in $\widetilde{M}_n(c)$. It is also well-known that there exist no *Einstein* real hypersurfaces in $\widetilde{M}_n(c)$. Thus, many geometers studied its weaker conditions and conditions related to the Ricci tensor of M^{2n-1} (See [3], [5], [7], [10], [11], [14], [15]).

In this paper, we focus on the structure tensor ϕ of M^{2n-1} and the Ricci tensor of M^{2n-1}. We define the notion of *ϕ-invariant Ricci tensor* of M^{2n-1} (for detail, see Section 5). This notion is divided into *strongly* ϕ-invariance of the Ricci tensor of M^{2n-1} or *weakly* ϕ-invariance of the Ricci tensor of M^{2n-1}. In particular, the latter is a weaker condition of Einstein real hypersurfaces.

In the theory of real hypersurfaces in $\widetilde{M}_n(c)$, Hopf hypersurfaces (namely, real hypersurfaces such that the characteristic vector ξ is a principal curvature vector at its each point) play an important role. We investigate Hopf hypersurfaces M^{2n-1} with ϕ-invariant Ricci tensors of M^{2n-1} in $\widetilde{M}_n(c)$. Note that there exist real hypersurfaces M^{2n-1} with weakly ϕ-invariant Ricci tensor of M^{2n-1} in $\widetilde{M}_n(c)$. In fact, the family of such real hypersurfaces includes real hypersurfaces of type (A) in $\widetilde{M}_n(c)$ (Theorem 1). It is known that real hypersurfaces of type (A) in $\widetilde{M}_n(c)$ have many nice geometric properties.

The purpose of this paper is to determine Hopf hypersurfaces M^{2n-1} having weakly ϕ-invariant Ricci tensor of M^{2n-1} in $\widetilde{M}_n(c)$. To do this, we shall prove that weakly ϕ-invariance of the Ricci tensor of M^{2n-1} is equivalent to the commutativity of the structure tensor ϕ of M^{2n-1} and the Ricci tensor Q of type (1, 1) of M^{2n-1} (that is, $\phi Q = Q\phi$) on a Hopf hypersurface M^{2n-1} in $\widetilde{M}_n(c)$. In addition, we shall show the non-existence of Hopf hypersurfaces M^{2n-1} with strongly ϕ-invariant Ricci tensor of M^{2n-1} in $\widetilde{M}_n(c)$.

In general, weakly ϕ-invariance of the Ricci tensor is *not* equivalent to the commutativity of the structure tensor ϕ and the Ricci tensor Q of type (1, 1) on a non-Hopf hypersurface in

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$\widetilde{M}_n(c)$. It is natural to consider non-Hopf hypersurfaces M^{2n-1} having weakly ϕ-invariant Ricci tensor of M^{2n-1} in $\widetilde{M}_n(c)$. Ruled real hypersurfaces are typical non-Hopf hypersurfaces in $\widetilde{M}_n(c)$. So, we shall also show the non-existence of ruled real hypersurfaces M^{2n-1} with weakly ϕ-invariant Ricci tensor of M^{2n-1} in $\widetilde{M}_n(c)$.

2 Preliminaries Let M^{2n-1} be a real hypersurface with a unit local vector field $\mathcal{N}$ of a complex n-dimensional non-flat complex space form $\widetilde{M}_n(c)$ of constant holomorphic sectional curvature c. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M^{2n-1} are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX,\, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for vector fields X and Y tangent to M^{2n-1}, where g denotes the induced metric from the standard Riemannian metric of $\widetilde{M}_n(c)$ and A is the shape operator of M^{2n-1} in $\widetilde{M}_n(c)$. (2.1) is called *Gauss's formula*, and (2.2) is called *Weingarten's formula*. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal vectors* of M^{2n-1} in $\widetilde{M}_n(c)$, respectively.

It is known that M^{2n-1} admits an *almost contact metric structure* (ϕ, ξ, η, g) induced from the Kähler structure J of $\widetilde{M}_n(c)$. *The characteristic vector field* ξ of M^{2n-1} is defined as $\xi = -J\mathcal{N}$ and this structure satisfies

$$\begin{aligned}\phi^2 = -I + \eta\otimes\xi,\ \eta(X) = g(X,\xi),\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi X) = 0,\\ g(\phi X, Y) = -g(X,\phi Y) \quad \text{and} \quad g(\phi X,\, \phi Y) = g(X\,,Y) - \eta(X)\eta(Y),\end{aligned} \tag{2.3}$$

where I denotes the identity map of the tangent bundle TM of M^{2n-1}. We call ϕ and η *the structure tensor* and *the contact form* of M^{2n-1}, respectively.

Let R be the curvature tensor of M^{2n-1} in $\widetilde{M}_n(c)$. We have the equation of Gauss given by:

$$\begin{aligned}R(X,Y)Z =&(c/4)\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y\\ &- 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY\end{aligned} \tag{2.4}$$

for all vectors X, Y and Z on M^{2n-1}.

The Ricci tensor S of type (0, 2) and the Ricci tensor Q of type (1, 1) of an arbitrary real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ $(n \geqq 2)$ is expressed as:

$$\begin{aligned}S(X,Y) = g(QX,Y) = (c/4)((2n+1)g(X,Y) - 3\eta(X)\eta(Y))\\ + (\mathrm{Trace}\,A)g(AX,Y) - g(A^2X,Y).\end{aligned} \tag{2.5}$$

3 Homogeneous Hopf hypersurfaces in $\widetilde{M}_n(c)$ We usually call M^{2n-1} a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M^{2n-1}. It is known that every tube of sufficiently small constant radius around each Kähler submanifold of $\widetilde{M}_n(c)$ is a Hopf hypersurface. This fact tells us that the notion of Hopf hypersurface is natural in the theory of real hypersurfaces in $\widetilde{M}_n(c)$ (see [15]).

The following lemma clarifies a fundamental property which is a useful tool in the theory of Hopf hypersurfaces in $\widetilde{M}_n(c)$ (cf. [15]).

Lemma 1. *For a Hopf hypersurface M^{2n-1} with the principal curvature δ corresponding to the characteristic vector field ξ in $\widetilde{M}_n(c)$, we have the following*:

(1) δ *is locally constant on* M^{2n-1};

(2) If X *is a tangent vector of* M^{2n-1} *perpendicular to* ξ *with* $AX = \lambda X$, *then* $(2\lambda - \delta)A\phi X = (\delta\lambda + (c/2))\phi X$.

In $\mathbb{C}P^n(c)\,(n \geqq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to a homogeneous real hypersurface (that is, real hypersurfaces which are expressed as orbits of some subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of $\widetilde{M}_n(c)$). Moreover, these real hypersurfaces are one of the following:

(A_1) A geodesic sphere $G(r)$ of radius r, where $0 < r < \pi/\sqrt{c}$;

(A_2) A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;

(B) A tube of radius r around a complex hyper quadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;

(C) A tube of radius r around a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n(\geqq 5)$ is odd;

(D) A tube of radius r around a complex Grassmann $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;

(E) A tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Summing up real hypersurfaces of type (A_1) and (A_2), we call them real hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows (cf. [15]):

	(A_1)	(A_2)	(B)	(C), (D), (E)
λ_1	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\right)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\right)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$
δ	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

The multiplicities of these principal curvatures are given as follows (cf. [15]):

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$m(\lambda_1)$	$2n-2$	$2n-2\ell-2$	$n-1$	2	4	6
$m(\lambda_2)$	—	2ℓ	$n-1$	2	4	6
$m(\lambda_3)$	—	—	—	$n-3$	4	8
$m(\lambda_4)$	—	—	—	$n-3$	4	8
$m(\delta)$	1	1	1	1	1	1

Remark 1. *A geodesic sphere* $G(r)$ *of radius* r $(0 < r < \pi/\sqrt{c}\,)$ *in* $\mathbb{C}P^n(c)$ *is congruent to a tube of radius* $(\pi/\sqrt{c}\,) - r$ *around totally geodesic* $\mathbb{C}P^{n-1}(c)$ *of* $\mathbb{C}P^n(c)$. *Indeed,* $\lim_{r\to\pi/\sqrt{c}} G(r) = \mathbb{C}P^{n-1}(c)$.

In $\mathbb{C}H^n(c)$ $(n \geqq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following:

(A_0) A horosphere in $\mathbb{C}H^n(c)$;

($A_{1,0}$) A geodesic sphere $G(r)$ of radius r, where $0 < r < \infty$;

($A_{1,1}$) A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;

(A_2) A tube of radius r around a totally geodesic $\mathbb{C}H^{\ell}(c)\,(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;

(B) A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A_0), ($A_{1,0}$), ($A_{1,1}$), (A_2) and (B). Here, type (A_1) means either type ($A_{1,0}$) or type ($A_{1,1}$). Summing up real hypersurfaces of types (A_0), (A_1) and (A_2), we call them hypersurfaces of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3}\,)$. Except for this real hypersurface, the numbers of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures are $2, 2, 2, 3, 3$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows (cf. [15]):

	(A_0)	($A_{1,0}$)	($A_{1,1}$)	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}\,r)$

The multiplicities of these principal curvatures are given as follows (cf. [15]):

	(A_0)	($A_{1,0}$)	($A_{1,1}$)	(A_2)	(B)
$m(\lambda_1)$	$2n-2$	$2n-2$	$2n-2$	$2n-2\ell-2$	$n-1$
$m(\lambda_2)$	—	—	—	2ℓ	$n-1$
$m(\delta)$	1	1	1	1	1

Remark 2. *The above Hopf hypersurfaces of type* (A) *and* (B) *in* $\mathbb{C}H^n(c)$ *are homogeneous real hypersurfaces. However, there exist non-Hopf homogeneous real hypersurfaces in* $\mathbb{C}H^n(c)$ (*for detail, see* [1]).

4 Ruled real hypersurfaces in $\widetilde{M}_n(c)$ Next we give ruled real hypersurfaces in a non-flat complex space form $\widetilde{M}_n(c)$, which are typical examples of non-Hopf hypersurfaces. A real hypersurface M^{2n-1} is called a *ruled real hypersurface* of a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$ if the holomorphic distribution T^0M defined by $T^0M(x) = \{X \in T_xM \mid X \perp \xi\}$ for $x \in M^{2n-1}$ is integrable and each of its maximal integral manifolds is a totally geodesic complex hypersurface $M_{n-1}(c)$ of $\widetilde{M}_n(c)$. A ruled real hypersurface is constructed in the following way. Given an arbitrary regular real smooth curve γ in $\widetilde{M}_n(c)$ which is defined on an interval I we have at each point $\gamma(t)$ $(t \in \mathrm{I})$ a totally geodesic complex hypersurface $M^{(t)}_{n-1}(c)$ that is orthogonal to the plane spanned by $\{\dot\gamma(t), J\dot\gamma(t)\}$. Then we see that $M^{2n-1} = \bigcup_{t\in I} M^{(t)}_{n-1}(c)$ is a ruled real hypersurface in $\widetilde{M}_n(c)$. The following is a well-known characterization of ruled real hypersurfaces in terms of the shape operator A.

Lemma 2. *For a real hypersurface M^{2n-1} in a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$, the following conditions are mutually equivalent*:

1. *M^{2n-1} is a ruled real hypersurface*;
2. *The shape operator A of M^{2n-1} satisfies the following equalities on the open dense subset $M_1 = \{x \in M^{2n-1} | \nu(x) \neq 0\}$ with a unit vector field U orthogonal to ξ : $A\xi = \mu\xi + \nu U$, $AU = \nu\xi$, $AX = 0$ for an arbitrary tangent vector X orthogonal to ξ and U, where μ, ν are differentiable functions on M_1 by $\mu = g(A\xi, \xi)$ and $\nu = \|A\xi - \mu\xi\|$*;
3. *The shape operator A of M^{2n-1} satisfies $g(Av, w) = 0$ for arbitrary tangent vectors $v, w \in T_xM$ orthogonal to ξ_x at each point $x \in M^{2n-1}$.*

We treat a ruled real hypersurface locally, because generally this hypersurface has singularities. When we study ruled real hypersurfaces, we usually omit points where ξ is principal and suppose that ν does not vanish everywhere, namely a ruled hypersurface M^{2n-1} is usually supposed $M_1 = M^{2n-1}$.

5 ϕ-invariances of the Ricci tensor and main theorem First, we define the notion of ϕ-invariance of the Ricci tensor S of M^{2n-1} in $\widetilde{M}_n(c)$. The Ricci tensor S of M^{2n-1} is called *strongly ϕ-invariant* if S satisfies

$$S(\phi X, \phi Y) = S(X, Y)$$

for all vectors X and Y on M^{2n-1}. Also it is called *weakly ϕ-invariant* if S satisfies

$$S(\phi X, \phi Y) = S(X, Y)$$

for all vectors X and Y on M^{2n-1} orthogonal to the characteristic vector ξ on M^{2n-1}.

Theorem 1. *Let M^{2n-1} be a real hypersurface in a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then the following holds*:

1. *Suppose that M^{2n-1} is a Hopf hypersurface in $\widetilde{M}_n(c)$. Then M^{2n-1} has weakly ϕ-invariant Ricci tensor S of M^{2n-1} if and only if M^{2n-1} satisfies $\phi Q = Q\phi$. Moreover, M^{2n-1} is locally congruent to one of the following*:
   (a) *A real hypersurface of type* (A) *in $\widetilde{M}_n(c)$*;
   (b) *A tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $\cot(\sqrt{c}\,r/2) = \sqrt{n-2} + \sqrt{n-1}$*;
   (c) *A tube of radius r around a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$, $n\ (\geqq 5)$ is odd and $\cot(\sqrt{c}\,r/2) = (\sqrt{n-1} + 1)/\sqrt{n-2}$*;
   (d) *A tube of radius r around a complex Grassmann $\mathbb{C}G_{2,5}$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$, $n = 9$ and $\cot(\sqrt{c}\,r/2) = (\sqrt{8} + \sqrt{3}\,)/\sqrt{5}$*;
   (e) *A tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$, $n = 15$ and $\cot(\sqrt{c}\,r/2) = (\sqrt{14} + \sqrt{5}\,)/3$*;
   (f) *A non-homogeneous real hypersurface which is a tube of radius r around an ℓ-dimensional non-totally geodesic Kähler submanifold $\widetilde{N}$ without principal curvatures $\pm(\sqrt{c}/2)\sqrt{(2\ell-1)/(2n-2\ell-1)}$, where the rank of every shape operator of $\widetilde{N}$ in the ambient space $\mathbb{C}P^n(c)$ is not greater than 2 and $\cot^2(\sqrt{c}\,r/2) = (2\ell-1)/(2n-2\ell-1)$ with $\ell = 1, \ldots, n-1$.*

2. *There does not exist a Hopf hypersurface M^{2n-1} with strongly ϕ-invariant Ricci tensor S of M^{2n-1}.*

3. *There does not exist a ruled real hypersurface M^{2n-1} with weakly ϕ-invariant Ricci tensor S of M^{2n-1}.*

Proof. From (2.5), we know that strongly ϕ-invariance of the Ricci tensor S of M^{2n-1} is equivalent to saying that

$$-\frac{c}{2}(n-1)\eta(X)\eta(Y) + (\mathrm{Trace}\,A)(g(A\phi X, \phi Y) - g(AX, Y)) - g(A^2\phi X, \phi Y) + g(A^2 X, Y) = 0 \tag{5.1}$$

for all vectors X, Y on M^{2n-1}. By this equation, we obtain that weakly ϕ-invariance of the Ricci tensor S of M^{2n-1} is equivalent to saying that

$$(\mathrm{Trace}\,A)(g(A\phi X, \phi Y) - g(AX, Y)) - g(A^2\phi X, \phi Y) + g(A^2 X, Y) = 0 \tag{5.2}$$

for all vectors X and Y orthogonal to ξ.

(1) First of all, we suppose that M^{2n-1} satisfies $\phi Q = Q\phi$. Then, we get

$$S(\phi X, \phi Y) = g(Q\phi X, \phi Y) = g(\phi QX, \phi Y) = -g(QX, \phi^2 Y) = g(QX, Y) = S(X, Y)$$

for any vectors X, Y orthogonal to ξ.

Next, we suppose that M^{2n-1} has weakly ϕ-invariant Ricci tensor S of M^{2n-1}. By (5.2), we have

$$(\mathrm{Trace}\,A)g(-\phi A\phi X - AX, Y) + g(\phi A^2\phi X + A^2 X, Y) = 0 \tag{5.3}$$

for any vectors X, Y orthogonal to ξ. Interchanging a vector $X(\perp \xi)$ with a vector $\phi X(\perp \xi)$ in Equation (5.3), we obtain

$$(\mathrm{Trace}\,A)g((\phi A - A\phi)X, Y) - g((\phi A^2 - A^2\phi)X, Y) = 0$$

for any vectors X, Y orthogonal to ξ. This implies that

$$g((\phi Q - Q\phi)X, Y) = 0 \tag{5.4}$$

for any vectors X, Y orthogonal to ξ. On the other hand, using assumption that M^{2n-1} is a Hopf hypersurface in $\widetilde{M}_n(c)$, we obtain $\phi Q\xi = 0 = Q\phi\xi$. This, combine with (5.4), implies $\phi Q = Q\phi$.

By the works of M. Kimura [8], [9] (the case of $n \geqq 3$ in $\mathbb{C}P^n(c)$), U-H. Ki and Y. J. Suh [6] (the case of $n \geqq 3$ in $\mathbb{C}H^n(c)$) and J. T. Cho [4] (the case of $\widetilde{M}_2(c)$), we know the classification of Hopf hypersurfaces with $\phi Q = Q\phi$ in $\widetilde{M}_n(c)$. Hence, we get the classification of Hopf hypersurfaces having weakly ϕ-invariant Ricci tensor in $\widetilde{M}_n(c)$.

(2) We suppose that M^{2n-1} is a Hopf hypersurface with $A\xi = \delta\xi$ in $\widetilde{M}_n(c)$. From (5.1), we find that M^{2n-1} has strongly ϕ-invariant Ricci tensor S of M^{2n-1} if and only if M^{2n-1} satisfies the following two conditions:

(i) The Hopf hypersurface M^{2n-1} has weakly ϕ-invariant Ricci tensor S of M^{2n-1};

(ii) The Hopf hypersurface M^{2n-1} satisfies the following equation:

$$\delta^2 - (\mathrm{Trace}\,A)\delta - \frac{c}{2}(n-1) = 0. \tag{5.5}$$

Now we shall check Equation (5.5) one by one for real hypersurfaces of (1) in our Theorem.

Let M^{2n-1} be a real hypersurface of type (A_1) in $\mathbb{C}P^n(c)$. Let $x = \cot(\sqrt{c}r/2)$, $0 < r < \pi/\sqrt{c}$. Then we have $\delta = (\sqrt{c}/2)(x - (1/x))$, $\delta^2 = (c/4)(x^2 - 2 + (1/x^2))$ and $\operatorname{Trace} A = (\sqrt{c}/2)((2n-1)x - (1/x))$. These, together with Equation (5.5) we get $n = 1$, which contradicts $n \geqq 2$. Hence M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}.

Let M^{2n-1} be a real hypersurface of type (A_2) in $\mathbb{C}P^n(c)$. Let $x = \cot(\sqrt{c}r/2)$, $0 < r < \pi/\sqrt{c}$. Then we have $\delta = (\sqrt{c}/2)(x - (1/x))$, $\delta^2 = (c/4)(x^2 - 2 + (1/x^2))$ and $\operatorname{Trace} A = (\sqrt{c}/2)((2n-2\ell-1)x - (2\ell+1)(1/x))$. These, together with Equation (5.5) we get $(n-\ell-1)x^4 + \ell = 0$. However, this equation can not occur. Hence M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}.

Let M^{2n-1} be a real hypersurface of type (A_0) in $\mathbb{C}H^n(c)$. Then we have $\delta = \sqrt{|c|}$, $\delta^2 = -c$ and $\operatorname{Trace} A = \sqrt{|c|} + (2n-2)(\sqrt{|c|}/2)$. These, together with Equation (5.5) we get $n = 1$, which contradicts $n \geqq 2$. Hence M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}.

Let M^{2n-1} be a real hypersurface of type ($A_{1,0}$) in $\mathbb{C}H^n(c)$. Let $x = \coth(\sqrt{|c|}r/2)$, $0 < r < \infty$. Then we have $\delta = (\sqrt{|c|}/2)(x + (1/x))$, $\delta^2 = -(c/4)(x^2 + 2 + (1/x^2))$ and $\operatorname{Trace} A = (\sqrt{|c|}/2)((2n-1)x + (1/x))$. These, together with Equation (5.5) we get $n = 1$, which contradicts $n \geqq 2$. Hence M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}. Similarly, we can show that real hypersurfaces of type ($A_{1,1}$) in $\mathbb{C}H^n(c)$ do not have strongly ϕ-invariant Ricci tensor.

Let M^{2n-1} be a real hypersurface of type (A_2) in $\mathbb{C}H^n(c)$. Let $x = \coth(\sqrt{|c|}r/2)$, $0 < r < \infty$. Then we have $\delta = (\sqrt{|c|}/2)(x + (1/x))$, $\delta^2 = -(c/4)(x^2 + 2 + (1/x^2))$ and $\operatorname{Trace} A = (\sqrt{|c|}/2)((2n-2\ell-1)x + (2\ell+1)(1/x))$. These, together with Equation (5.5) we get $(n-\ell-1)x^4 + \ell = 0$. However, this equation can not occur. Hence M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}.

Let M^{2n-1} be a real hypersurface of the case of (b) in our Theorem. Then we have $\delta = \sqrt{c(n-2)}$, $\delta^2 = c(n-2)$ and $\operatorname{Trace} A = -\sqrt{c}/\sqrt{n-2}$. These, together with Equation (5.5) we get $n = 1$, which contradicts $n \geqq 3$.

Let M^{2n-1} be a real hypersurface of the case of (c) in our Theorem. Then we have $\delta = \sqrt{c}/\sqrt{n-2}$, $\delta^2 = c/(n-2)$ and $\operatorname{Trace} A = -\sqrt{c(n-2)}$. These, together with Equation (5.5) we get $n^2 - 5n + 4 = 0$, so that $n = 1, 4$, which contradicts $n \geqq 5$.

Let M^{2n-1} be a real hypersurface of the case of (d) in our Theorem. Then we have $\delta = \sqrt{3c}/\sqrt{5}$, $\delta^2 = 3c/5$ and $\operatorname{Trace} A = -\sqrt{5c}/\sqrt{3}$. These, together with Equation (5.5) we get $n = 21/5$, which contradicts $n = 9$.

Let M^{2n-1} be a real hypersurface of the case of (e) in our Theorem. Then we have $\delta = \sqrt{5c}/3$, $\delta^2 = 5c/9$ and $\operatorname{Trace} A = -3\sqrt{5c}/5$. These, together with Equation (5.5) we get $n = 37/9$, which contradicts $n = 15$.

Let M^{2n-1} be a real hypersurface of the case of (f) in our Theorem. Then M^{2n-1} has at most five distinct principal curvatures as follow: $\sqrt{c}\cot(\sqrt{c}r)$ with multiplicity 1, $(\sqrt{c}/2)\cot(\sqrt{c}r/2)$ with multiplicity $2n-2\ell-2$, $-(\sqrt{c}/2)\tan(\sqrt{c}r/2)$ with multiplicity $2\ell-2$, $(\sqrt{c}/2)\cot((\sqrt{c}r/2)-\theta)$ with multiplicity 1 and $(\sqrt{c}/2)\cot((\sqrt{c}r/2)+\theta)$ with multiplicity 1, where $(\sqrt{c}/2)\cot\theta$ is a principal curvature of the Kähler submanifold $\widetilde{N}$ (see [3], [9], [10], [12]). In this case, M^{2n-1} has either the case of $\delta = 0$ or the case of $\delta \neq 0$. When $\delta = 0$ (that is, the case of $n = 2\ell$), we have $(c/2)(n-1) = 0$, which is a contradiction. When $\delta \neq 0$, we have

$$\operatorname{Trace} A = \delta + \frac{c}{2\delta}(n-1). \tag{5.6}$$

It follows from (1) of Lemma 1 that the right side of Equation (5.6) is constant on M^{2n-1}.

On the other hand, the left side of Equation (5.6) is non-constant. Indeed, Trace A of M^{2n-1} is expressed as:

$$\begin{aligned}\text{Trace } A = \delta + (2n-2\ell-2)\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell-2)\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right) \\ + \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r-\theta\right) + \frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r+\theta\right).\end{aligned}$$

Note that $(\sqrt{c}/2)\cot((\sqrt{c}r/2)-\theta)+(\sqrt{c}/2)\cot((\sqrt{c}r/2)+\theta)$ is non-constant on M^{2n-1}. Thus, we have a contradiction. Hence, M^{2n-1} does not have strongly ϕ-invariant Ricci tensor S of M^{2n-1}.

Therefore, there exist no Hopf hypersurface M^{2n-1} with strongly ϕ-invariant Ricci tensor S of M^{2n-1} in $\widetilde{M}_n(c)$.

(3) We suppose that M^{2n-1} is a ruled real hypersurface with weakly ϕ-invariant Ricci tensor S of M^{2n-1} in $\widetilde{M}_n(c)$. It follows from (5.2) and (3) of Lemma 2 that we obtain

$$-g(A^2\phi X, \phi Y) + g(A^2X, Y) = 0$$

for all vectors X, Y orthogonal to ξ. Setting $X = Y = U$, by using Lemma 2 we have

$$0 = -g(A^2\phi U, \phi U) + g(A^2U, U) = \nu^2 \neq 0,$$

which is a contradiction. Hence, M^{2n-1} does not have weakly ϕ-invariant Ricci tensor S of M^{2n-1}. □

Remark 3. *Note that the commutativity of the structure tensor ϕ and the Ricci tensor Q of type $(1,1)$ always implies weakly ϕ-invariance of the Ricci tensor. However, in general, we do not know whether the converse holds or not.*

6 Concluding remarks

6.1 In general, there exist contact metric manifolds with strongly ϕ-invariant Ricci tensor.

For example, $\mathbb{R}^3$ with coordinates (x^1, x^2, x^3) and the contact form $\eta = (1/2)(\cos x^3\, dx^1 + \sin x^3\, dx^2)$. The characteristic vector filed ξ is defined by $\xi = 2(\cos x^3(\partial/\partial x^1)+\sin x^3(\partial/\partial x^2))$ and the metric g is given by $g_{ij} = (1/4)\delta_{ij}$, where g_{ij} are components of g. Then $\mathbb{R}^3$ has a *flat* contact metric structure (cf. [2]). Hence clearly this example admits strongly ϕ-invariant Ricci tensor.

6.2 In [13], S. Maeda and H. Naitoh investigated real hypersurfaces with *ϕ-invariant shape operators* in $\mathbb{C}P^n(c)$. The shape operator A of a real hypersurface M^{2n-1} is called *strongly ϕ-invariant* if A satisfies

$$g(A\phi X, \phi Y) = g(AX, Y)$$

for all vectors X and Y on M^{2n-1}. Also, it is called *weakly ϕ-invariant* if A satisfies

$$g(A\phi X, \phi Y) = g(AX, Y)$$

for all vectors X and Y orthogonal to the characteristic vector ξ on M^{2n-1}.

S. Maeda and H. Naitoh [13] obtained the following results:

Proposition 1. *Let M^{2n-1} be a real hypersurface M^{2n-1} with strongly ϕ-invariant shape operator A of M^{2n-1} in $\mathbb{C}P^n(c)$. Then M^{2n-1} is locally congruent to a real hypersurface of type* (A) *of radius $\pi/(2\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$.*

Proposition 2. *Let M^{2n-1} be a real hypersurface M^{2n-1} with weakly ϕ-invariant shape operator A of M^{2n-1} in $\mathbb{C}P^n(c)$. Then the following holds*:

1. *If M^{2n-1} is a Hopf hypersurface in $\mathbb{C}P^n(c)$, then M^{2n-1} is locally congruent to a real hypersurface of type* (A) *in $\mathbb{C}P^n(c)$.*
2. *If the holomorphic distribution $T^0M = \{X \in TM : X \perp \xi\}$ is integrable, then M^{2n-1} is locally congruent to a ruled real hypersurface in $\mathbb{C}P^n(c)$.*

By using the discussion of [13], we know that there exists no real hypersurface M^{2n-1} in $\mathbb{C}H^n(c)$ such that the shape operator A of M^{2n-1} is strongly ϕ-invariant. In addition, for real hypersurfaces in $\mathbb{C}H^n(c)$, Proposition 2 also holds.

From our theorem, ruled real hypersurfaces do not have weakly ϕ-invariant Ricci tensor in $\widetilde{M}_n(c)$. However, ruled real hypersurfaces have weakly ϕ-invariant shape operator in $\widetilde{M}_n(c)$.

6.3 We shall consider the notion of *ϕ-invariant curvature tensor* R of M^{2n-1} in $\widetilde{M}_n(c)$. The curvature tensor R of a real hypersurface M^{2n-1} is called *strongly ϕ-invariant* if R satisfies

$$R(\phi X, \phi Y) = R(X, Y)$$

for all vectors X and Y on M^{2n-1}. Also, it is called *weakly ϕ-invariant* if R satisfies

$$R(\phi X, \phi Y) = R(X, Y)$$

for all vectors X and Y orthogonal to the characteristic vector ξ on M^{2n-1} .

From our theorem and S. Maeda and H. Naitoh's work [13], real hypersurfaces of type (A) in $\widetilde{M}_n(c)$ have both weakly ϕ-invariant Ricci tensor and weakly ϕ-invariant shape operator. Now we investigate whether there exists a real hypersurface of type (A) in $\widetilde{M}_n(c)$ having weakly ϕ-invariant curvature tensor R or not.

Proposition 3. *There does not exist a real hypersurface M^{2n-1} of type* (A) *admitting weakly ϕ-invariant curvature tensor R of M^{2n-1} in $\widetilde{M}_n(c)$ $(n \geqq 3)$.*

Proof. We suppose that a real hypersurface M^{2n-1} admitting weakly ϕ-invariant curvature tensor R of M^{2n-1}. By (2.4), we know that weakly ϕ-invariance of the curvature tensor R of M^{2n-1} is equivalent to saying that

$$g(A\phi Y, Z)A\phi X - g(A\phi X, Z)A\phi Y - g(AY, Z)AX + g(AX, Z)AY = 0 \tag{6.1}$$

for $\forall X, Y \perp \xi$ and $\forall Z \in TM$.

Let M^{2n-1} be a real hypersurface of type (A_1) in $\mathbb{C}P^n(c)$ $(n \geqq 3)$. We take a local field of orthogonal frame $\{e_1, e_2, \ldots, e_{n-1}, \phi e_1, \phi e_2, \ldots \phi e_{n-1}, \xi\}$ in M^{2n-1} such that

$$Ae_i = (\sqrt{c}/2)\,\cot(\sqrt{c}\,r/2)e_i, \quad A\phi e_i = (\sqrt{c}/2)\,\cot(\sqrt{c}\,r/2)\phi e_i \quad (1 \leqq i \leqq n-1).$$

We can put $X = e_i, Y = e_j, Z = e_j$ in Equation (6.1) satisfying $e_i \neq e_j, \phi e_i \neq e_j$. Then we have $\cot^2(\sqrt{c}\,r/2) = 0$, which is a contradiction. Hence M^{2n-1} does not have weakly ϕ-invariant curvature tensor R of M^{2n-1}. Similarly, real hypersurfaces M^{2n-1} of types (A_0) and (A_1) in $\mathbb{C}H^n(c)$ $(n \geqq 3)$ do not admit ϕ-invariant curvature tensor R of M^{2n-1}.

Let M^{2n-1} be a real hypersurface of type (A_2) in $\mathbb{C}P^n(c)$ $(n \geqq 3)$. We take a local field of orthogonal frame $\{e_1, e_2, \ldots, e_{2n-2}, \xi\}$ in M^{2n-1} such that

$$Ae_i = (\sqrt{c}/2)\,\cot(\sqrt{c}\,r/2)e_i \quad (1 \leqq i \leqq 2n-2\ell-2),$$
$$Ae_j = -(\sqrt{c}/2)\,\tan(\sqrt{c}\,r/2)e_j \quad (2n-2\ell-1 \leqq j \leqq 2n-2).$$

We set $X = e_i, Y = e_j, Z = e_j$ $(1 \leqq i \leqq 2n-2\ell-2, 2n-2\ell-1 \leqq j \leqq 2n-2)$ in Equation (6.1). Note that $\phi V_{\lambda_1} = V_{\lambda_1} = \{X \in TM : AX = \lambda_1 X\}, \phi V_{\lambda_2} = V_{\lambda_2} = \{X \in TM : AX = \lambda_2 X\}$ and $V_{\lambda_1} \oplus V_{\lambda_2} = T^0 M = \{X \in TM : X \perp \xi\}$, where $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2), \lambda_2 = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$. Then we obtain $\cot(\sqrt{c}\,r/2)\tan(\sqrt{c}\,r/2) = 0$, which is a contradiction. Hence M^{2n-1} does not have weakly ϕ-invariant curvature tensor R of M^{2n-1}. Similarly, real hypersurfaces M^{2n-1} of type (A_2) in $\mathbb{C}H^n(c)\,(n \geqq 3)$ does not have ϕ-invariant curvature tensor R of M^{2n-1}.

Therefore real hypersurfaces of type (A) in $\widetilde{M}_n(c)\,(n \geqq 3)$ do not admit ϕ-invariant curvature tensor. □

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