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DEFORMATIONS OF THE CHEBYSHEV HYPERGROUPS

Tatsuya Tsurii

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Abstract. In the present paper we introduce q-deformations of the Chebyshev hypergroups of the first kind and of the second kind as models of q-deformations of countable discrete hypergroups. Moreover we study q-deformations of character hypergroups $\mathcal{K}(\hat{G})$ of certain compact groups G.

1 Introduction

The notion of compact quantum groups is introduced in [14] and [15] by S. L. Woronowicz. Especially, he studied the structure of $SU_q(2)$ which is obtained by a q-deformation of $SU(2)$ in the category of Hopf algebras. Compact quantum groups play an important role not only in mathematics but also in theoretical physics.

Deformations of groups and hypergroups are investigated in [16] by K. A. Ross and D. Xu and our previous paper [6] in the category of hypergroups. Many new hypergroups are produced by deforming groups and hypergroups. The notion of q-deformations of groups and hypergroups is one of the way to understand hypergroup structures.

The structure of countable discrete hypergroups arising from orthogonal polynomials has been studied by many authors (for example [7], [8] and [9]). But there is no notion of q-deformations of countable discrete hypergroups in the category of hypergroups. In the present paper, we consider q-deformations of countable discrete commutative hypergroups, mainly of the Chebyshev hypergroups $\mathcal{T}$ of the first kind and $F_d(\mathcal{U})$ of the second kind. In the present paper the q-deformation K_q of a countable discrete hypergroup K is to deform continuously structures of K by a parameter q $(0 < q \leq 1)$ and $K_1 = K$ in the category of hypergroups. A notion of dimension functions of countable discrete hypergroups and of fusion rule algebras plays an essential role in our discussions.

In section 3, we consider dimension functions of countable discrete hypergroups as well as of fusion rule algebras. In section 4, we discuss q-deformations $\mathcal{T}_q$ of the Chebyshev hypergroup $\mathcal{T}$ of the first kind. Moreover we consider q-deformations $\mathcal{K}_q(\hat{G})$ of a character hypergroup $\mathcal{K}(\hat{G})$ of the compact group $G = \mathbb{T} \rtimes_\alpha \mathbb{Z}_2$ as an application of q-deformations of $\mathcal{T}$. In section 5, we discuss q-deformations $\mathcal{U}_q$ of the Chebyshev hypergroup $F_d(\mathcal{U})$ of the second kind which is obtained by normalization of the fusion rule algebra $\mathcal{U}$ by the dimension function d of $\mathcal{U}$. Moreover we investigate q-deformations $\mathcal{K}_q(\widehat{SU(2)})$ of a character hypergroup $\mathcal{K}(\widehat{SU(2)})$ of the compact Lie group $SU(2)$.

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2 Preliminary

For a countable discrete set $K = \{X_0, X_1, X_2, \cdots, X_n, \cdots\}$ we denote the algebraic complex linear space based on K by $\mathbb{C}K$, namely

$$\mathbb{C}K = \left\{ X = \sum_{k=0}^{\infty} a_k X_k : a_k \in \mathbb{C},\ |\mathrm{supp}(X)| < +\infty \right\},$$

where $|\mathrm{supp}(X)|$ is the cardinal number of $\mathrm{supp}(X)$ and the *support* of X is

$$\mathrm{supp}(X) := \{k : a_k \neq 0\}.$$

A *countable discrete hypergroup* $(K, \mathbb{C}K, \circ, *)$ consists of the set $K = \{X_0, X_1, \cdots, X_n, \cdots\}$ together with a product (called convolution) $\circ$ and an involution $*$ in the complex linear space $\mathbb{C}K$ satisfying the following conditions.

(1) For $X_m, X_n \in K$, the convolution $X_m \circ X_n$ belongs to $\mathbb{C}K$ and

$$X_m \circ X_n = \sum_{k \in S(m,n)} a_{mn}^k X_k,$$

where

$$S(m,n) := \mathrm{supp}(X_m \circ X_n),\ a_{mn}^k \geq 0 \text{ and } \sum_{k \in S(m,n)} a_{mn}^k = 1.$$

(2) The space $(\mathbb{C}K, \circ, *)$ is an associative $*$-algebra with unit X_0.

(3) The map $X_n \mapsto X_n^*$ is a bijection on K. Moreover for all $X_m, X_n \in K$, $X_n = X_m^*$ if and only if $0 \in \mathrm{supp}(X_m \circ X_n)$.

We denote the hypergroup $(K, \mathbb{C}K, \circ, *)$ by K. A hypergroup K is called *commutative* if the convolution $\circ$ on $\mathbb{C}K$ is commutative and be called *hermitian* if $X_n^* = X_n$.

If the given countable discrete hypergroup K is commutative, its dual $\hat{K}$ can be introduced as the set of all bounded functions $\chi \neq 0$ on $\mathbb{C}K$ satisfying

$$\chi(X_m \circ X_n) = \chi(X_m)\chi(X_n), \quad \chi(X_n^*) = \overline{\chi(X_n)}$$

for all $X_i, X_j \in K$. This set of characters $\hat{K}$ of K becomes a compact space with respect to the topology of uniform convergence on compact sets, but generally fails to be a hypergroup. If $\hat{K}$ is a hypergroup, then K is called a strong hypergroup or a hypergroup of strong type.

Let G be a compact group and $\hat{G}$ the set of all equivalence classes of irreducible representations of G. Put

$$\mathcal{K}(\hat{G}) := \{ch(\pi) : \pi \in \hat{G}\},$$

where

$$ch(\pi)(g) := \frac{1}{\dim \pi} \mathrm{tr}(\pi(g)) \quad (g \in G).$$

Then $\mathcal{K}(\hat{G})$ always becomes a discrete commutative hypergroup which is called the character hypergroup of G. (Refer to [1] for details.)

Let $K = (K, \mathbb{C}K, \circ, *)$ be a countable discrete hypergroup where $K = \{X_0, X_1, \cdots, X_n, \cdots\}$. For q $(0 < q \leq 1)$, put $K_q = \{X_0(q), X_1(q), \cdots, X_n(q), \cdots\}$ a new basis in $\mathbb{C}K$. Then the convolution $X_m(q)$ and $X_n(q)$ of $\mathbb{C}K$ is defined by

$$X_m(q) \circ X_n(q) := \sum_{k=0}^{\infty} a_{mn}^k(q) X_k(q)$$

where $a_{mn}^k(q)$ is continuous with respect to q. The involution $*$ of K_q is given by

$$X_n(q)^* = X_m(q)^* \quad \text{when} \quad X_n^* = X_m.$$

For the hypergroup $K_q = (K_q, \mathbb{C}K, \circ, *)$ satisfies the following conditions

$$X_n(1) = X_n \ \text{ and } \ X_n(q) \to X_n \ \text{ as } \ q \to 1,$$

we call K_q a q-deformation of K.

A *fusion rule algebra* $(F, \mathbb{C}F, \diamond, ^-)$ consists of the set $F = \{Y_0, Y_1, \cdots, Y_n, \cdots\}$ together with a product (called convolution) $\diamond$ and an involution $^-$ in the complex linear space $\mathbb{C}F$ based on F satisfying the following conditions.

(1) For $Y_m, Y_n \in F$, the convolution $Y_m \diamond Y_n$ belongs to $\mathbb{C}F$ and

$$Y_m \diamond Y_n = \sum_{k \in S(m,n)} a_{mn}^k Y_k \quad (a_k \in \mathbb{Z}_+ = \{0, 1, 2, \cdots\}),$$

$$Y_n^- \diamond Y_n = Y_0 + \sum_{\substack{k \in S(m,n) \\ k \neq 0}} a_{mn}^k Y_k.$$

where $S(m, n) := \mathrm{supp}(Y_m \diamond Y_n)$.

(2) The space $(\mathbb{C}F, \diamond, ^-)$ is an associative involutive algebra with unit Y_0.

We denote the fusion rule algebra $(F, \mathbb{C}F, \diamond, ^-)$ by F.

For the dual $\hat{G}$ of a compact group G, put

$$\mathcal{F}(\hat{G}) := \{Ch(\pi) : \pi \in \hat{G}\},$$

where

$$Ch(\pi)(g) := \mathrm{tr}(\pi(g)) \quad (g \in G).$$

Then $\mathcal{F}(\hat{G})$ always becomes a fusion rule algebra.

3 A dimension function

In this section, we discuss a dimension function of a countable discrete hypergroup and a fusion rule algebra.

For a countable discrete hypergroup K, the mapping d from K to $\mathbb{R}_+^\times = \{x \in \mathbb{R} : x > 0\}$ is called a *dimension function* of K if d is a homomorphism in the sense that

$$X_m \circ X_n = \sum_{k \in S(m,n)} a_{mn}^k X_k \Rightarrow d(X_m) d(X_n) = \sum_{k \in S(m,n)} a_{mn}^k d(X_k).$$

The dimension function d of K is uniquely extendable as a linear mapping from $\mathbb{C}K$ to $\mathbb{C}$ and satisfies

$$d(X_m \circ X_n) = d(X_m)d(X_n).$$

Proposition 3.1 Let K be a countable discrete hypergroup where $K = \{X_0, X_1, \cdots, X_n, \cdots\}$. For the dimension function d of K, put

$$c_n := \frac{1}{d(X_n)} X_n \quad \text{and} \quad K_d := \{c_0, c_1, \cdots, c_n, \cdots\}.$$

Then K_d is a hypergroup.

Proof By the axiom (1) of a countable discrete hypergroup, the structure equation of K is written by

$$X_m \circ X_n = \sum_{k \in S(m,n)} a_{mn}^k X_k.$$

Hence, the structure equation of K_d is

$$\begin{aligned} c_m \circ c_n &= \frac{1}{d(X_m)d(X_n)} X_m \circ X_n \\ &= \frac{1}{d(X_m)d(X_n)} \sum_{k \in S(m,n)} a_{mn}^k X_k \\ &= \sum_{k \in S(m,n)} \frac{a_{mn}^k d(X_k)}{d(X_m)d(X_n)} c_k. \end{aligned}$$

Here we note that

$$\sum_{k \in S(m,n)} \frac{a_{mn}^k d(X_k)}{d(X_m)d(X_n)} = 1$$

by the fact

$$d(X_m)d(X_n) = \sum_{k \in S(m,n)} a_{mn}^k d(X_k).$$

It is clear that the coefficients of the convolution $c_m \circ c_n$ are non-negative. It is easy to check other conditions of axiom of a countable discrete hypergroup. Hence, K_d is a countable discrete hypergroup. □

Remark If K is a finite hypergroup, the dimension function d of K is known to be unique such that $d(X_k) = 1$ for all $X_k \in K$.

For a fusion rule algebra F, the dimension function d of F is defined in a similar way to the above.

Proposition 3.2 Let F be a fusion rule algebra where $F = \{Y_0, Y_1, \cdots, Y_n, \cdots\}$. For the dimension function d of a fusion rule algebra F, put

$$b_n := \frac{1}{d(Y_n)} Y_n \quad \text{and} \quad F_d := \{b_0, b_1, \cdots, b_n, \cdots\}.$$

Then F_d becomes a hypergroup.

Proof The desired assertion is obtained in a similar way to the proof of Proposition 3.1. □

4 Q-deformations of the Chebyshev hypergroup of the first kind

Let $T_n(x)$ be the Chebyshev polynomial of the first kind of degree n, then $T_n(x)$ ($n = 0, 1, 2, \cdots$) satisfy the following equation.

$$T_m(x)T_n(x) = \frac{1}{2}T_{|m-n|}(x) + \frac{1}{2}T_{m+n}(x).$$

Then, for the set $\mathcal{T} = \{T_0, T_1, \cdots, T_n, \cdots\}$, $(\mathcal{T}, \mathbb{C}\mathcal{T}, \circ, *)$ is a countable discrete hypergroup by the product

$$T_m \circ T_n := T_m(x)T_n(x).$$

The hypergroup $\mathcal{T} = (\mathcal{T}, \mathbb{C}\mathcal{T}, \circ, *)$ is called the Chebyshev hypergroup of the first kind.

Next, we consider a mapping d_q from $\mathcal{T}$ to $\mathbb{R}_+^\times$. For $T_n \in \mathcal{T}$ the mapping d_q defined by

$$d_q(T_n) := T_n\left(\frac{q+q^{-1}}{2}\right) = \frac{q^n + q^{-n}}{2} \geq 1.$$

Proposition 4.1 The mapping d_q from $\mathcal{T}$ to $\mathbb{R}_+^\times$ is a dimension function of $\mathcal{T}$.

Proof Put $x(q) := \frac{q+q^{-1}}{2}$ and $d_q(T_n) = T_n(x(q))$. Then

$$\begin{aligned} d_q(T_m)d_q(T_n) &= T_m(x(q))T_n(x(q)) \\ &= \frac{1}{2}T_{|m-n|}(x(q)) + \frac{1}{2}T_{m+n}(x(q)) \\ &= \frac{1}{2}d_q(T_{|m-n|}) + \frac{1}{2}d_q(T_{m+n}) \end{aligned}$$

Hence, d_q is a dimension function of $\mathcal{T}$. □

Next, put

$$X_n(q) := \frac{1}{d_q(T_n)}T_n \quad \text{and} \quad \mathcal{T}_q := \{X_0(q), X_1(q), \cdots, X_n(q), \cdots\}.$$

Then the following theorem holds.

Theorem 4.2 $\mathcal{T}_q$ becomes a hypergroup which is a q-deformation of $\mathcal{T}$. The structure equation is

$$X_m(q) \circ X_n(q) = \sum_{k \in S(m,n)} a_{mn}^k(q) X_k(q)$$

where $S(m,n) = \{|m-n|, m+n\}$ and

$$a_{mn}^k(q) = \frac{q^k + q^{-k}}{(q^m + q^{-m})(q^n + q^{-n})}.$$

Proof By Proposition 3.1, $\mathcal{T}_q$ is a hypergroup. The convolution $X_m(q)$ and $X_n(q)$ is

$$X_m(q) \circ X_n(q) = \frac{1}{d_q(T_m)d_q(T_n)}T_m \circ T_n = \sum_{k \in S(m,n)} \frac{d_q(T_k)}{2d_q(T_m)d_q(T_n)} X_k(q)$$

where $S(m,n)=\{|m-n|, m+n\}$. Then

$$a^k_{mn}(q)=\frac{d_q(T_k)}{2d_q(T_m)d_q(T_n)}=\frac{q^k+q^{-k}}{(q^m+q^{-m})(q^n+q^{-n})}.$$

Hence, we see that $a^k_{mn}(q)$ is continuous with respect to q. The involution $*$ of $\mathcal{T}_q$ is an identity map by the fact that

$$X_n(q)=\frac{1}{d_q(T_n)}T_n \quad \text{and} \quad T_n^*=T_n.$$

When $q=1$, it is clear that $X_n(1)=T_n$. Since $d_q(T_n)=\frac{q^n+q^{-n}}{2}$ is continuous, $X_n(q)\to T_n$ as $q\to 1$. Hence, $\mathcal{T}_q$ is a q-deformation of $\mathcal{T}$. □

Let α be an action of $\mathbb{Z}_2=\{e,g\}$ $(g^2=e)$ on the torus $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$ defined by

$$\alpha_g(z)=\overline{z}.$$

Then we have a compact group $G=\mathbb{T}\rtimes_\alpha\mathbb{Z}_2$ in the form of a semi-direct product. We consider q-deformations of the character hypergroup $\mathcal{K}(\hat{G})$ of $G=\mathbb{T}\rtimes_\alpha\mathbb{Z}_2$. The dual of $\hat{\mathbb{T}}$ of $\mathbb{T}$ and $\widehat{\mathbb{Z}_2}$ of $\mathbb{Z}_2$ are given by

$$\begin{aligned}&\hat{\mathbb{T}}=\{\chi_n : n\in\mathbb{Z}\}, \text{ where } \chi_n(z)=z^n \quad \text{for } z\in\mathbb{T},\\ &\widehat{\mathbb{Z}_2}=\{\tau_0,\tau_1\}, \text{ where } \tau_1^2=\tau_0.\end{aligned}$$

For $\chi\in\hat{\mathbb{T}}$ and $\tau\in\widehat{\mathbb{Z}_2}$, the irreducible representations ρ_0, ρ_1 and π_n $(n=1,2,\cdots)$ of $G=\mathbb{T}\rtimes_\alpha\mathbb{Z}_2$ are written by

$$\begin{aligned}&\rho_0((z,h))=\tau_0(h)=1, \quad \rho_1((z,h))=\tau_1(h),\\ &\pi_n=\mathrm{ind}_{\mathbb{T}}^G\chi_n \quad (n=1,2,\cdots).\end{aligned}$$

Then the dual $\hat{G}$ of G is determined by $\hat{G}=\{\rho_0,\rho_1,\pi_1,\pi_2,\cdots,\pi_n,\cdots\}$ by Mackey Machine. For the irreducible representations π_n $(n=1,2,\cdots)$, put

$$Ch(\pi_n)(g):=\mathrm{tr}(\pi_n(g)) \quad (g\in G)$$

and

$$\mathcal{F}(\hat{G}):=\{\rho_0,\rho_1,Ch(\pi_1),Ch(\pi_2),\cdots,Ch(\pi_n),\cdots\}.$$

Then $\mathcal{F}(\hat{G})$ becomes a fusion rule algebra with unit ρ_0. The structure equations are

$$\begin{aligned}&\rho_1^2=\rho_0, \quad \rho_1 Ch(\pi_n)=Ch(\pi_n),\\ &Ch(\pi_m)Ch(\pi_n)=Ch(\pi_{|m-n|})+Ch(\pi_{m+n}) \quad (m\neq n),\\ &Ch(\pi_n)^2=\rho_0+\rho_1+Ch(\pi_{2n}).\end{aligned}$$

Moreover, put

$$ch(\pi_n):=\frac{1}{\dim\pi_n}Ch(\pi_n)=\frac{1}{2}Ch(\pi_n)$$

and

$$\mathcal{K}(\hat{G})=\{\rho_0,\rho_1,ch(\pi_1),ch(\pi_2),\cdots,ch(\pi_n),\cdots\}.$$

Then $\mathcal{K}(\hat{G})$ becomes a hypergroup with unit ρ_0 by Proposition 3.2. This hypergroup $\mathcal{K}(\hat{G})$ is the character hypergroup of G. The hypergroup structure of $\mathcal{K}(\hat{G})$ is the hypergroup join of $\mathbb{Z}_2$ by $\mathcal{T}$ which is written by

$$\mathcal{K}(\hat{G}) = \mathbb{Z}_2 \vee \mathcal{T}.$$

Hence, we obtain a q-deformation $\mathcal{K}_q(\hat{G})$ of the countable discrete hypergroup $\mathcal{K}(\hat{G})$ as follows.

Theorem 4.3 The hypergroup $\mathcal{K}_q(\hat{G}) = \mathbb{Z}_2 \vee \mathcal{T}_q$ is a q-deformation of $\mathcal{K}(\hat{G})$.

The hypergroups $\mathcal{T}$ and $\mathcal{K}(\hat{G})$ are strong hypergroup. Since $\hat{\mathcal{T}} = \mathcal{K}^\alpha(\mathbb{T})$ and $\widehat{\mathcal{K}(\hat{G})} = \mathcal{K}(G)$ where $\mathcal{K}^\alpha(\mathbb{T})$ is the orbital hypergroup of the action α of $\mathbb{Z}_2$ on $\mathbb{T}$ and $\mathcal{K}(G)$ is the conjugacy class hypergroup of G.

Conjecture When $q \neq 1$, the hypergroups $\mathcal{T}_q$ and $\mathcal{K}_q(\hat{G})$ are not strong.

5 Q-deformations of the Chebyshev hypergroup of the second kind

Let $U_n(x)$ be the Chebyshev polynomial of the second kind of degree n, then $U_n(x)$ $(n = 0, 1, 2, \cdots)$ satisfy the following equation.

$$U_m(x)U_n(x) = U_{|m-n|}(x) + U_{|m-n|+2}(x) + \cdots + U_{m+n}(x).$$

Hence, for the set $\mathcal{U} = \{U_0, U_1, \cdots, U_n, \cdots\}$, $(\mathcal{U}, \mathbb{C}\mathcal{U}, \diamond, ^-)$ has the structure of a fusion rule algebra by the product

$$U_m \diamond U_n := U_m(x)U_n(x).$$

The canonical dimension function d of $\mathcal{U}$ is given by

$$d(U_n) = n + 1.$$

Put

$$c_n := \frac{1}{d(U_n)} U_n \quad \text{and} \quad F_d(\mathcal{U}) := \{c_0, c_1, \cdots, c_n, \cdots\}.$$

Then $F_d(\mathcal{U})$ becomes a hypergroup by the product

$$c_m \circ c_n := \frac{1}{d(U_m)} U_m \diamond \frac{1}{d(U_n)} U_n.$$

This hypergroup is called the Chebyshev hypergroup of the second kind. The structure equation is

$$\begin{aligned} c_m \circ c_n = & \frac{|m-n|+1}{(m+1)(n+1)} c_{|m-n|} + \frac{|m-n|+3}{(m+1)(n+1)} c_{|m-n|+2} + \cdots \\ & + \frac{m+n+1}{(m+1)(n+1)} c_{m+n}, \end{aligned}$$

where c_0 is the unit element and $c_n^* = c_n$.

Next, we consider a mapping d_q from $\mathcal{U}$ to $\mathbb{R}_+^\times$. For $U_n \in \mathcal{U}$, the mapping d_q defined by

$$d_q(U_n) := U_n \left(\frac{q + q^{-1}}{2} \right) = \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} = q^n + q^{n-2} + \cdots + q^{-n} \quad (0 < q \leq 1).$$

Proposition 5.1 The mapping d_q from $\mathcal{U}$ to $\mathbb{R}_+^\times$ is a dimension function of $\mathcal{U}$.

Proof The proof is obtained in a similar way to Proposition 4.1. □

Put

$$X_n(q) := \frac{1}{d_q(U_n)} U_n \text{ and } \mathcal{U}_q := \{X_0(q), X_1(q), \cdots, X_n(q), \cdots\}.$$

Then the following theorem holds.

Theorem 5.2 $\mathcal{U}_q$ becomes a hypergroup which is a q-deformation of $F_d(\mathcal{U})$. The structure equation is

$$X_m(q) \circ X_n(q) = \sum_{k \in S(m,n)} a_{mn}^k(q) X_k(q)$$

where $S(m,n) = \{|m-n|, |m-n|+2, \cdots, m+n\}$ and

$$a_{mn}^k(q) = \frac{(q-q^{-1})(q^{k+1}-q^{-(k+1)})}{(q^{m+1}-q^{-(m+1)})(q^{n+1}-q^{-(n+1)})}.$$

Proof By Proposition 3.1, $\mathcal{U}_q$ is a hypergroup. The convolution $X_m(q)$ and $X_n(q)$ is

$$X_m(q) \circ X_n(q) = \frac{1}{d_q(U_m)d_q(U_n)} U_m \diamond U_n = \sum_{k \in S(m,n)} \frac{d_q(U_k)}{d_q(U_m)d_q(U_n)} X_k(q)$$

where $S(m,n) = \{|m-n|, |m-n|+2, \cdots, m+n\}$. Then,

$$a_{mn}^k(q) = \frac{d_q(U_k)}{d_q(U_m)d_q(U_n)} = \frac{(q-q^{-1})(q^{k+1}-q^{-(k+1)})}{(q^{m+1}-q^{-(m+1)})(q^{n+1}-q^{-(n+1)})}.$$

The coefficients $a_{mn}^k(q)$ can also write

$$a_{mn}^k(q) = \frac{(q^k+q^{k-2}+\cdots+q^{-k})}{(q^m+q^{m-2}+\cdots+q^{-m})(q^n+q^{n-2}+\cdots+q^{-n})}.$$

Hence, we see that $a_{mn}^k(q)$ is continuous with respect to q. The involution of $\mathcal{U}_q$ is an identity map by the fact that

$$X_n(q) = \frac{1}{d_q(U_n)} U_n \text{ and } U_n^* = U_n.$$

When $q = 1$, it is clear that $X_n(1) = \frac{1}{n+1} U_n$. Since $d_q(U_n) = q^n + q^{n-2} + \cdots + q^{-n}$ is continuous, $X_n(q) \to \frac{1}{n+1} U_n$ as $q \to 1$. Hence, $\mathcal{U}_q$ is a q-deformation of $F_d(\mathcal{U})$. □

Next, we consider the relation with the dual $\widehat{SU(2)} = \{\pi_0, \pi_1, \cdots, \pi_n, \cdots\}$, where

$$\dim \pi_n = n+1 \text{ and } \pi_m \otimes \pi_n \cong \pi_{|m-n|} \oplus \pi_{|m-n|+2} \oplus \cdots \oplus \pi_{m+n}.$$

The character ρ_n of $\pi_n \in \widehat{SU(2)}$ is given by

$$\rho_n(g) = \mathrm{tr}(\pi_n(g)) \quad (g \in SU(2)).$$

Then,

$$\rho_m \rho_n = \rho_{|m-n|} + \rho_{|m-n|+2} + \cdots + \rho_{m+n}$$

as a function on $SU(2)$. Put $\mathcal{F}(\widehat{SU(2)}) = \{\rho_0, \rho_1, \cdots, \rho_n, \cdots\}$. Then $\mathcal{F}(\widehat{SU(2)})$ becomes a fusion rule algebra and isomorphic to $\mathcal{U}$.

For the representative element $g = g_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SU(2)$ in the conjugacy class of $SU(2)$,

$$\rho_n(g_\theta) = \frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta).$$

Put

$$\chi_n = \frac{1}{d_q(U_n)}\rho_n \quad (0 < q \leq 1)$$

and

$$\mathcal{K}_q(\widehat{SU(2)}) = \{\chi_0, \chi_1, \cdots, \chi_n, \cdots\}.$$

Then, $\mathcal{K}_q(\widehat{SU(2)})$ is a hypergroup which is isomorphic to $\mathcal{U}_q$.

Remark The hypergroups $F_d(\mathcal{U})$ and $\mathcal{K}(\widehat{SU(2)})$ are strong hypergroups.

Conjecture The q-deformations $\mathcal{U}_q$ of $F_d(\mathcal{U})$ and $\mathcal{K}_q(\widehat{SU(2)})$ of $\mathcal{K}(\widehat{SU(2)})$ are not strong when $q \neq 1$.

Conjecture The character hypergroup $\mathcal{K}(\widehat{SU_q(2)})$ of the quantum group $SU_q(2)$ is well defined and

$$\mathcal{K}_q(\widehat{SU(2)}) \cong \mathcal{K}(\widehat{SU_q(2)}).$$

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Address

Tatsuya Tsurii : Osaka Prefecture University
Graduate School of Science
1-1 Gakuen-cho, Nakaku, Sakai
Osaka, 599-8531
Japan

e-mail : `tatsuya_t_99@yahoo.co.jp`

Current address

Nara University of Education
Center for New Education of Science and Mathematics
Takabatake-cho, Nara, 630-8528
Japan

e-mail : `tatsuya_tsurii@nara-edu.ac.jp`