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A CONDITION FOR REDUCING EXPANSIVE VARIATIONS OF OPTIMAL POLICY IN RESTAURANT REVENUE MANAGEMENT

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Abstract. An industry which is recently applied to revenue management is restaurant. The revenue management for restaurant is called *restaurant revenue management*. The restaurant revenue management has a problem by which state space enormously expands because of multi-dimensional resources and customers. This problem gives rise to some practical difficulty: computation complexity increases, required data size for optimal policy becomes larger and etc.. This paper presents a sufficient condition for substantially reducing data size of optimal policy.

1 Introduction There are many scenes at which a business manager controls the limited resources for variable demand to aim to maximize his(her) company's benefit. For companies with fixed capacity, dealing with perishable products and large fixed cost, how to manage the demand (e.g. setting variable terms and prices for each product and etc.) significantly affects their benefit. This management is widely known as *revenue management* or *yield management*. Traditional applications of the revenue management are airline, hotel and car rental industries.

In theory of the revenue management, there is a problem in which threshold price is solved by using dynamic programming. This problem is used to decide whether a revenue manager should accept for a request of reservation in a certain period to maximize revenue. This control by using the threshold price is called *bid price control*. Lee and Hersh(1993) suggested a bid price control model for airline industry with single resource, multiple booking classes and multiple seat booking. Further, they indicated monotonicity of threshold price for their model. However, the model did not include assumptions of cancellation and overbooking. Subramanian et al.(1999) considered a model with cancellation and overbooking, and added some assumptions to declare monotonicity of threshold price. Researches, problems, traditional models, and a glossary of revenue management for airline can be found in McGill and Ryzin(1999).

Recently, for non-traditional industries, the bid price control models have been widely researched. Chiang, Chen and Xu(2007) reviewed recent application and techniques of revenue management. One of the non-traditional industries which is applicable to the theory of revenue management is restaurant industry. The revenue management for restaurant is called *restaurant revenue management*. The bid price control model for the restaurant revenue management additionally need to decide which table a party should be allocated if the party should be accepted. The policy is called *seating policy* in Guerriero et al.(2014). There are not many researches which deals with the seating policy. Bertsimas and Shioda(2003) presented some models: an integer programming, a stochastic programming, and an approximate dynamic programming model. Guerriero et al.(2014) suggested a dynamic programming model with no waiting line, reservation, and meal duration by using the

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techniques of *network revenue management.* These studies have focused on making models and algorithms for solving expected total revenue because the bid price control approach in restaurant revenue management is difficult for solving. The difficulty is due largely to *the curse of dimensionality.*

1.1 The curse of dimensionality in restaurant revenue management The bid price control model in restaurant revenue management is referred as a model in *network revenue management* because restaurants have multi-dimensional capacity which is the different size of tables. It is known that a model in the network revenue management is more complex than a model with single-resource. A part of reasons for the complexity is that state space enormously expands. Furthermore, in restaurant revenue management, state space of a bid price control model needs to enlarge more than ordinary models(seeing as an example in Sec.3.2 of Talluri and Ryzin(2005)) in network revenue management. Because the bid price control model in restaurant revenue management must include departure process of parties which implies cancellation process in the airline or hotel industry. Fig.1 shows states for cases with no-cancellation and with cancellation. The case without cancellation process is Case 1 and the another case with cancellation process is Case 2 in Fig.1.

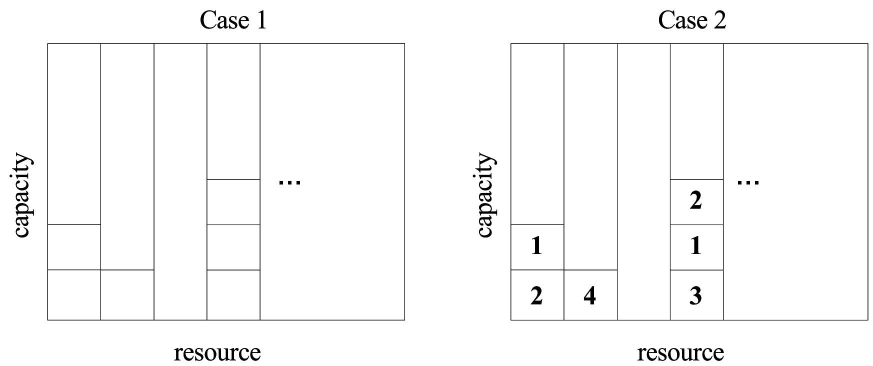

Figure 1: States in the cases without cancellation process(Case 1) and with cancellation process(Case 2).

In revenue management, the departure process commonly depends on a customer class. (See p.500 in Talluri and Ryzin(2005).) It is actually intuitive that the departure process depends on the customer class which implies size of party in restaurant revenue management. The state of the Case 1 in Fig.1 does not need to preserve the customer classes which have arrived until a certain period because of an assumption of the no-departure process. Hence, the state in the Case 1 is shown as a vector for capacity. In contrast, the state in the Case 2 needs to preserve the customer classes that have arrived until a certain period. This means that each resource in the Case 2 have a vector for the customer classes. Thus, the state space of Case 2 is much larger than the one of the Case 1. Additionally, If meal duration for each customer which is stated in Kimes et al.(1999)(2002), Guerriero et al.(2014) and etc. is considered, then an information about how long each customer has been in the state must add to the state and solving the seating problem as exact dynamic programming approach is practically impossible.

To broach this argument, in section 2, this paper presents an exact dynamic programming model for seating policy, given some conditions to simplify. Furthermore, some monotonicities are indicated by setting some realistic assumptions. From the monotonicities, this paper shows a sufficient condition for reducing varieties of optimal policy, and its structural property. In section 3, the structural property is confirmed by numerical examples.

2 A model and its property

2.1 Conditions and notation To simplify a model, some conditions are given to parties and tables. The conditions are that a composition of the tables can not be modified to suit the arriving party, size of the parties can not be divided to suit the tables, and the size of the parties does not exceed a maximum of the tables in the restaurant. Further, tables of the same size and seats are not distinguished. Suppose sets $P = \{1, \cdots, \overline{P}\}$ and $I = \{1, \cdots, \overline{I}\}$ for notations. The notations about the party and the table are shown as

- $\overline{P}$: the number of different party sizes,
- $\overline{I}$: the number of different table sizes,
- g_p: the party size for $p \in P$,
- t_i: the table size for $i \in I$,
- m_i: the number of the table for $i \in I$.

To simplify, we regard $p \in P$ as a party with party size g_p, and $i \in I$ as a table with table size t_i, respectively. Throughout this paper, a party p and a table i are indexed as $g_1 < g_2 < \cdots < g_p$ and $t_1 < t_2 < \cdots < t_i$, respectively. In addition, subsets for $p \in P$ and $i \in I$ are indicated as

- $P_i = \{p \in P : g_p \leq t_i\}, i \in I$: the party set which is able to be allocated to a table $i \in I$ with the number of the different party sizes $\overline{P}_i$,
- $I_p = \{i \in I : g_p \leq t_i\}, p \in P$: the table set to which a party $p \in P$ is able to be allocated with the number of the different table sizes $\overline{I}_p$.

The opening horizon is sufficiently divided into the $N + 1$ periods $n = 0, 1, \cdots, N$. One event of the customer's arrival or departure occurs in the period n. A period N corresponds to opening of the restaurant and a period 0 corresponds to closing of the restaurant. Parties arrive according to time-dependent Poisson process while the restaurant is opening. All of them are walk-in customers, without reservation. Departure process of the parties depends on not their length of staying time, but the state of restaurant and the period. Notations about the state space, the arrival and departure rate, and expected revenue are shown as

- $\overline{X}_i = \{\boldsymbol{x}_i = (x^i_p) : x^i_p \geq 0, p \in P_i\,; \sum_p x_p \leq m_i\}, i \in I$: state space for a table $i \in I$ where x^i_p is the number of parties who are sitting in a table $i \in I_p$,
- $X_n = \{X = (\boldsymbol{x}_1 | \cdots | \boldsymbol{x}_{\overline{I}}) : \boldsymbol{x}_i \in \overline{X}_i, i \in I; \sum_i \sum_p x^i_p \leq N - n\}, n = 0, \cdots N$: state space for a restaurant with a submatrix $\boldsymbol{x_i}$ in a period n,
- r^n_p: the expected revenue for a party $p \in P$ in a period n,
- $\lambda^n_p(X)$: the arrival rate for a party $p \in P$ and a state $X \in X_n$ in a period n, where $\lambda^n_p(X) > 0$,
- $q^n_{ip}(X)$: the departure rate for a party $p \in P_i$ where $i \in I$, and a state $X \in X_n$ in a period n,

- λ_0^n: a probability of a null event in period n.

Suppose that $|X_n|$ corresponds to the number of elements of the state space X_n for n. Referring p.15 in Stanley(1997), we can obtain a maximum of $|X_n|$ for n: $\chi = \max_n\{|X_n|\}$ as

$$\chi = \prod_{i=1}^{\overline{I}} \binom{m_i + \overline{P}_i}{\overline{P}_i}. \tag{1}$$

The eq.(1) is helpful to roughly estimate size of state space for a restaurant. From the assumption of the arrival and the departure process in a period n, the equation

$$\sum_{p=1}^{\overline{P}} \lambda_p^n(X) + \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} q_{ip}^n(X) + \lambda_0^n(X) = 1 \tag{2}$$

is obtained.

2.2 A formulation of model Let $U_n(X)$ be the maximal expected revenue from operating over periods n to 0. Firstly, Suppose the maximal expected revenue in a general form as follows.

$$\begin{aligned} U_n(X) = \sum_{p=1}^{\overline{P}} \lambda_p^n(X) \left\{ \left(r_p^n - \min_{i \in I_p} \Delta_p^i U_{n-1}(X) \right)^+ + U_{n-1}(X) \right\} \\ + \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} q_{ip}^n(X) U_{n-1}(X - \boldsymbol{e}_p^i) \\ + \left(1 - \sum_{p=1}^{\overline{P}} \lambda_p^n(X) - \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} q_{ip}^n(X) \right) U_{n-1}(X), \\ X \in X_n, n \geq 1, \end{aligned} \tag{3}$$

where $\boldsymbol{e}_p^i = (\boldsymbol{x}_1|\cdots|\boldsymbol{x}_{\overline{I}})$ in which $x_p^i = 1$ and otherwise 0, $(a)^+ = \max\{a, 0\}$, and $\Delta_p^i U_n(X) = U_n(X) - U_n(X + \boldsymbol{e}_p^i)$. Boundary conditions are that $U_n(X) = -\infty$ for $X \notin X_n$, and $U_0(X) = 0$ for $X \in X_0$. The $\min_{i \in I_p} \Delta_p^i U_n(X)$ means a threshold price for a party $p \in P_i$, such that the party p who arrives for the state X in n is acceptable if r_p^n exceeds the threshold price $\min_{i \in I_p} \Delta_p^i U_n(X)$ and not acceptable if r_p^n is less than the threshold price $\min_{i \in I_p} \Delta_p^i U_n(X)$(See pp.31-32 in Talluri and Ryzin(2005).). $\Delta_p^i U_n(X)$ is an opportunity cost of accepting the party p for the table $i \in I_p$ in $n+1$. Note that $\lambda_0^n(X) = 1 - \sum_{p=1}^{\overline{P}} \lambda_p^n(X) - \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} q_{ip}^n(X)$ from eq.(2). The first member of the right hand in (3) indicates a expected value in a case where a party arrives at a restaurant in a period n. If a p is accepted in the table $i \in I_p$, then a expected value for the case is $r_p^n - \Delta_p^i U_{n-1}(X)$ in n. The second member indicates a expected value in a case where a party sitting in a restaurant leaves in a period n. The third member is for a case where no event occurs in a period n. From eq.(3), optimal policy is indicated as below.

Optimal policy: An optimal policy for a party $p \in P$ and a state $X \in X_n$ is that if

$r_p^n - \min_{i\in I_p} \Delta_p^i U_{n-1}(X) \geq 0$, then a party p is accepted in a table $\arg\min_{i\in I_p} \Delta_p^i U_{n-1}(X)$, and if $r_p^n - \min_{i\in I_p} \Delta_p^i U_{n-1}(X) < 0$, then a party p is denied.

Then, Some assumptions are supposed to simplify the eq.(3).

Assumption 1. assume $\lambda_p^n(X) = \lambda_p^n$ for $p \in P$ and $X \in X_n$ in $n = 0, \cdots, N$.

Assumption 2. assume $q_{ip}^n(X) = x_p^i q_{ip}^n$ for $p \in P_i$ where $i \in I$ and $X \in X_n$ in $n = 0, \cdots, N$.

The Assumption 1 indicates that arrival rates do not depend on states, which means that congestion level of a restaurant does not affect the arrival rates. The Assumption 2 indicates that a party p in a table i and a period n departs independently of other parties sitting in other table, which implies that a party leaves from a restaurant according to exponential distribution. Let t be $\Delta_N + \Delta_{N-1} + \cdots + \Delta_{n+1}$ where Δ_n is the length of the nth period. Suppose $t = 0$ for Nth period. λ_p^n indicates $f_p(t)\Delta_n$ where $f_p(t), 0 \leq t \leq \Delta_N + \Delta_{N-1} + \cdots + \Delta_1$ is a mean of time-dependent Poisson distribution for a p. q_{ip}^n indicates $\mu_{ip}(t)\Delta_n$ where $\mu_{ip}(t), 0 \leq t \leq \Delta_N + \Delta_{N-1} + \cdots + \Delta_1$ is a parameter of exponential distribution at time t for a p sitting in a table $i \in I_p$. For detail of this method, Subramanian et al.(1999) explained in Appendex A.

Under these assumptions, the eq.(3) can be rewritten as the equation

$$
\begin{aligned}
U_n(X) = \sum_{p=1}^{\overline{P}} \lambda_p^n \left\{ \left(r_p^n - \min_{i\in I_p} \Delta_p^i U_{n-1}(X) \right)^+ + U_{n-1}(X) \right\} \\
+ \sum_{p=1}^{\overline{P}} \sum_{i\in I_p} x_p^i q_{ip}^n U_{n-1}(X - \boldsymbol{e}_p^i) \\
+ \left(1 - \sum_{p=1}^{\overline{P}} \lambda_p^n - \sum_{p=1}^{\overline{P}} \sum_{i\in I_p} x_p^i q_{ip}^n \right) U_{n-1}(X), \\
X \in X_n, n \geq 1.
\end{aligned} \tag{4}
$$

Boundary conditions are not modified. The eq.(4) is close to a equation which is extended by cancellation process for the model with upgrades which is suggested as eq.(1) in Steinhardt and Gönsch(2012). However, state space of the model in Steinhardt and Gönsch(2012) is different from the one which is defined in this paper as previously shown in Sec.1.1. Note that the first member of eq.(4) is a case of the one of eq.(1) in Steinhardt and Gönsch(2012) because of physical bundles between parties and tables, and the condition on which composition of the tables and size of the parties are fixed. For proofs as following sections, policy vector $\boldsymbol{d}$ is defining.

Let the policy vector be $\boldsymbol{d} = (d_p)$ where $p \in P$. An element of the policy vector d_p is a table $i \in I_p$ $(d_p = i)$ if a party p is accepted into the table, or 0 $(d_p = 0)$ if a party p is denied. Assume that if there are some acceptable tables, then the smallest i is selected. As the result, a set of policy vector is defined as

$$D_n(X) = \{\boldsymbol{d} = (d_p) : (d_p = 0) \vee \left((X + \boldsymbol{e}_p^{d_p} \in X_n) \wedge (d_p \in I_p) \right), p \in P\}, X \in X_n, n = 1, \cdots, N.$$

2.3 Property of $\Delta_p^i U_n(X)$ and the optimal policy Supposing the Assumption 3 as below, a monotonicity which is similar to the monotonicity suggested as Proposition 1 in Steinhardt and Gönsch(2012) is obtained for $\Delta_p^i U_n(X)$ in eq.(4).

Assumption 3. assume $q^n_{\delta p} = q^n_{\delta' p}$ for $p \in P$ and $\delta, \delta' \in I_p$ in $n = 0, \cdots, N$ where $\overline{I}_p \geq 2$ and $\delta \neq \delta'$.

Lemma 1. Under assumuption 1 to 3, for a given $p \in P$ and $X \in X_n$ in $n = 0, \cdots, N$,

$$\Delta^{\delta}_p U_n(X) \leq \Delta^{\delta'}_p U_n(X) \tag{5}$$

where δ, $\delta' \in I_p$, $t_\delta < t_{\delta'}$, $\sum_p x^\delta_p < m_\delta$, and $\sum_p x^{\delta'}_p < m_{\delta'}$.

Proof. $U_n(X + \boldsymbol{e}^\delta_p) \geq U_n(X + \boldsymbol{e}^{\delta'}_p)$ should be indicated by induction for $\Delta^\delta_p U_n(X) \leq \Delta^{\delta'}_p U_n(X)$. For $n = 0$, It is obvious that $U_0(X+\boldsymbol{e}^\delta_p) = U_0(X+\boldsymbol{e}^{\delta'}_p) = 0$. Then, assume that $U_{n-1}(X + \boldsymbol{e}^\delta_p) \geq U_{n-1}(X + \boldsymbol{e}^{\delta'}_p)$. Let the first member, the second member, and the third member of the equation (4) call *arrival part, departure part*, and *null part*, respectively. In the following, we are indicating the orderings of each part.

Firstly, an order of the arrival part is indicated. The arrival part of eq.(4) is rewritten using the optimal vector as

$$\max_{\boldsymbol{d} \in D_n(X)} \left\{ \sum_{p|d_p \neq 0} \lambda^n_p (r^n_p + U_{n-1}(X + \boldsymbol{e}^{d_p}_p)) + \sum_{p|d_p = 0} \lambda^n_p U_{n-1}(X) \right\}.$$

Let optimal policy vectors for $U_n(X+\boldsymbol{e}^\delta_p)$ and $U_n(X+\boldsymbol{e}^{\delta'}_p)$ be $\boldsymbol{d}^{(\delta)*}$ and $\boldsymbol{d}^{(\delta')*}$, respectively.

For a given $p \in P$, there are four cases for $d^{(\delta)*}_p$ and $d^{(\delta')*}_p$ as follows.

i) In the case: $d^{(\delta)*}_p \neq 0$ and $d^{(\delta')*}_p \neq 0$, we should make a comparison between $r^n_p + U_{n-1}(X + \boldsymbol{e}^\delta_p + \boldsymbol{e}^{d^{(\delta)*}_p}_p)$ and $r^n_p + U_{n-1}(X + \boldsymbol{e}^{\delta'}_p + \boldsymbol{e}^{d^{(\delta')*}_p}_p)$ for the arrival parts of $U_n(X+\boldsymbol{e}^\delta_p)$ and $U_n(X+\boldsymbol{e}^{\delta'}_p)$. Further, this case is divided into two cases for ordering between $d^{(\delta)*}_p$ and $d^{(\delta')*}_p$.

i-1)In the case: $d^{(\delta)*}_p \leq d^{(\delta')*}_p$, from the inductive hypothesis, $r^n_p + U_{n-1}(X + \boldsymbol{e}^\delta_p + \boldsymbol{e}^{d^{(\delta)*}_p}_p) \geq r^n_p + U_{n-1}(X + \boldsymbol{e}^{\delta'}_p + \boldsymbol{e}^{d^{(\delta)*}_p}_p) \geq r^n_p + U_{n-1}(X + \boldsymbol{e}^{\delta'}_p + \boldsymbol{e}^{d^{(\delta')*}_p}_p)$ is obtained

i-2)In the case: $d^{(\delta)*}_p > d^{(\delta')*}_p$, from the inductive hypothesis and number of capacities of tables, $d^{(\delta)*}_p \leq \delta'$ and $d^{(\delta')*}_p = \delta$ is obtained. Thus, $r^n_p + U_{n-1}(X + \boldsymbol{e}^\delta_p + \boldsymbol{e}^{d^{(\delta)*}_p}_p) \geq r^n_p + U_{n-1}(X + \boldsymbol{e}^{\delta'}_p + \boldsymbol{e}^\delta_p) = r^n_p + U_{n-1}(X + \boldsymbol{e}^{\delta'}_p + \boldsymbol{e}^{d^{(\delta')*}_p}_p)$.

ii)In the case: $d^{(\delta)*}_p = 0$ and $d^{(\delta')*}_p \neq 0$, we should make a comparison between $U_{n-1}(X+\boldsymbol{e}^\delta_p)$ and $r^n_p + U_{n-1}(X + \boldsymbol{e}^{\delta'}_p + \boldsymbol{e}^{d^{(\delta')*}_p}_p)$. From the inductive hypothesis and $d^{(\delta)*}_p = 0$, $U_{n-1}(X + \boldsymbol{e}^\delta_p) \geq r^n_p + U_{n-1}(X + \boldsymbol{e}^\delta_p + \boldsymbol{e}^{d^{(\delta')*}_p}_p) \geq r^n_p + U_{n-1}(X + \boldsymbol{e}^{\delta'}_p + \boldsymbol{e}^{d^{(\delta')*}_p}_p)$.

iii)In the case: $d^{(\delta)*}_p \neq 0$ and $d^{(\delta')*}_p = 0$, we should make a comparison between $r^n_p + U_{n-1}(X + \boldsymbol{e}^\delta_p + \boldsymbol{e}^{d^{(\delta)*}_p}_p)$ and $U_{n-1}(X + \boldsymbol{e}^{\delta'}_p)$. From the inductive hypothesis and $d^{(\delta)*}_p \neq 0$, $r^n_p + U_{n-1}(X + \boldsymbol{e}^\delta_p + \boldsymbol{e}^{d^{(\delta)*}_p}_p) \geq U_{n-1}(X + \boldsymbol{e}^\delta_p) \geq U_{n-1}(X + \boldsymbol{e}^{\delta'}_p)$.

iv)In the case: $d^{(\delta)*}_p = d^{(\delta')*}_p = 0$, from the inductive hypothesis, It is obvious that $U_{n-1}(X+\boldsymbol{e}^\delta_p) \geq U_{n-1}(X + \boldsymbol{e}^{\delta'}_p)$.

Next, we consider the departure parts. To simplify the notation, suppose that $q^n_{ip} = q^n_p$. For the p, the departure parts of $U_n(X + \boldsymbol{e}^\delta_p)$ and $U_n(X + \boldsymbol{e}^{\delta'}_p)$ are

$$\sum_{i \in I_p} (x^i_p + e^{\delta i}_p) q^n_p U_{n-1}(X + \boldsymbol{e}^\delta_p - \boldsymbol{e}^i_p) \tag{6}$$

and

$$\sum_{i\in I_p}(x_p^i + e_p^{\delta' i})q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i), \tag{7}$$

respectively, where $e_p^{ki} = 1$ if $i = k$ and otherwise $e_p^{ki} = 0$. The eq.(6) and (7) can stand for

$$\begin{aligned} &q_p^n \left\{\cdots + (x_p^\delta + 1)U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^\delta) + \cdots + x_p^{\delta'} U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^{\delta'}) + \cdots\right\} \\ &\qquad\qquad = q_p^n \left\{U_{n-1}(X) + x_p^1 U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^1) + \cdots\right\} \end{aligned} \tag{8}$$

and

$$\begin{aligned} &q_p^n \left\{\cdots + x_p^\delta U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^\delta) + \cdots + (x_p^{\delta'} + 1)U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^{\delta'}) + \cdots\right\} \\ &\qquad\qquad = q_p^n \left\{U_{n-1}(X) + x_p^1 U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^1) + \cdots\right\}, \end{aligned} \tag{9}$$

respectively. Therefore, from the inductive hypothesis, $\sum_{i\in I_p}(x_p^i + e_p^{\delta i})q_p^n U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^i) \geq \sum_{i\in I_p}(x_p^i + e_p^{\delta' i})q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i)$ is obtained.

Finally, we consider the null parts. For the p, the null parts of $U_n(X+\boldsymbol{e}_p^\delta)$ and $U_n(X+\boldsymbol{e}_p^{\delta'})$ are

$$\left(1 - \lambda_p^n - \sum_{i\in I_p}(x_p^i + e_p^{\delta i})q_p^n\right) U_{n-1}(X + \boldsymbol{e}_p^\delta) \tag{10}$$

and

$$\left(1 - \lambda_p^n - \sum_{i\in I_p}(x_p^i + e_p^{\delta' i})q_p^n\right) U_{n-1}(X + \boldsymbol{e}_p^{\delta'}), \tag{11}$$

respectively. In these equations, the coefficients of the $U_{n-1}(X + \boldsymbol{e}_p^\delta)$ and $U_{n-1}(X + \boldsymbol{e}_p^{\delta'})$ are the same. Thus,

$$\left(1 - \lambda_p^n - \sum_{i\in I_p}(x_p^i + e_p^\delta)q_p^n\right) U_{n-1}(X + \boldsymbol{e}_p^\delta) \geq \left(1 - \lambda_p^n - \sum_{i\in I_p}(x_p^i + e_p^{\delta'})q_p^n\right) U_{n-1}(X + \boldsymbol{e}_p^{\delta'})$$

is obtained from the inductive hypothesis.

From these ordering of the arrival parts, the departure parts, and the null parts of $U_n(X + \boldsymbol{e}_p^\delta)$ and $U_n(X + \boldsymbol{e}_p^{\delta'})$, the eq.(5) is indicated. □

The Assumption 3 means that departure rate depends on only a period and a party size. Thus, q_{ip}^n stands for q_p^n to simplify in the following. For this assumption, Kimes et al.(2004) suggested that meal duration which relates to the departure rate did not depend on position, configuration, and size of tables while it depended on the size of a party. Therefore, the Assumption 3 can be considered as realistic one.

For the submatrix $\boldsymbol{x}_i$ of $X \in X_n$, suppose $\sum_p x_p^i := x^i$. Furthermore, let $X \in X_n$ and $\hat{X} \in X_n$ be the states with submatrices $\boldsymbol{x}_i$ and $\hat{\boldsymbol{x}}_i$, respectively, where $X \neq \hat{X}$ and $x^i = \hat{x}^i$ for $i \in I$. This assumption for X and $\hat{X}$ is used in the following this section.

The Claim 1 is obtained from the Lemma 1.

Claim 1. If optimal policy vectors $\boldsymbol{d}^*$ and $\hat{\boldsymbol{d}}^*$ for the states X and $\hat{X}$, respectively, are $d_p^* \neq 0$ and $\hat{d}_p^* \neq 0$, then $d_p^* = \hat{d}_p^*$.

Proof. From $d_p^* \neq 0$ and $\hat{d}_p^* \neq 0$, arrival parts of $U_n(X)$ and $U_n(\hat{X})$ are

$$\lambda_p^n(r_p^n + U_{n-1}(X + \boldsymbol{e}^{d_p^*})) \tag{12}$$

and

$$\lambda_p^n(r_p^n + U_{n-1}(\hat{X} + \boldsymbol{e}^{\hat{d}_p^*})), \tag{13}$$

respectively. From $x^i = \hat{x}^i$, the table sets which are able to be d_p^* and $\hat{d}_p^*$ for $p \in P$ are the same. Then, $d_p^* = \hat{d}_p^*$ is obtained. $\square$

Suppose an assumption for the ordering of departure process of parties $p \in P$, and a proposition about a monotonicity of $\Delta_p^i U_n(X)$ for $p \in P$ as below.

Assumption 4. For $\psi \in P$ and $\psi' \in P$ where $\psi < \psi'$, assume $q_\psi^n \geq q_{\psi'}^n$ in $n = 0, \cdots, N$.

Proposition 1. Under the Assumption 1 to 4, for a given $\sigma \in I$ at which $\overline{P}_\sigma \geq 2$,

$$\Delta_\psi^\delta U_n(X) \leq \Delta_{\psi'}^\delta U_n(X), \tag{14}$$

where $\psi, \psi' \in P_\sigma$ and $\psi < \psi'$, in $n = 0, \cdots, N$.

Proof. It is obtained by induction. $U_n(X + \boldsymbol{e}_\psi^\delta) \geq U_n(X + \boldsymbol{e}_{\psi'}^\delta)$ should be indicated for $\Delta_\psi^\delta U_n(X) \leq \Delta_{\psi'}^\delta U_n(X)$. In the case $n = 0$, $U_0(X + \boldsymbol{e}_\psi^\delta) = U_0(X + \boldsymbol{e}_{\psi'}^\delta)$ is clear. Then, assume that $\Delta_\psi^\delta U_{n-1}(X) \leq \Delta_{\psi'}^\delta U_{n-1}(X)$.

Firstly, we consider about the arrival parts. Let the optimal vectors for the states $X + \boldsymbol{e}_\psi^\delta$ and $X + \boldsymbol{e}_{\psi'}^\delta$ be $\boldsymbol{d}^{(\psi)^*}$ and $\boldsymbol{d}^{(\psi')^*}$, respectively.

i)In the case: $d_p^{(\psi)^*} \neq 0$ and $d_p^{(\psi')^*} \neq 0$, we make a comparison between $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}})$ and $r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$. The optimal vectors for the states $X + \boldsymbol{e}_\psi^\delta$ and $X + \boldsymbol{e}_{\psi'}^\delta$ are $d_p^{(\psi)^*} = d_p^{(\psi')^*}$ from the Claim 1 because capacities of the states are the same. Hence, $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$ is indicated.

ii)In the case: $d_p^{(\psi)^*} \neq 0$ and $d_p^{(\psi')^*} = 0$, we compare $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}})$ to $U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta)$. From the inductive hypothesis and $d_p^{(\psi')^*} = 0$, $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}}) \geq U_{n-1}(X + \boldsymbol{e}_\psi^\delta) \geq U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta)$ is obtained.

iii)In the case: $d_p^{(\psi)^*} = 0$ and $d_p^{(\psi')^*} \neq 0$, we make a comparison between $U_{n-1}(X + \boldsymbol{e}_\psi^\delta)$ and $r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$. From the inductive hypothesis and $d_p^{(\psi)^*} = 0$, $U_{n-1}(X + \boldsymbol{e}_\psi^\delta) > r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$ is obtained.

iv)In the case: $d_p^{(\psi)^*} = 0$ and $d_p^{(\psi')^*} = 0$, it is obvious.

Then, we consider the departure parts of $U_n(X + \boldsymbol{e}_\psi^\delta)$ and $U_n(X + \boldsymbol{e}_{\psi'}^\delta)$ which are

$$\sum_{p=1}^{\overline{P}} \sum_{i \in I_p} (x_p^i + e_{\psi p}^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^i) \tag{15}$$

and

$$\sum_{p=1}^{\overline{P}} \sum_{i \in I_p} (x_p^i + e_{\psi' p}^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i), \tag{16}$$

respectively, where $e_{lp}^{ki} = 1$ if $i = k$ and $p = l$, otherwise $e_{lp}^{ki} = 0$.

We should consider only the cases $p = \psi$, $i = \sigma$ and $p = \psi'$, $i = \delta$ for the eq.(15) and eq.(16) as

$$\begin{aligned} \cdots + q_\psi^n U_{n-1}(X) + x_\psi^\delta q_\psi^n U_{n-1}(X + \boldsymbol{e}_\psi^\delta - \boldsymbol{e}_\psi^\delta) + \cdots \\ \cdots + x_{\psi'}^\delta q_{\psi'}^n U_{n-1}(X + \boldsymbol{e}_\psi^\delta - \boldsymbol{e}_{\psi'}^\delta) + \cdots \end{aligned} \tag{17}$$

and

$$\begin{aligned} \cdots + x_\psi^\delta q_\psi^n U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta - \boldsymbol{e}_\psi^\delta) + \cdots \\ \cdots + q_{\psi'}^n U_{n-1}(X) + x_{\psi'}^\delta q_{\psi'}^n U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta - \boldsymbol{e}_{\psi'}^\delta) + \cdots . \end{aligned} \tag{18}$$

From the inductive hypothesis and the Assumption 4, it is indicated that

$$\sum_{p=1}^{\overline{P}} \sum_{i \in I_p} (x_p^i + e_{\psi p}^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta} - \boldsymbol{e}_p^i) \geq \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} (x_p^i + e_{\psi' p}^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i).$$

Finally, we consider the null parts. It is clear that coefficients of the null parts of $U_n(X + \boldsymbol{e}_\psi^\delta)$ and $U_n(X + \boldsymbol{e}_{\psi'}^\delta)$ are the same.

From the ordering of the each part, we obtain that $\Delta_\psi^\delta U_n(X) \leq \Delta_{\psi'}^\delta U_n(X)$. □

The Assumption 4 means that a party stochastically stays longer than the smaller one. Thompson(2009) applied this assumption to his simulation study. Furthermore, the researches in Kimes et al.(2003) and Bell and Pliner(2004) showed that a correlation between the size of a party and meal duration is significantly positive for real restaurants. Therefore, the Assumption 4 is considered as realistic one.

The Remark 1 for the Proposition 1 is indicated as follows.

Remark 1. Note that the monotonicity of the Proposition 1 does not depend on the expected revenue r_p^n, which is same to the Lemma 1. Seeing the proof for the Proposition1, we can recognize that the Assumption 4 is used in only the members of $U_{n-1}(X)$ in the eq.(17) and (18). Further, the orderings for the each part expect the the members of $U_{n-1}(X)$ in eq.(17) and (18) is conditioned by the inductive hypothesis and facts of the cases. Thus, the ordering of the Proposition 1 is conditioned by only the ordering of departure rates between the parties.

Thus, from the Proposition 1 and its character, a difference between the maximal expected revenues $U_n(X)$ and $U_n(\hat{X})$ stems from differences for departure rates among parties. If there are differences for departure rates among parties, then they are affected by all factors; arrival rates, rewards, and etc. as a matter of course. However, If there are not the differences for departure rates among parties, then there is not the difference between $U_n(X)$ and $U_n(\hat{X})$, nevertheless the parties have difference parameters each other.

From the monotonicities which is indicated in this paper, a sufficient condition which is able to reduce variations of optimal policies can be obtained. The sufficient condition is shown as Theorem 1. For given a party $p \in P$, let $\overline{d}_p^*$ be the minimum $i \in I_p$ where the $m_i - \sum_{k \in P_i} x_k^i > 0$.

Theorem 1. If the condition

$$r_p^n \notin \left[\min(\Delta_p^{\overline{d}_p^*} U_{n-1}(X), \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})), \max(\Delta_p^{\overline{d}_p^*} U_{n-1}(X), \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X}))\right) \tag{19}$$

is satisfied for a given $p \in P$ and n, then the optimal vectors $\boldsymbol{d}^*$ and $\hat{\boldsymbol{d}}^*$ for the states X and $\hat{X}$, respectively is that $d_p^* = \hat{d}_p^*$ for the $p \in P$ in the period n.

Proof. The d_p^* and $\hat{d}_p^*$ are divided in four cases.
i)In the case: $d_p^* \neq 0$ and $\hat{d}_p^* \neq 0$, from the Claim 1, $d_p^* = \hat{d}_p^*$.
ii)In the case: $d_p^* = 0$ and $\hat{d}_p^* \neq 0$, from $d_p^* = 0$, we obtain that

$$\lambda_p^n U_{n-1}(X) > \max_{d_p | d_p \neq 0} \left\{ \lambda_p^n (r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{d_p})) \right\}. \tag{20}$$

In addition, the eq.(20) can be rewritten to

$$\lambda_p^n U_{n-1}(X) > \lambda_p^n (r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\hat{d}_p^*})) \tag{21}$$

from the condition $x^i = \hat{x}^i$. We also obtain that

$$\lambda_p^n U_{n-1}(\hat{X}) \leq \lambda_p^n (r_p^n + U_{n-1}(\hat{X} + \boldsymbol{e}_p^{\hat{d}_p^*})) \tag{22}$$

because of $\hat{d}_p^* \neq 0$. From the eq.(21) and (22), we indicate

$$\Delta_p^{\hat{d}_p^*} U_{n-1}(\hat{X}) \leq r_p^n < \Delta_p^{\hat{d}_p^*} U_{n-1}(X) \tag{23}$$

as a condition for $d_p^* = 0$ and $\hat{d}_p^* \neq 0$.
iii)In the case: $d_p^* \neq 0$ and $\hat{d}_p^* = 0$, calculating this case similar to the case ii), we can obtain

$$\Delta_p^{d_p^*} U_{n-1}(X) \leq r_p^n < \Delta_p^{d_p^*} U_{n-1}(\hat{X}) \tag{24}$$

as a condition for $d_p^* \neq 0$ and $\hat{d}_p^* = 0$.
iv)In the case: $d_p^* = 0$ and $\hat{d}_p^* = 0$, it is clearly.

Then, the relation between $\hat{d}_p^*$ and d_p^* in eq.(23) and (24) is $\hat{d}_p^* = d_p^* = \overline{d}_p^*$ due to $x^i = \hat{x}^i$, Lemma 1, and $\hat{d}_p^*, d_p^* \neq 0$. Therefore, if a range which does not include the ranges (23) and (24):

$$r_p^n \notin \left[\min(\Delta_p^{\overline{d}_p^*} U_{n-1}(X), \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})), \max(\Delta_p^{\overline{d}_p^*} U_{n-1}(X), \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X}))\right) \tag{25}$$

is satisfied for a $p \in P$ and n, then $d_p^* = \hat{d}_p^*$. □

The remark of the Theorem 1 is below.

Remark 2. The range(19) indicates a sufficient condition which makes the same optimal policy for the state X and $\hat{X}$. The width of the range $|\Delta_p^{\overline{d}_p^*} U_{n-1}(X) - \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})|$ stands for difficulty of reducing variety of the optimal policies. If the width becomes narrower, then it is more difficult to insert the expected revenue r_p^n into the range and optimal policy goes to depend only capacities for tables.

The width of the range $|\Delta_p^{\overline{d}_p^*} U_{n-1}(X) - \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})|$ can be rewritten $|U_{n-1}(\hat{X}) - U_{n-1}(X) + U_{n-1}(X + \boldsymbol{e}_p^{\overline{d}_p^*}) - U_{n-1}(\hat{X} + \boldsymbol{e}_p^{\overline{d}_p^*})|$ where the Proposition 1 is applicable to $U_{n-1}(\hat{X}) - U_{n-1}(X)$ and $U_{n-1}(X + \boldsymbol{e}_p^{\overline{d}_p^*}) - U_{n-1}(\hat{X} + \boldsymbol{e}_p^{\overline{d}_p^*})$. If there are not differences in departure rates among parties, then the width is effected by nothing because the width is zero, regardless of existing differences in arrival rates or expected revenues among the parties. As a consequence of this property, existing the differences in departure rates among parties is an only trigger for expanding varieties of optimal policy.

3 Numerical Examples In this section, we confirm the feature which is stated in the Remark 2. Numerical examples are computed using an equation which is applied the Assumptions 1 to 4 to the eq.(4). Configurations for tables and parties are which $\overline{P} = 2$, $\overline{I} = 2$, $g_1 = 1, g_2 = 2$, $t_1 = 1, t_2 = 2$, $m_1 = 2$, and $m_2 = 2$. From this parameters sets, χ is 18 by using eq.(1). Arrival rates, departure rates, and expected revenues for each party $p \in P$ in a period n are shown in Table 1.

The parameters set in Table 1 is named Sample 1. The Sample 1 has a single peak for the arrival rates, departure rates, and expected revenues. The peak time is likely lunch time. The expected revenues in the Sample 1 are set to increase as they get closer to the peak time since a restaurant which is considered for this section also serves as a cafe except in lunch time. Optimal policies for $p = 1$ which is computed from the Sample 1 are shown in Table 2. The values in cells of the Table 2 stand for policy vectors.

Seeing optimal policies for states (2|1,0) and (2|0,1), we can find that the optimal policies in $n = 16$ and 17 are difference between the states; nevertheless capacities for the states are the same. Let the states (2|1,0) and (2|0,1) be X and $\hat{X}$, respectively. To confirm the Theorem 1 for the states, $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$ and $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$ where $\overline{d}_1^* = 2$, are shown in Table 3 which also includes the expected revenue r_1^n to make a comparison easily.

Table 1: Arrival rates, departure rates, and expected revenues of Sample 1.

	Arrival Rate		Departure Rate		Reward	
n	λ_1^n	λ_2^n	q_1^n	q_2^n	r_1^n	r_2^n
0-5	.021	.014	.018	.014	3	6
6-7	.105	.070	.088	.070	4	8
8-11	.150	.100	.125	.100	5	10
12-13	.105	.070	.088	.070	4	8
14-20	.021	.014	.018	.014	3	6

r_1^{16} and r_1^{17} are put in the ranges between $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$ and $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$ for $n = 16$ and $n = 17$. Then, for a case where there is not difference in departure rates between parties, we have computed the range. Sample 2 is the case in which the departure rates for $p = 2$ become the same to the ones for $p = 1$ for the Sample 1. $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$, $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$, and the width of the range are shown as Table 4.

Table 2: Optimal policies for $p = 1$ in period n.

$n \backslash X$	$2 \mid 0,2$	$2 \mid 2,0$	$2 \mid 0,0$	$2 \mid 1,1$	$2 \mid 0,1$	$2 \mid 1,0$	$0 \mid 0,2$	$0 \mid 2,0$	$0 \mid 0,0$	$0 \mid 1,1$	$0 \mid 0,1$	$0 \mid 1,0$	$1 \mid 0,2$	$1 \mid 2,0$	$1 \mid 0,0$	$1 \mid 1,1$	$1 \mid 0,1$	$1 \mid 1,0$
0	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
1	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
3	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
4	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
5	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
6	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
7	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
8	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
9	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
10	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
11	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
12	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
13	0	0	2	0	2	2	1	1	1	1	1	1	1	1	1	1	1	1
14	0	0	2	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
15	0	0	2	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
16	0	0	2	0	0	2	1	1	1	1	1	1	1	1	1	1	1	1
17	–	–	2	–	0	2	1	1	1	1	1	1	1	1	1	1	1	1
18	–	–	2	–	–	–	1	1	1	1	1	1	–	–	1	–	1	1
19	–	–	–	–	–	–	–	–	1	–	1	1	–	–	1	–	–	–
20	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–

Table 3: The range in Theorem 1 for the states X and $\hat{X}$.

n	r_1^n	$\Delta_1^{\overline{d_1^*}} U_{n-1}(X)$	$\Delta_1^{\overline{d_1^*}} U_{n-1}(\hat{X})$
0	3	–	–
1	3	0.000	0.000
2	3	0.147	0.147
3	3	0.282	0.282
4	3	0.405	0.406
5	3	0.518	0.521
6	4	0.622	0.626
7	4	1.348	1.360
8	5	1.785	1.814
9	5	2.551	2.601
10	5	2.932	3.006
11	5	3.174	3.262
12	4	3.337	3.434
13	4	3.172	3.272
14	3	3.095	3.193
15	3	3.040	3.140
16	3	2.989	3.090
17	3	2.941	3.043

Table 4: The range and the difference for Sample 2.

n	$\Delta_1^{\overline{d_1^*}} U_{n-1}(\hat{X})$	$\Delta_1^{\overline{d_1^*}} U_{n-1}(X)$	Dif.
0	–	–	–
1	0.000	0.000	0.000
2	0.147	0.147	0.000
3	0.282	0.282	0.000
4	0.405	0.405	0.000
5	0.518	0.518	0.000
6	0.622	0.622	0.000
7	1.348	1.348	0.000
8	1.786	1.786	0.000
9	2.555	2.555	0.000
10	2.940	2.940	0.000
11	3.189	3.189	0.000
12	3.359	3.359	0.000
13	3.199	3.199	0.000
14	3.128	3.128	0.000
15	3.075	3.075	0.000
16	3.026	3.026	0.000
17	2.980	2.980	0.000

We can confirm that the width of eq.(19) is zero since there is not difference in the departure rates between the parties. Remember that there is difference in the arrival rates and the expected revenues between the parties. Additionally, how the range has influence on the difference for departure rates is indicated. Let additional datasets in where the departure rate for $p = 2$ is multiplied by 0.75, 0.5, and 0.25 for the Sample 1 be Sample 3, 4, and 5, respectively. The widths of the ranges for the states X and $\hat{X}$ which are computed from the Sample 1 to 5 are shown in Table 5. We can recognize that the widths enlarge for all n if the differences for the departure rates enlarge. Thus, what increasing difference for the departure rates enlarges the width of the range is suggested.

Table 5: The widths of the ranges for the samples.

n	Sample2	Sample1	Sample3	Sample4	Sample5
1	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000
3	0.000	0.001	0.001	0.002	0.002
4	0.000	0.001	0.003	0.004	0.006
5	0.000	0.003	0.005	0.008	0.011
6	0.000	0.004	0.008	0.013	0.017
7	0.000	0.013	0.025	0.038	0.052
8	0.000	0.029	0.058	0.088	0.119
9	0.000	0.050	0.101	0.154	0.209
10	0.000	0.074	0.150	0.229	0.310
11	0.000	0.088	0.180	0.275	0.374
12	0.000	0.097	0.198	0.302	0.410
13	0.000	0.101	0.204	0.311	0.421
14	0.000	0.098	0.199	0.302	0.407
15	0.000	0.100	0.202	0.306	0.411
16	0.000	0.101	0.204	0.308	0.411
17	0.000	0.101	0.204	0.308	0.410

4 Conclusion This study has presented the formulation which is modeled seating problem as bid price control by dynamic programming(Markov decision process). Further, the sufficient condition which makes variations of optimal policy reduce and its property have been indicated. It is meaningful to investigate the sufficient condition because reducing variations of optimal policies leads requisite data capacity to reduce.

This paper's result indicates that we should pay attention to difference for departure rates among parties. Specially, if there is not difference in departure rates among the parties for big scale problem, then results of the Theorem 1 and Proposition 1 have significance. If parameters sets are that $\overline{P} = 4$, $\overline{I} = 2$, $g_p = p$, $t_1 = 2, t_2 = 4$, $m_1 = 6$, and $m_2 = 7$ where $p \in P$, then $\chi = 9240$. However, if there is not difference in departure rates for the case, a maximum of the variations of optimal policies is reduced to 56.

However, This study's result is based on the assumption which is that departure rates depend on exponential distribution. It is mystery that what kinds of restaurant; first-food restaurant, traditional restaurant, cafeteria restaurant, cafe restaurant, and etc.. can be approximately applied to this assumption. This question is a big future issue for this study.

Although this study's model has the problem for a restaurant, the model also corresponds to a upgrade model with departure of parties where resources are rooms or tables. Some other future issues are mentioned that for example, considering meal duration as probability distribution, investigating effect of elements; arrival rate, reward, and etc. for the width of the sufficient condition, and making a relation between this results and heuristic calculation method of existing researches clear.

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