ON THE FIRST-PASSAGE TIME OF AN INTEGRATED GAUSS-MARKOV PROCESS

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ABSTRACT. It is considered the integrated process $X(t) = x + \int_0^t Y(s)ds$, where $Y(t)$ is a Gauss-Markov process starting from y. The first-passage time (FPT) of X through a constant boundary and the first-exit time of X from an interval (a, b) are investigated, generalizing some results on FPT of integrated Brownian motion. An essential role is played by a useful representation of X, which allows to reduces the FPT of X to that of a time-changed Brownian motion. Some explicit examples are reported; when theoretical calculation is not available, the quantities of interest are estimated by numerical computation.

1   Introduction First-passage time (FPT) problems for integrated Markov processes arise both in theoretical and applied Probability. For instance, in certain stochastic models for the movement of a particle, its velocity, $Y(t)$, is modeled as Ornstein-Uhlenbeck (OU) process, which is indeed suitable to describe the velocity of a particle immersed in a fluid; as the friction parameter approaches zero, $Y(t)$ becomes Brownian motion B_t (BM). More generally, the particle velocity $Y(t)$ can be modeled by a diffusion. Thus, particle position turns out to be the integral of $Y(t)$, and any question about the time at which the particle first reaches a given place leads to the FPT of integrated $Y(t)$. This kind of investigation is complicated by the fact that the integral of a Markov process such as $Y(t)$, is no longer Markovian; however, the two-dimensional process $\mathcal{Y}(t) = \left(\int_0^t Y(s)ds, Y(t)\right)$ is Markovian, so the FPT of integrated $Y(t)$ can be studied by using Kolmogorov's equations approach. The first apparition in the literature of $\mathcal{Y}(t)$, with $Y(t) = B_t$, dates back to the beginning of the twentieth century (see [24]), in modeling a harmonic oscillator excited by a Gaussian white noise (see [25] and references therein).

The study of $\int_0^t Y(s)ds$ has interesting applications in Biology, in the framework of diffusion models for neural activity; if one identifies $Y(t)$ with the neuron voltage at time t, then $\frac{1}{t}\int_0^t Y(s)ds$ represents the time average of the neural voltage in the interval $[0, t]$. Moreover, integrated Brownian motion arises naturally in stochastic models for particle sedimentation in fluids (see [22]). Another application can be found in Queueing Theory, if $Y(t)$ represents the length of a queue at time t; then $\int_0^t Y(s)ds$ represents the cumulative waiting time experienced by all the "users" till the time t. Furthermore, as an application in Economy, one can suppose that $Y(t)$ represents the rate of change of a commodity's price, i.e. the current inflation rate; hence, the price of the commodity at time t is $X(t) = X(0) + \int_0^t Y(s)ds$. Finally, integrated diffusions also play an important role in connection with the so-called realized stochastic volatility in Finance (see e.g. [8], [17], [20]).

FPT problems of integrated BM (namely, when $Y(t) = B_t$) through one or two boundaries, attracted the interest of a lot of authors (see e.g. [10], [18], [22], [26], [27], [29], [35] for single boundary, and [25], [32], [33] for double boundary); the FPT of integrated Ornstein-Uhlenbeck process was studied in [10], [30]. Motivated by these works, our aim is to extend to integrated Gauss-Markov processes the literature's results concerning FPT of integrated BM.

Let $m(t)$, $h_1(t)$, $h_2(t)$ be C^1-functions of $t \geq 0$, such that $h_2(t) \neq 0$ and $\rho(t) = h_1(t)/h_2(t)$ is a non-negative and monotonically increasing function, with $\rho(0) = 0$.
If $B(t) = B_t$ denotes standard Brownian motion (BM), then

$$(1.1) \qquad Y(t) = m(t) + h_2(t)B(\rho(t)), \ t \geq 0,$$

is a continuous Gauss-Markov process with mean $m(t)$ and covariance $c(s,t) = h_1(s)h_2(t)$, for $0 \le s \le t$.

Throughout the paper, Y will denote a Gauss-Markov process of the form (1.1), starting from $y = m(0)$.

Besides BM, a noteworthy case of Gauss-Markov process is the Ornstein-Uhlenbeck (OU) process, and in fact any continuous Gauss-Markov process can be represented in terms of a OU process (see e.g. [36]).

Given a continuous Gauss-Markov process Y, we consider its integrated process, starting from $X(0)$:

$$(1.2) \qquad\qquad X(t) = X(0) + \int_0^t Y(s)ds.$$

For a given boundary a, we study the FPT of X through a, with the conditions that $X(0) = x < a$ and $Y(0) = y$, that is:

$$(1.3) \qquad\qquad \tau_a(x,y) = \inf\{t > 0 : X(t) = a | X(0) = x, Y(0) = y\};$$

moreover, for $b > a$ and $x \in (a,b)$, we also study the first-exit time of X from the interval (a,b), with the conditions that $X(0) = x$ and $Y(0) = y$, that is:

$$(1.4) \qquad\qquad \tau_{a,b}(x,y) = \inf\{t > 0 : X(t) \notin (a,b) | X(0) = x, Y(0) = y\}.$$

In our investigation, an essential role is played by the representation of X in terms of BM, which was previously obtained by us in [1]. By using this, we avoid to address the FPT problem by Kolmogorov's equations approach, namely to study the equations associated to the two-dimensional process $(X(t), Y(t))$; many authors (see the references cited above) followed this analytical approach to study the distribution and the moments of the FPT of integrated BM, and they obtained explicit solutions, in terms of special functions. On the contrary, our approach is based on the properties of Brownian motion and continuous martingales and it has the advantage to be quite simple, since the problem is reduced to the FPT of a time-changed BM. Actually, for $Y(0) = y = 0$ we present explicit formulae for the density and the moments of the FPT of the integrated Gauss-Markov process X, both in the one-boundary and two-boundary case; in particular, in the two-boundary case, we are able to express the nth order moment of the first-exit time as a series involving only elementary functions.

2 Preliminary Results on Integrated Gauss-Markov Processes
We recall from [1] the following:

Theorem 2.1 *Let Y be a Gauss-Markov process of the form (1.1); then $X(t) = x + \int_0^t Y(s)ds$ is normally distributed with mean $x + M(t)$ and variance $\gamma(\rho(t))$, where $M(t) = \int_0^t m(s)ds$, $\gamma(t) = \int_0^t (R(t) - R(s))^2 ds$ and $R(t) = \int_0^t h_2(\rho^{-1}(s))/\rho'(\rho^{-1}(s))ds$. Moreover, if $\gamma(+\infty) = +\infty$, then there exists a BM $\widehat{B}$ such that $X(t) = x + M(t) + \widehat{B}(\widehat{\rho}(t))$, where $\widehat{\rho}(t) = \gamma(\rho(t))$. Thus, the integrated process X can be represented as a Gauss-Markov process with respect to a different BM.*

$\square$

Remark 2.2 Notice that, if $\gamma(+\infty) = +\infty$, though X is represented as a Gauss-Markov process for a suitable BM $\widehat{B}$, X is not Markov with respect to its natural filtration $\mathcal{F}_t$ (i.e. the $\sigma-$field generated by X up to time t). In fact, a Gaussian process X enjoys this property if and only if its covariance $K(s,t) = cov(X(s), X(t))$ satisfies the condition (see e.g. [16], [31], [34]) $K(u,t) = \frac{K(u,s)K(s,t)}{K(s,s)}$, $u \le s \le t$. Really, if X is e.g. integrated BM with $y = 0$, $x = 0$ (that is, $X(t) = \int_0^t B_s ds$), one has $K(s,t) = cov\left(\int_0^s B_u du, \int_0^t B_u du\right) = \frac{s^2}{6}(3t - s)$ (see e.g. [39], pg. 654 or [23], pg. 105), and so the above condition does not hold. On the other hand, the two-dimensional process $\left(\int_0^s B_u du, \int_0^t B_u du\right)$ has not the same joint distribution as $\left(\widehat{B}(\widehat{\rho}(s)), \widehat{B}(\widehat{\rho}(t))\right)$, because $cov\left(\widehat{B}(\widehat{\rho}(s)), \widehat{B}(\widehat{\rho}(t))\right) = E[\widehat{B}(\widehat{\rho}(s)) \cdot \widehat{B}(\widehat{\rho}(t))] = \widehat{\rho}(s) = s^3/3$, for $s \le t$ (see Example 1 below), which is different from $K(s,t)$. However, the process (X, B) is Markov, and the marginal distributions of the random vector $\left(X(s), X(t)\right)$ are equal to the distributions of $\widehat{B}(\widehat{\rho}(s))$ and $\widehat{B}(\widehat{\rho}(t))$, respectively; this is enough for the FPT problems we aim to investigate.

Remark 2.3 By Theorem 2.1 the FPT problem for integrated Gauss-Markov process is reduced to the FPT problem for another suitable Gauss-Markov process. Thus, to compute the FPT densities involving one or two boundaries, one can use the methods (both analytical and numerical) developed in [15] and [37] for general Gauss-Markov processes, in which the FPT densities are obtained as solutions to non-singular second-kind Volterra integral equations.

Example 1 (integrated Brownian motion)
Let be $Y(t) = y + B_t$, then $m(t) = y$, $h_1(t) = t$, $h_2(t) = 1$ and $\rho(t) = t$. Moreover, $R(t) = \int_0^t ds = t$ and $\gamma(t) = \int_0^t (t-s)^2 ds = t^3/3$. Thus, $\widehat{\rho}(t) = t^3/3$, $\gamma(+\infty) = +\infty$, and so there exists a BM $\widehat{B}$ such that $X(t) = x + yt + \widehat{B}(t^3/3)$ (see [4]).

The following three examples are taken from Section 3 of [1]; notice that the process here denoted by Y was there indicated by X.

Example 2 (integrated OU process)
Let $Y(t)$ be the solution of the SDE (Langevin equation):

$$dY(t) = -\mu(Y(t) - \beta)dt + \sigma dB_t, \ Y(0) = y,$$

where $\mu, \sigma > 0$ and $\beta \in \mathbb{R}$. The explicit solution is (see e.g. [2]):

$$(2.1) \qquad Y(t) = \beta + e^{-\mu t}[y - \beta + \widetilde{B}(\rho(t)],$$

where $\widetilde{B}$ is Brownian motion and $\rho(t) = \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right)$. Thus, Y is a Gauss-Markov process with $m(t) = \beta + e^{-\mu t}(y - \beta)$, $h_1(t) = \frac{\sigma^2}{2\mu}\left(e^{\mu t} - e^{-\mu t}\right)$, $h_2(t) = e^{-\mu t}$ and $c(s,t) = h_1(s)h_2(t)$. The functions $M(t)$, $R(t)$ and $\gamma(t)$ defined in the statement of Theorem 2.1 can be easily obtained in closed form (see Example 2 of [1] for calculations and more details). Then, by Theorem 2.1, we get that $X(t) = x + \int_0^t Y(s)ds$ is normally distributed with mean $x + M(t)$ and variance $\widehat{\rho}(t) = \gamma(\rho(t))$. Moreover, as easily seen, $\lim_{t \to +\infty} \gamma(t) = +\infty$, so there exists a BM $\widehat{B}$ such that $X(t) = x + M(t) + \widehat{B}\left(\widehat{\rho}(t)\right)$.

Notice that in both Example 1 and 2 it holds $\rho(+\infty) = +\infty$, so the condition $\gamma(+\infty) = +\infty$ is equivalent to $\widehat{\rho}(+\infty) = +\infty$.

Example 3 (integrated Brownian bridge)
For $T > 0$ and $\alpha, \beta \in \mathbb{R}$, let $Y(t)$ be the solution of the SDE:

$$dY(t) = \frac{\beta - Y(t)}{T - t}\,dt + dB_t, \ 0 \le t \le T, \ Y(0) = y = \alpha.$$

This is a transformed BM with fixed values at each end of the interval $[0, T]$, $Y(0) = y = \alpha$ and $Y(T) = \beta$. The explicit solution is (see e.g. [38]):

$$Y(t) = \alpha\left(1 - t/T\right) + \beta t/T + (T - t)\int_0^t \frac{1}{T - s}dB(s)$$

$$(2.2) \qquad = \alpha\left(1 - t/T\right) + \beta t/T + (T - t)\widetilde{B}\left(\frac{t}{T(T - t)}\right), \ 0 \le t \le T,$$

where $\widetilde{B}$ is BM. So, for $0 \le t \le T$, Y is a Gauss-Markov process with:

$$m(t) = \alpha\left(1 - t/T\right) + \beta t/T, \ h_1(t) = t/T, \ h_2(t) = T - t, \ \rho(t) = \frac{t}{T(T - t)}, \ c(s,t) = h_1(s)h_2(t).$$

Notice that now $\rho(t)$ is defined only in $[0, T)$. The functions $M(t)$, $R(t)$ and $\gamma(t)$ defined in the statement of Theorem 2.1 can be easily obtained in closed form (see Example 3 of [1] for

calculations and more details). Then, by Theorem 2.1, we get that $X(t) = x + \int_0^t Y(s)ds$ is normally distributed with mean $x + M(t)$ and variance $\widehat{\rho}(t) = \gamma(\rho(t))$. As easily seen, $\lim_{t \to T^-} \rho(t) = +\infty$; moreover, by a straightforward, but boring calculation, we get that $\lim_{t \to T^-} \widehat{\rho}(t) = \gamma_1(+\infty) = +\infty$, so there exists a BM $\widehat{B}$ such that $X(t) = x + M(t) + \widehat{B}(\widehat{\rho}(t))$, $t \in [0, T]$.

Example 4 (the integral of a generalized Gauss-Markov process)
Let us consider the diffusion $Y(t)$ which is the solution of the SDE:

$$dY(t) = m'(t)dt + \sigma(Y(t))dB_t, \ Y(0) = y,$$

where $\sigma(y) > 0$ is a smooth deterministic function. In this Example, we denote by $\rho(t)$ the quadratic variation of $Y(t)$, that is, $\rho(t) := \langle Y \rangle_t = \int_0^t \sigma^2(Y(s))ds$, and suppose that $\rho(+\infty) = +\infty$; then, it follows (see Example 4 of [1]) that $Y(t) = m(t) + \widehat{B}(\rho(t))$, $t \geq 0$ ($m(0) = y$), where $\widehat{B}$ is BM; here, $\rho(t)$ is an increasing, but not necessarily deterministic function, namely it can be a random function. For this reason, we call Y a generalized Gauss-Markov process. By using the arguments leading to the proof of Theorem 2.1, we conclude that, under certain conditions, for fixed t the law of $\int_0^t Y(s)ds$, conditional to $\rho(t)$, is normal with mean $M(t) = \int_0^t m(s)ds$ and variance $\widehat{\rho}(t)$, where $\widehat{\rho}(t)$ is increasing and bounded between two certain deterministic functions (see [1] for more details).

In the sequel, we suppose that all the assumptions of Theorem 2.1 hold, and $\gamma(+\infty) = +\infty$; we limit ourselves to consider the special case when $m(t)$ is a constant (that is, $m(t) \equiv Y(0) = y$, $\forall t$), thus $Y(t) = y + h_2(t)B(\rho(t))$ and $X(t) = x + yt + \int_0^t h_2(s)B(\rho(s))ds$. Our aim is to investigate the FPT problem of X, for one or two boundaries. One approach to the FPT problem of X consists in considering the two-dimensional process $(X(t), Y(t))$ given by:

$$\begin{cases} X(t) = x + \int_0^t Y(s)ds \\ Y(t) = y + h_2(t)B(\rho(t))dt \ , \end{cases}$$

or, in differential form:

$$\begin{cases} dX(t) = Y(t)dt \\ dY(t) = h_2'(t)B(\rho(t))dt + h_2(t)\sqrt{\rho'(t)}dB_t \ , \end{cases}$$

and to study the associated Kolmogorov's equations.
Many authors (see e.g. [19], [25], [26], [27], [29]) followed this way in the case of integrated BM, namely for $Y(t) = y + B_t$. In fact, for $\tau = \tau_a$ or $\tau = \tau_{a,b}$, the law of the couple $(\tau(x,y), B_{\tau(x,y)})$ was investigated. Let us denote by $\mathcal{G}$ the generator of (X, B), that is:

$$\mathcal{G}f(x,y) = \frac{\partial f}{\partial x} \cdot y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2} \ , \ f \in C^2;$$

if one considers, for instance, the one boundary case, then the Laplace transform of $(\tau_a(x,y), B_{\tau_a(x,y)})$, defined for $x \leq a$, $y \in \mathbb{R}$, by $u(\lambda, \nu) := E\left[\exp\left(-\lambda \tau_a(x,y) - \nu B_{\tau_a(x,y)}\right)\right]$ (λ, $\nu \geq 0$), is the solution of the problem with boundary conditions:

(2.3)
$$\begin{cases} \mathcal{G}u(x,y) = \lambda u(x,y), \ x \leq a, \ y \in \mathbb{R} \\ u(a^-, y) = e^{-\nu y}, \ y \geq 0 \\ u(a^+, y) = e^{\nu y}, \ y < 0 \end{cases}$$

(see e.g. [27], Lemma 3, or ref. [4], [5], [7], therein). Moreover, for $n = 1, 2, \ldots$ the nth order moments $T_n(x,y) = E(\tau_a^n(x,y))$ are solutions to the equations $\mathcal{G}T_n = -nT_{n-1}$ ($T_0 \equiv 1$), subjected to certain boundary conditions; however, these boundary value problems are not well-posed (see [22], where some numerical methods to estimate T_n were also considered).
  Notice that, in the case of integrated BM, explicit, rather complicated formulae for the joint distribution of $(\tau_a(x,y), B_{\tau_a(x,y)})$ (and therefore for the density of $\tau_a(x,y)$) were found in [18],

[26], [35]). In order to avoid not convenient formulae, we propose an alternative approach, based on the representation of the integrated process X as a Gauss-Markov process, with respect to the BM $\widehat{B}$ (see Theorem 2.1); this way works very simply, almost in the case when $y = 0$. Thus, in the following, we suppose that $Y(t) = y + h_2(t)B(\rho(t))$ and $\gamma(+\infty) = +\infty$, so the integrated process is of the form $X(t) = x + yt + \widehat{B}(\widehat{\rho}(t))$, where $\widehat{\rho}(t) = \gamma(\rho(t))$ and $\widehat{B}$ is a suitable BM. Notice however, that the integrated OU process and the integrated Brownian bridge belong to this class only if $y = \beta$, and $\alpha = \beta = y$, respectively; this easily follows from the explicit expressions of $M(t)$ given in Examples 2 and 3 in [1].

3 FPT through one boundary Under the previous assumptions, let a be a fixed constant boundary; for $x < a$ and $y \in \mathbb{R}$, the FPT of X through a can be written as follows:

$$(3.1) \qquad \tau_a(x,y) = \inf\{t > 0 : x + yt + \widehat{B}(\widehat{\rho}(t)) = a\}.$$

Thus, if we set $\widehat{\tau}_a(x,y) = \widehat{\rho}(\tau_a(x,y))$, we get:

$$(3.2) \qquad \widehat{\tau}_a(x,y) = \inf\{t > 0 : \widehat{B}_t = h(t)\},$$

where $h(t) = a - x - y\widehat{\rho}^{-1}(t)$, and so we reduce to consider the FPT of BM through a curved boundary. Since, for $x < a$ and $y \geq 0$ the function $h(t)$ is not increasing, we are able to conclude that $\tau_a(x,y)$ is finite with probability one, if $y \geq 0$. In fact, as it is well-known, the FPT of BM $\widehat{B}_t$ through the constant barrier $h(0) = a - x$, say $\bar{\tau}(x)$, is finite with probability one; then, if $y \geq 0$, from $h(t) \leq h(0)$ we get that $\widehat{\tau}_a(x,y) \leq \bar{\tau}(x)$ and therefore also $\widehat{\tau}_a(x,y)$ is finite with probability one. Finally, if $y \geq 0$, we obtain that $P(\tau_a(x,y) < +\infty) = 1$, because $\tau_a(x,y) = \widehat{\rho}^{-1}(\widehat{\tau}_a(x,y)) \leq \widehat{\rho}^{-1}(\bar{\tau}_a(x))$. Note, however, that this argument does not work for $y < 0$.

A more difficult problem is to find the distribution of $\widehat{\tau}_a(x,y)$, and then that of $\tau_a(x,y)$. However, if $h(t)$ is either convex or concave, then lower and upper bounds to the distribution of $\widehat{\tau}_a(x,y)$ can be obtained by considering a "polygonal approximation" of $h(t)$ by means of a piecewise-linear function (see e.g. [3], [6]), but in general, it is not possible to find the distribution of $\widehat{\tau}_a(x,y)$ exactly.

Remark 3.1 Actually, it is possible to find explicitly the density of the FPT of X through certain moving boundaries, by using results on the FPT-density of BM (see [7], [9], [11], [13], [14], [21], [40], [41]). Let $v(t)$, $t \geq 0$, be a curved boundary for which the FPT-density $\widehat{f}_v(t|x)$ of BM through v, when starting from $x < v(0)$, is explicitly known; if $S(t) = v(\widehat{\rho}(t)) + yt$, one can easily find the density of the FPT of X through S, with the condition that $x < S(0) = v(0)$. In fact, if $\tau_S(x,y) = \inf\{t > 0 : X(t) = S(t) | X(0) = x, Y(0) = y\}$, one gets $\tau_S(x,y) = \inf\{t > 0 : x + ty + \widehat{B}(\widehat{\rho}(t)) = S(t)\}$; then, $\widehat{\tau}_v(x,y) := \widehat{\rho}(\tau_S(x,y)) = \inf\{t > 0 : x + \widehat{B}(t) = v(t)\}$ has density $\widehat{f}_v$ and so the density of $\tau_S(x,y)$ turns out to be

$$(3.3) \qquad f_S(t|x) = \widehat{f}_v(\widehat{\rho}(t)|x)\widehat{\rho}'(t).$$

For instance, if X is integrated BM ($\widehat{\rho}(t) = t^3/3$), and we consider the cubic boundary $S(t) = a + ty + bt^3$ ($a > 0$, $b < 0$), it results $S(t) = v(\widehat{\rho}(t)) + yt$, with $v(t) = a + 3bt$ and so, for $x < a$ we reduce to consider the FPT of BM starting from x through the linear boundary $a + 3bt$. Since this has the inverse Gaussian density i.e. $\frac{a-x}{\sqrt{2\pi}\ t^{3/2}}e^{-(3bt+a-x)^2/2t}$ (see e.g. [6]), the density of $\tau_S(x,y)$ can be easily recovered from (3.3).

Formula (3.2), with $y = 0$, allows to find the density of $\tau_a(x,0)$ in closed form; in fact, $\widehat{\tau}_a(x,0)$ is the FPT of BM $\widehat{B}$ through the level $a - x > 0$, and so its density is:

$$(3.4) \qquad \widehat{f}_a(t|x) := \frac{d}{dt}P(\widehat{\tau}_a(x,0) \leq t) = \frac{a-x}{\sqrt{2\pi}\ t^{3/2}}e^{-(a-x)^2/2t},$$

from which the density of $\tau_a(x,0) = \widehat{\rho}^{-1}(\widehat{\tau}_a(x,0))$ follows:

$$(3.5) \qquad f_a(t|x) := \frac{d}{dt}P(\tau_a(x,0) \leq t) = \widehat{f}_a(\widehat{\rho}(t)|x)\widehat{\rho}'(t) = \frac{(a-x)\ \widehat{\rho}'(t)}{\sqrt{2\pi}\ \widehat{\rho}(t)^{3/2}}e^{-(a-x)^2/2\widehat{\rho}(t)}.$$

If X is integrated BM, we have $X(t) = x + \widehat{B}(\widehat{\rho}(t))$, with $\widehat{\rho}(t) = t^3/3$, so we get (cf. [18]):

$$(3.6) \qquad f_a(t|x) = \frac{3^{3/2}(a-x)}{\sqrt{2\pi}\; t^{5/2}} e^{-3(a-x)^2/2t^3}.$$

If X is integrated OU process, the density of $\tau_a(x,0)$ can be obtained by inserting in (3.5) the function $\widehat{\rho}(t)$ deducible from Example 2, but it takes a more complex form.

Remark 3.2 Formula (3.5) implies that the nth order moment of the FPT, $E(\tau_a^n(x,0))$, is finite if and only if the function $t^n \widehat{\rho}'(t)/\widehat{\rho}(t)^{3/2}$ is integrable in $(0, +\infty)$.
Now, let us suppose that there exists $\alpha > 0$ such that $\widehat{\rho}(t) \sim const \cdot t^\alpha$, as $t \to +\infty$; then, in order that $E(\tau_a^n(x,0)) < \infty$, it must be $\alpha = 2(n+\delta)$, for some $\delta > 0$. For integrated BM, we have $\alpha = 3$, then for $n = 1$ the last condition holds with $\delta = 1/2$, so we obtain the finiteness of $E(\tau_a(x,0))$ (notice that the mean FPT of BM through a constant barrier is instead infinite). Of course, this is not always the case; in fact, if X is integrated OU process, we have $\rho(t) \sim const \cdot e^{2\mu t}$, $\gamma(t) \sim const \cdot \ln(2\mu t/\sigma^2)$, as $t \to +\infty$, and so $\widehat{\rho}(t) = \gamma(\rho(t)) \sim const \cdot t$, as $t \to +\infty$, namely $\alpha = 1$ and the condition above is not satisfied with $n = 1$; therefore $E(\tau_a(x,0)) = +\infty$. Not even $E((\tau_a(x,0))^{1/2})$ is finite, but $E((\tau_a(x,0))^{1/4})$ is so. Notice that the moments of any order of the FPT of (non integrated) OU through a constant barrier are instead finite.

As for the second order moment of the FPT of integrated BM, instead, we obtain $E\left[(\tau_a(x,0))^2\right] = +\infty$, since the equality $\alpha = 2(n+\delta)$ with $\alpha = 3$ and $n = 2$ is not satisfied, for any $\delta > 0$.

From (3.4) we get that the nth order moment of $\tau_a(x,0)$, if it exists finite, is explicitly given by:

$$E\left[(\tau_a(x,0))^n\right] = E\left[(\widehat{\rho}^{\,-1}(\widehat{\tau}_a(x,0)))^n\right]$$

$$(3.7) \qquad = \int_0^{+\infty} (\widehat{\rho}^{\,-1}(t))^n \frac{a-x}{\sqrt{2\pi}t^{3/2}} e^{-(a-x)^2/2t} dt.$$

For instance, if X is integrated BM, one has:

$$E(\tau_a(x,0)) = E((3\,\widehat{\tau}_a(x,0))^{1/3}) = \int_0^{+\infty} (3t)^{1/3} \frac{a-x}{\sqrt{2\pi}t^{3/2}} e^{-(a-x)^2/2t} dt$$

$$= \frac{3^{1/3}(a-x)}{\sqrt{2\pi}} \int_0^{+\infty} \frac{1}{t^{7/6}} e^{-(a-x)^2/2t} dt.$$

By the variable's change $z = 1/t$, the integral can be written as:

$$\int_0^{+\infty} \frac{1}{z^{5/6}} e^{-(a-x)^2 z/2} dz = \frac{\Gamma\left(\frac{1}{6}\right) 2^{1/6}}{(a-x)^{1/3}} \int_0^{+\infty} \left(\frac{(a-x)^2}{2}\right)^{1/6} \frac{1}{\Gamma\left(\frac{1}{6}\right)} z^{1/6-1} e^{-\frac{(a-x)^2}{2} z} dz$$

$$= \frac{\Gamma\left(\frac{1}{6}\right) 2^{1/6}}{(a-x)^{1/3}},$$

where we have used that the last integral equals one, because the integrand is a Gamma density. Thus, for integrated BM, we finally obtain:

$$(3.8) \qquad E(\tau_a(x,0)) = \left(\frac{3}{2}\right)^{1/3} \Gamma\left(\frac{1}{6}\right) \frac{(a-x)^{2/3}}{\sqrt{\pi}}.$$

Notice that an asymptotic expression of $E(\tau_a(x,y))$ for large $y > 0$ was obtained in [22].

3.1 Random starting point Until now we have supposed that the starting point $X(0) < a$ is given and fixed. We can introduce a randomness in the starting point, replacing $X(0)$ with a random variable x, having density $g(u)$ whose support is the interval $(-\infty, a)$; the corresponding FPT problem is particularly relevant in contexts such as neuronal modeling, where the reset value of the membrane potential is usually unknown (see e.g. [28]). In fact, the quantity of interest becomes now the unconditional FPT through the boundary a, that is, $\inf\{t > 0 : X(t) = a | Y(0) =$

y}. Notice that results on FPT problems for general Gauss-Markov processes in the presence of random initial position are available in [15].

In particular, if X is integrated BM and $y = 0$, one gets from (3.8) that the average FPT through the boundary a, over all initial positions $x < a$, is:

$$(3.9) \qquad \overline{T}_a = \int_{-\infty}^{a} E(\tau_a(u,0))g(u)du = \left(\frac{3}{2}\right)^{1/3} \frac{\Gamma\left(\frac{1}{6}\right)}{\sqrt{\pi}} \int_{-\infty}^{a} (a-u)^{2/3}g(u)du.$$

For instance, suppose that $a - x$ has Gamma distribution with parameters α, $\lambda > 0$, namely, x has density

$$g(u) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda(a-u)}(a-u)^{\alpha-1} \cdot \mathbb{I}_{(-\infty,a)}(u).$$

Then, by the change of variable $z = a - u$ one obtains that the above integral is nothing but $E\left(Z^{2/3}\right)$, where Z is a random variable with the same distribution of $a - x$; then, recalling the expressions of the moments of the Gamma distribution, one obtains $E\left(Z^{2/3}\right) = \frac{\Gamma(\alpha+\frac{2}{3})}{\lambda^{2/3}\Gamma(\alpha)}$. Finally, by inserting this quantity in (3.9), it follows that:

$$\overline{T}_a = \frac{\left(\frac{3}{2\lambda^2}\right)^{1/3}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\alpha+\frac{2}{3}\right)}{\Gamma(\alpha)}.$$

Remark 3.3 For $y = Y(0) = 0$, we have considered the FPT of X through the boundary a from "below", with the condition $x = X(0) < a$; if one considers the FPT of X through the barrier a from "above", with the condition $X(0) > a$ (namely, $\inf\{t > 0 : X(t) \leq a|X(0) = x, Y(0) = 0\}$), then in all formulae $a - x$ has to be replaced with $x - a$. More generally, if one considers the first hitting time of X to a (from above or below), $a - x$ must be replaced by $|a - x|$.

4   First exit time from an interval

Assume, as always, that $\gamma(+\infty) = +\infty$; for $x \in (a,b)$ and $y \in \mathbb{R}$, the first-exit time of X from the interval (a,b) is:

$$(4.1) \qquad \tau_{a,b}(x,y) = \inf\{t > 0 : x + yt + \widehat{B}(\widehat{\rho}(t)) \notin (a,b)\}.$$

Set $\widehat{\tau}_{a,b}(x,y) = \widehat{\rho}(\tau_{a,b}(x,y))$, then:

$$(4.2) \qquad \widehat{\tau}_{a,b}(x,y) = \inf\{t > 0 : x + \widehat{B}_t \leq a - y\widehat{\rho}^{-1}(t) \ \text{ or } \ x + \widehat{B}_t \geq b - y\widehat{\rho}^{-1}(t)\}.$$

If $\widehat{\tau}_{a,b}(x,y)$ is finite with probability one, also $\tau_{a,b}(x,y)$ is so. In the sequel, we will focus on the case when $y = 0$, namely we will consider $\tau_{a,b}(x,0) = \widehat{\rho}^{-1}(\widehat{\tau}_{a,b}(x,0))$, where $\widehat{\tau}_{a,b}(x,0) = \inf\{t > 0 : x + \widehat{B}_t \notin (a,b)\}$; as it is well-known, $\widehat{\tau}_{a,b}(x,0)$ is finite with probability one and its moments are solutions of Darling and Siegert's equations (see [12]).

First, we will find sufficient conditions so that the moments of $\tau_{a,b}(x,0)$ are finite; then, we will carry on explicit computations of them, in the case of integrated BM.

Proposition 4.1 *If $\widehat{\rho}$ is convex, then $E\left(\tau_{a,b}(x,0)\right) < \infty$; moreover, if there exist constants c, $\delta > 0$, such that $0 \leq \widehat{\rho}^{-1}(t) \leq c \cdot t^\delta$, then $E\left(\tau_{a,b}(x,0)\right)^n < \infty$, for any integer n.*

Proof. If $\widehat{\rho}$ is convex, then $\widehat{\rho}^{-1}$ is concave, and the finiteness of $E\left(\tau_{a,b}(x,0)\right)$ follows by Jensen's inequality written for concave functions. Next, denote by $\widehat{f}_{-\alpha,\alpha}(t|x)$ the density of the first-exit time of $x + \widehat{B}_t$ from the interval $(-\alpha, \alpha)$, $\alpha > 0$; we recall from [12] that the Laplace transform of $\widehat{f}_{-\alpha,\alpha}(t|x)$, namely, $\int_0^{+\infty} e^{-\theta t}\widehat{f}_{-\alpha,\alpha}(t|x)dt$ is:

$$(4.3) \qquad \mathcal{L}\left[\widehat{f}_{-\alpha,\alpha}\right](\theta|x) = \frac{\cosh(\sqrt{2\theta}x)}{\cosh(\sqrt{2\theta}\alpha)} \ , \ -\alpha < x < \alpha, \ \theta \geq 0.$$

By inverting this Laplace transform, one obtains (see [12]):

$$(4.4) \qquad \widehat{f}_{-\alpha,\alpha}(t|x) = \frac{\pi}{\alpha^2} \sum_{k=0}^{\infty} (-1)^k \left(k+\frac{1}{2}\right) \cos\left[\left(k+\frac{1}{2}\right)\frac{\pi x}{\alpha}\right] \exp\left[-\left(k+\frac{1}{2}\right)^2 \frac{x^2 t}{2\alpha^2}\right].$$

The case of an interval (a, b), $b > a$, is reduced to the previous one; in fact, as easily seen, if $\alpha = (b-a)/2$ one has:

$$\widehat{f}_{a,b}(t|x) = \widehat{f}_{-\alpha,\alpha}\left(t\Big|x - \frac{a+b}{2}\right).$$

Of course, the density of $\tau_{a,b}(x, 0)$ turns out to be $\widehat{f}_{a,b}(\widehat{\rho}(t)|x)\widehat{\rho}'(t)$. For the sake of simplicity, we take $a = -\alpha$, $b = \alpha$, $\alpha > 0$; then, for $x \in (-\alpha, \alpha)$ and an integer n:

$$(4.5) \qquad E\left[(\tau_{a,b}(x,0))^n\right] = E\left[(\tau_{-\alpha,\alpha}(x,0))^n\right] = E\left[\left(\widehat{\rho}^{-1}(\widehat{\tau}_{-\alpha,\alpha}(x,0))\right)^n\right] = \sum_{k=0}^{\infty} A_k(x),$$

where

$$(4.6) \qquad A_k(x) = \frac{\pi}{\alpha^2}(-1)^k\left(k + \frac{1}{2}\right)\cos\left(\left(k + \frac{1}{2}\right)\frac{\pi x}{\alpha}\right)\int_0^{+\infty} e^{-(k+1/2)^2\pi^2 t/2\alpha^2}\left(\widehat{\rho}^{-1}(t)\right)^n dt.$$

The integral can be written as:

$$\frac{2\alpha^2}{\pi^2(k+1/2)^2}E\left(\widehat{\rho}^{-1}(Z_k)\right)^n,$$

where Z_k is a random variable with exponential density of parameter $\lambda_k = (k+1/2)^2\pi^2/2\alpha^2$; so:

$$A_k(x) = (-1)^k\cos\left(\left(k + \frac{1}{2}\right)\frac{\pi x}{\alpha}\right)\frac{2}{\pi(k+1/2)}E\left(\widehat{\rho}^{-1}(Z_k)\right)^n.$$

Recalling that $E[(Z_k)^{n\delta}] = \frac{\Gamma(1+n\delta)}{(\lambda_k)^{n\delta}}$, by the hypotheses we get $E\left((\widehat{\rho}^{-1}(Z_k))^n\right) \leq c^n E[(Z_k)^{n\delta}] = const \cdot \frac{\Gamma(1+n\delta)}{(k+1/2)^{2n\delta}}$; thus:

$$|A_k(x)| \leq \frac{const'}{(k+1/2)^{1+2n\delta}},$$

from which it follows that the series $\sum_k A_k(x)$ is absolutely convergent for every $x \in (-\alpha, \alpha)$, and therefore $E\left[(\tau_{-\alpha,\alpha}(x,0))^n\right] < +\infty$. The finiteness of $E\left[(\tau_{a,b}(x,0))^n\right]$ in the general case is easily obtained.

$\square$

Remark 4.2 The condition $0 \leq \widehat{\rho}^{-1}(t) \leq c \cdot t^\delta$ is satisfied e.g. for integrated BM, since $\widehat{\rho}^{-1}(t) = 3^{1/3}t^{1/3}$ (see Example 1), and for integrated OU process, because from the expression of $\widehat{\rho}(t)$ deducible from Example 2, it can be shown that $c_1 t \leq \widehat{\rho}(t) \leq c_2 t$ for suitable $c_1, c_2 > 0$ which depend on μ and σ, and therefore $\frac{1}{c_2}t \leq \widehat{\rho}^{-1}(t) \leq \frac{1}{c_1}t$.

Now, we carry on explicit computations of $E[\tau_{a,b}(x,0)]$ and $E\left[(\tau_{a,b}(x,0))^2\right]$, in the case of integrated BM. Inserting $\widehat{\rho}(t) = t^3/3$, $(\widehat{\rho}^{-1}(y) = (3y)^{1/3})$, and $n = 1, 2$ in (4.5), (4.6), after some calculations we obtain:

$$(4.7) \quad E\left[\tau_{a,b}(x,0)\right] = \frac{3^{1/3}2^{7/3}\Gamma(\frac{4}{3})(b-a)^{2/3}}{\pi^{5/3}}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^{5/3}}\cos\left[\frac{\pi(2k+1)}{b-a}\left(x - \frac{a+b}{2}\right)\right].$$

$$(4.8) \qquad E\left[(\tau_{a,b}(x,0))^2\right] = \frac{12(b-a)^4}{\pi^4}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^4}\cos\left[\frac{\pi(2k+1)}{b-a}\left(x - \frac{a+b}{2}\right)\right].$$

Notice that it is arduous enough to express the sums of the Fourier-like series above in terms of elementary functions of $x \in (a, b)$, and then to obtain the moments of $\tau_{a,b}(x,0)$ in a simple closed form; actually, by using the Kolmogorov's equations approach, in [32], [33], it was obtained a formula for $E(\tau_{a,b}(x,0))$ in terms of hypergeometric functions. This kind of difficulty does not
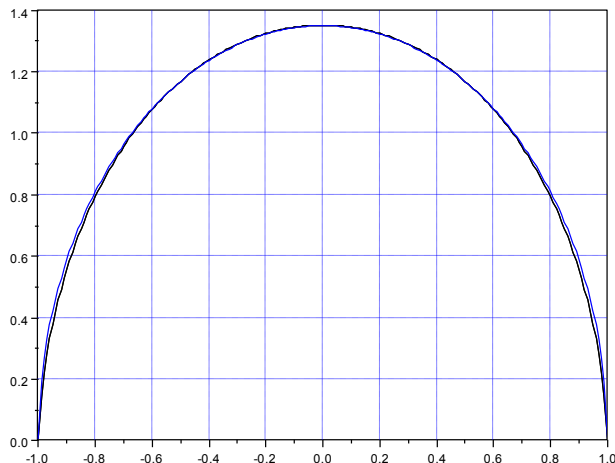
Figure 1: Plots of the mean exit time, $E(\tau_{-1,1}(x,0))$, of integrated BM from the interval $(-1,1)$ (lower curve), and of the function $z(x) = 1.35 \cdot (1-x^2)^{1/2}$ (upper curve), as functions of $x \in (-1,1)$.

arise, for instance, in the case of (non-integrated) BM; in fact, by using formula (4.5) with $\widehat{\rho}(t) = t$ and $n = 1$, one obtains:

$$E\left[\tau_{-\alpha,\alpha}(x)\right] = \frac{32\alpha^2}{\pi^3} \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)^3} \cos\left[(2k+1)\frac{\pi}{2\alpha}x\right];$$

on the other hand, the well-known formula for the mean first-exit time of BM from the interval $(-\alpha,\alpha)$, provides that the sum of the series must be $\alpha^2 - x^2$.

However, (4.7) and (4.8) turn out to be very convenient to estimate the first two moments of $\tau_{a,b}(x,0)$ for integrated BM; in fact the two series converge fast enough, so to obtain "good" estimates of the moments, it suffices to consider a few terms of them. As for $E\left[\tau_{a,b}(x,0)\right]$, it appears to be fitted very well by the square root of a quadratic function; this was obtained by least square interpolation implemented in MATLAB. In the Figure 1, for integrated BM, we compare the graphs of $E(\tau_{a,b}(x,0))$, calculated by replacing the series in (4.7) with a finite summation over the first 20 addends, and that of $C \cdot [(b-x)(x-a)]^{1/2}$, as functions of $x \in (a,b)$, for $a = -1$, $b = 1$, and $C = 1.35$; the two curves appear to be almost undistinguishable.

We have also calculated the second order moment of the first-exit time of integrated BM, by summing the first 20 addends of the series in (4.8). In the Figure 2, we plot $E\left[(\tau_{a,b}(x,0))^2\right]$, $E^2\left[\tau_{a,b}(x,0)\right]$ and the variance $Var\left[\tau_{a,b}(x,0)\right]$, as a function of $x \in (-1,1)$, for $a = -1$, $b = 1$; as we see, the maximum of $Var\left[\tau_{a,b}(x,0)\right]$ is about 10% times the maximum of $E(\tau_{-1,1}(x,0))$.

4.1 Random starting point As in the one boundary case, if we introduce a randomness in the starting point, replacing $X(0) \in (a,b)$ with a random variable x, having density $g(u)$ whose support is the interval (a,b), we can consider the average exit time over all initial positions $x \in (a,b)$. If $y = 0$, this quantity is:

$$\overline{T}_{a,b} = \int_a^b E(\tau_{a,b}(u,0))g(u)du.$$

Notice that results on first-exit times for general Gauss-Markov processes, in the presence of random initial position, are available in [37].

In the case of integrated BM, $\overline{T}_{a,b}$ can be calculated by using the expression of $E(\tau_{a,b}(x,0))$ given
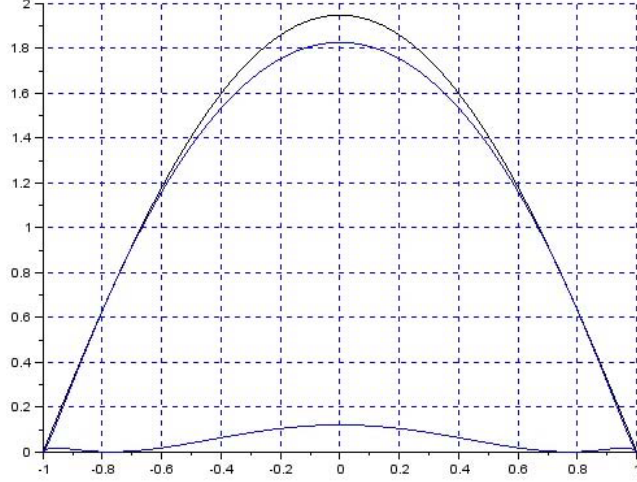
Figure 2: From top to bottom: plot of the second moment (first curve), the square of the first moment (second curve), and the variance of the first-exit time $\tau_{-1,1}(x,0)$ (third curve) of integrated BM from the interval $(-1,1)$, as functions of $x \in (-1.1)$.

by (4.7). We obtain:
(4.9)
$$\overline{T}_{a,b} = \frac{3^{1/3}2^{7/3}\Gamma(\frac{4}{3})(b-a)^{2/3}}{\pi^{5/3}} \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)^{5/3}} \int_a^b \cos\left[\frac{\pi(2k+1)}{b-a}\left(u - \frac{a+b}{2}\right)\right] g(u)du$$

(it has been possible to exchange the integral of the sum with the sum of the integrals, thanks to the dominated convergence theorem); the integral in (4.9) equals $E(U_k)$, where
$U_k = \cos\left[\frac{\pi(2k+1)}{b-a}\left(\eta - \frac{a+b}{2}\right)\right] \leq 1$. Therefore:

$$(4.10) \qquad \overline{T}_{a,b} = \frac{3^{1/3}2^{7/3}\Gamma(\frac{4}{3})(b-a)^{2/3}}{\pi^{5/3}} \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)^{5/3}} E(U_k).$$

In the special case when g is the uniform density in the interval (a,b), we get by calculation:

$$\overline{T}_{a,b} = \frac{3^{1/3}2^{7/3}\Gamma(\frac{4}{3})(b-a)^{2/3}}{\pi^{5/3}} \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)^{5/3}} \int_a^b \cos\left[\frac{\pi(2k+1)}{b-a}\left(x - \frac{a+b}{2}\right)\right] \frac{1}{b-a} dx$$

$$(4.11) \qquad = \frac{3^{1/3}2^{10/3}\Gamma(\frac{4}{3})(b-a)^{2/3}}{\pi^{8/3}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{8/3}} \cdot$$

Thus, $\overline{T}_{a,b} = const \cdot (b-a)^{2/3}$. This confirms the result by Masoliver and Porrà (see [32], [33]), obtained by the Kolmogorov's equations approach in the case of integrated BM, with $y=0$ and uniform distribution of the $X-$ starting point, according to which, the dependence of $\overline{T}_{a,b}$ on the size $L = (b-a)$ of the interval, is $L^{2/3}$.

As far as integrated OU process is concerned, the moments of $\tau_{a,b}(x,0)$ can be found again by formula (4.5), where $\widehat{\rho}(t)$ can be deduced from Example 2; however, it is not possible to calculate explicitly the integral which appears in the expression of $A_k(x)$, so it has to be numerically computed. Since the integrand function decreases exponentially fast, it suffices to calculate the integral over the interval $(0,10)$, to obtain precise enough estimates. In the Figure 3 we have
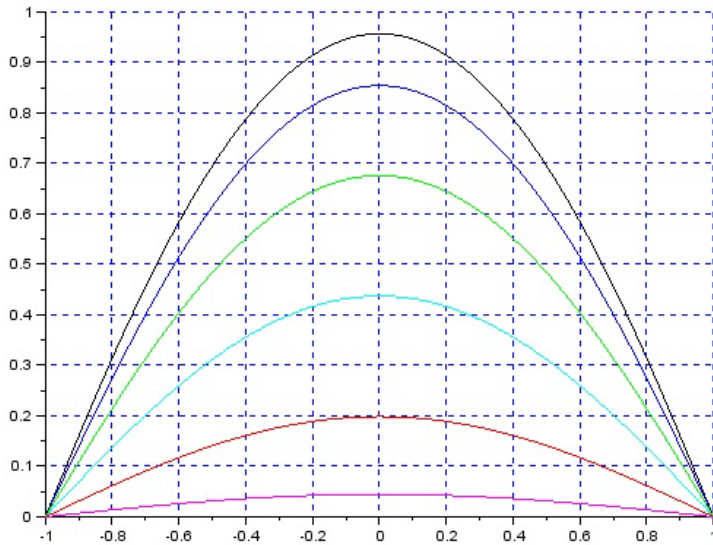
Figure 3: Plot of numerical evaluation of the mean exit time, $E\left(\tau_{-1,1}(x,0)\right)$, of integrated OU with $\beta = y = 0$, from the interval $(-1,1)$, as a function of $x \in (-1,1)$, for $\sigma = 1$ and several values of μ. From top to bottom, with respect to the peak of the curve: $\mu = 2; 1.8; 1.6; 1.4; 1.2; 1$.

plotted, for comparison, the numerical evaluation of the mean exit time of integrated OU process with $y = \beta = 0$, from the interval $(-1,1)$, as a function of $x \in (-1,1)$, for $\sigma = 1$ and several values of μ; in the Figure 4 we have plotted the numerical evaluation of $E\left[(\tau_{-1,1}(x,0))^2\right]$, $E^2\left[\tau_{-1,1}(x,0)\right]$ and the variance $Var\left[\tau_{-1,1}(x,0)\right]$ of the first exit time of integrated OU process, for $\sigma = 1$ and $\mu = 1$. As we see, the maximum of $Var\left[\tau_{-1,1}(x,0)\right]$ is about 5% times the maximum of $E\left(\tau_{-1,1}(x,0)\right)$.

Finally, we mention the exit probabilities of the integrated Gauss-Markov process X through the ends of the interval (a,b), namely:

$$\pi_a(x,y) = P\left(\tau_a(x,y) < \tau_b(x,y)\right) = P\left(X(\tau_{a,b}(x,y)) = a\right),$$

and

$$\pi_b(x,y) = P\left(\tau_b(x,y) < \tau_a(x,y)\right) = P\left(X(\tau_{a,b}(x,y)) = b\right).$$

Recalling the well-known formulae for exit probabilities of BM, we get, for $y = 0$ and $x \in (a,b)$:

$$\pi_a(x,0) = P\left(x + \widehat{B}(\widehat{\tau}_{a,b}(x,0)) = a\right) = \frac{b-x}{b-a}, \quad \pi_b(x,0) = P\left(x + \widehat{B}(\widehat{\tau}_{a,b}(x,0)) = b\right) = \frac{x-a}{b-a}.$$

Notice that, in the case of integrated BM, several probability laws related to the couple $\left(\tau_{a,b}, B_{\tau_{a,b}}\right)$ were evaluated in [25] (in particular, explicit formulae for $\pi_a(x,0)$ and $\pi_b(x,0)$ were obtained), but they are written in terms of special functions.
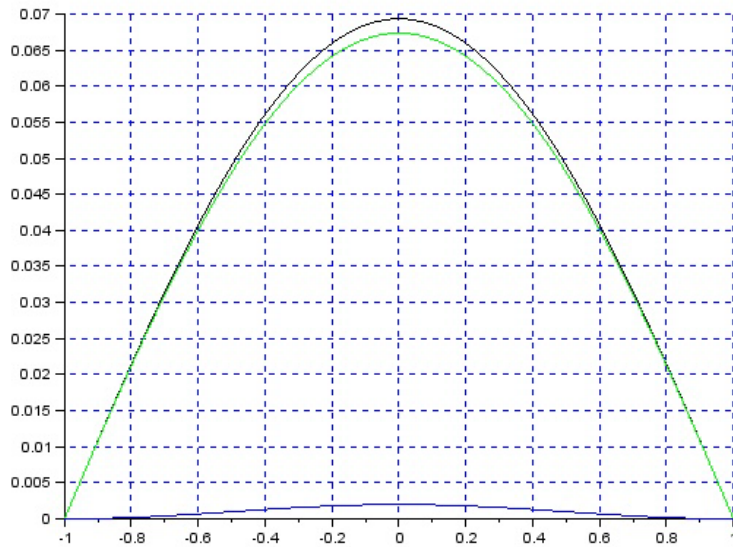
Figure 4: From top to bottom: plot of the second moment (first curve), the square of the first moment (second curve), and the variance of the first-exit time $\tau_{-1,1}(x,0)$ (third curve) of integrated OU with $y = \beta = 0$, from the interval $(-1,1)$, as a function of $x \in (-1,1)$, for $\sigma = 1, \mu = 1$.

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