JOINT TOPOLOGICAL DIVISORS AND NONREMOVABLE IDEALS IN A COMMUTATIVE REAL BANACH ALGEBRA

H. S. MEHTA, R. D. MEHTA AND A. N. ROGHELIA

Received August 6, 2015

Abstract.

The concept of joint topological zero divisors (JTZD) in a real Banach algebra was discussed in [4]. In this paper we study the concepts of cortex, Šilov boundary and non-removable ideals and relating them with ideals consisting of JTZD.

1 Introduction and Preliminaries The concepts of ideals consisting of JTZD, cortex and non removable ideals for a complex Banach algebra are studied in detail [5, 6, 7, 8]. Here we extend some of these results for a real Banach algebra. We have modified certain concepts and used the complexification technique to prove some results which was applied effectively in [3].

Throughout the paper, A denotes a real commutative Banach algebra with identity, $Car\,(A)$ and $\mathfrak{M}\,(A)$ denote the space of all nonzero (real) homomorphisms from A to $\mathbb{C}$ called the carrier space and the space of all maximal ideals of A respectively. We refer to [5] and [3] for the basic definitions.

Definition 1.1. Let A be a real Banach algebra with identity 1 and $cxA = \{(a,b) : a, b \in A\}$. Then with the following operations, cxA becomes a complex algebra with identity $(1, 0)$.

$$\left.\begin{array}{l}(a,b)+(c,d)=(a+c,b+d)\\(\alpha+i\beta)\,(a,b)=(\alpha a-\beta b,\alpha b+\beta a)\\(a,b)\,(c,d)=(ac-bd,ad+bc)\end{array}\right\}\begin{array}{c}for\ \ all\ \ a,b,c,d\in A\\ \alpha,\beta\in\mathbb{R}\end{array}$$

It is called the *complixification* of A. Further, there exists a norm $\|\cdot\|_{cxA}$ on cxA [3], making cxA a Banach algebra and satisfying,

(i) $\max\left(\|a\|,\|b\|\right) \leq \|\,(a,b)\|_{\mathrm{cx}A} \leq 2\max\left(\|a\|,\|b\|\right)\ \ for\ all\ a,b\in A.$

(ii) $\|\,(a,0)\|_{\mathrm{cx}A} = \|a\|\ \ for\ all\ a \in A.$

2010 *Mathematics Subject Classification.* 46J10, 46J20 .

Key words and phrases. Joint TZD, Cortex, Non-removable ideals .

Note that $a \to (a, 0)$ embeds A into cxA isometrically. Now onwards we use $\|(a, b)\|$ instead of $\|(a, b)\|_{\mathrm{cx}A}$.

We associate $Car\,(cxA)$ and $\mathfrak{M}\,(cxA)$ with A. The following diagram (Figure 1) shows their interrelations.

$$\begin{tikzcd}
Car(A) \arrow[r, "ker"] & \mathfrak{M}(A) \\
Car(cxA) \arrow[u, "R"] \arrow[r, "ker"] & \mathfrak{M}(cxA) \arrow[u, "cx*"']
\end{tikzcd}$$

Figure 1:

We list the properties of the maps shown in the diagram.

(i) $R : Car\,(cxA) \to Car\,(A)$ defined as $R\,(\psi) = \psi_{/A}$, is a one-to-one, onto, continuous and open mapping.

(ii) $cx^* : \mathfrak{M}\,(cxA) \to \mathfrak{M}\,(A)$ defined by $cx^*\,(M) = M \bigcap A$ is a two to one, onto continuous and open mapping. Also, $cx^*\,(\Gamma\,(cxA)) = \Gamma\,(A)$ where, $\Gamma\,(A)$ denote the Šilov boundary of A [3].

(iii) $\ker : Car\,(A) \to \mathfrak{M}\,(A)$ defined by $\psi \mapsto \ker\psi$ is a two to one, onto, continuous mapping [3].

(iv) If A is a complex Banach algebra, then the map ker is a one to one mapping.

Further, we define, $\sigma : cxA \to cxA$ by $\sigma\,(f, g) = (f, -g)$. Then σ is a linear map which is also isometry.

We shall need the next proposition to prove the main result.

Proposition 1.2. *If N is a closed ideal in A, then N_{cxA} is a closed ideal in cxA where, $N_{cxA} = \{(x, y) : x, y \in N\}$. Further if N is maximal, then N_{cxA} is contained in exactly two maximal ideals of cxA namely $\ker\psi$ and $\ker\left(\bar{\psi} \circ \sigma\right)$, where $\psi = R^{-1}\,(\phi)$, $\overline{\psi}(x) = \overline{\psi(x)}$ and $N = \ker\phi$.*

Proof. It is easy to verify that N_{cxA} is a closed ideal in cxA. Let $N \in \mathfrak{M}\,(A)$. Then, $N = \ker\phi$ for some $\phi \in Car\,(A)$. Note that $\ker\phi = \ker\bar{\phi}$ and if $R^{-1}\,(\phi) = \psi$, then

$R^{-1}\left(\bar{\phi}\right) = \bar{\psi} \circ \sigma$.

Claim 1: $N_{cxA} = \ker\psi \bigcap \ker\left(\bar{\psi} \circ \sigma\right)$.

Let $(x, y) \in N_{cxA}$ with $x, y \in N$. Then $\phi(x) = \phi(y) = 0 = \bar{\phi}(x) = \bar{\phi}(y)$, which implies $\psi(x, y) = \phi(x) + i\phi(y) = 0$ and $\left(\bar{\psi} \circ \sigma\right)(x, y) = \bar{\phi}(x) + i\bar{\phi}(y) = 0$. Hence, (x, y) $\in \ker\psi \bigcap \ker\left(\bar{\psi} \circ \sigma\right)$. Thus, $N_{cxA} \subset \ker\psi \bigcap \ker\left(\bar{\psi} \circ \sigma\right)$.

Conversely, if $(x, y) \in \ker\psi \bigcap \ker\left(\bar{\psi} \circ \sigma\right)$, then $0 = \psi(x, y) = \phi(x) + i\phi(y)$ and $0 = \left(\bar{\psi} \circ \sigma\right)(x, y) = \bar{\phi}(x) + i\bar{\phi}(y)$. So, $\phi(x) - i\phi(y) = 0$. Therefore, $\phi(x) = 0 = \phi(y)$. Hence, $x, y \in N$ and so, $(x, y) \in N_{cxA}$. Therefore, $\ker\psi \bigcap \ker\left(\bar{\psi} \circ \sigma\right) \subset N_{cxA}$. Hence, $N_{cxA} = \ker\psi \bigcap \ker\left(\bar{\psi} \circ \sigma\right)$.

Claim 2: N_{cxA} is contained in only two maximal ideals namely $\ker\psi$ and $\ker\left(\bar{\psi} \circ \sigma\right)$.

Suppose $N_{cxA} \subset M'$, where $M' \in \mathfrak{M}(cxA)$, then $M' = \ker\psi'$ for some $\psi' \in Car(cxA)$. Let $\phi' = \psi'_{|A} = R(\psi')$. Then, we show that $\ker\phi = \ker\phi'$.

Let $x \in \ker\phi = N$. Then $(x, x) \in N_{cxA} \subset M'$. So, $\psi'(x, x) = 0$, i.e., $\phi'(x) + i\phi'(x) = 0$. Hence, $\phi'(x) = 0$. Thus, $x \in \ker\phi'$. Hence, $\ker\phi \subset \ker\phi'$. Therefore, $\ker\phi = \ker\phi'$ as both of them are maximal ideals in A. So, $\phi = \phi'$ or $\bar{\phi} = \phi'$. Hence, $\psi = \psi'$ or $\bar{\psi} \circ \sigma = \psi'$. $\square$

2 Joint topological zero divisor

In this section, we have defined joint topological zero divisor for a real Banach algebra. Also, we have proved some results similar to that of complex Banach algebras [6].

Definition 2.1. Let A be a real commutative Banach algebra. A subset S of A is said to be consisting of *joint topological zero divisors* (JTZD) if for every finite subset $\{x_1, ..., x_n\}$ of S

$$d(x_1, ..., x_n) = \inf\left\{\sum_{i=1}^{n} \|x_i z\| : z \in A, \|z\| = 1\right\} = 0.$$

Equivalently, there exists a net (z_α) in A with $\|z_\alpha\| = 1$ such that $\lim\limits_{\alpha} xz_\alpha = 0$ for each $x \in S$ [4]. In particular, if S is an ideal, then it is called an ideal consisting of JTZD. Note that if $S = \{x\}$, then the above definition coincides with topological zero divisor.

Theorem 2.2. *If A is a real commutative Banach algebra and $I \subset A$ is a nonzero ideal consisting of JTZD, then there exists a maximal ideal N in A consisting of JTZD and $I \subset N$.*

To prove the above result we need the following lemmas.

Lemma 2.3. *If I is an ideal in A consisting of JTZD, then*
$I_{cxA} = \{(x, y) : x, y \in I\}$ is an ideal in cxA consisting of JTZD.

Proof. As we have noted in Proposition 1.2, I_{cxA} is an ideal in cxA. To show that I_{cxA} consists of JTZD, let $(x, y) \in I_{cxA}$. Then $x, y \in I$. Since, I consists of JTZD, there exists a net (x_α) in A with $\|x_\alpha\| = 1$ such that $\|xx_\alpha\| < \frac{\varepsilon}{2}$ for $\alpha \geq \alpha_x$ and $\|yx_\alpha\| < \frac{\varepsilon}{2}$ for $\alpha \geq \alpha_y$. Let $\alpha_\varepsilon \geq \alpha_x$ and $\alpha_\varepsilon \geq \alpha_y$. Then $\|xx_\alpha\| < \frac{\varepsilon}{2}$ and $\|yx_\alpha\| < \frac{\varepsilon}{2}$ for $\alpha \geq \alpha_\varepsilon$.

Consider $z_\alpha = (x_\alpha, 0)$. Then, (z_α) is a net in cxA. Also, $\|z_\alpha\| = \|(x_\alpha, 0)\| = \|x_\alpha\| = 1$ and $\|z_\alpha (x, y)\| = \|(x_\alpha x, x_\alpha y)\| \leq 2 \max (\|x_\alpha x\|, \|x_\alpha y\|) < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. So, $\lim_\alpha z_\alpha (x, y) = 0$ for each $(x, y) \in I_{cxA}$. Hence, I_{cxA} consists of JTZD. $\square$

Lemma 2.4. *If J is an ideal in cxA consisting of JTZD, then $J \bigcap A$ is an ideal in A consisting of JTZD.*

Proof. Clearly, $I = J \bigcap A$ is an ideal in A. Let $x \in I$. Then, $(x, 0) \in J$. Therefore, there exists a net $(z_\alpha)_{\alpha \in \Lambda}$ in cxA with $\|z_\alpha\| = 1$ such that $\|z_\alpha (x, 0)\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$.

Let $z_\alpha = (x_\alpha, y_\alpha)$. Then $\|(x_\alpha, y_\alpha)(x, 0)\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. Therefore, $\|(x_\alpha x, y_\alpha x)\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. So, $\max (\|x_\alpha x\|, \|y_\alpha x\|) \leq \|(x_\alpha x, y_\alpha x)\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. Hence, $\|x_\alpha x\| < \varepsilon$ and $\|y_\alpha x\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. So, $\lim_\alpha x_\alpha x = 0$ and $\lim_\alpha y_\alpha x = 0$.

Now, $\max (\|x_\alpha\|, \|y_\alpha\|) \leq \|z_\alpha\| = 1 \leq 2 \max (\|x_\alpha\|, \|y_\alpha\|)$ for each α. Therefore,
$\frac{1}{2} \leq \max (\|x_\alpha\|, \|y_\alpha\|) \leq 1$ for each $\alpha \in \Lambda$.

Let

$$z_{\alpha'} = \begin{cases} \frac{x_\alpha}{\|x_\alpha\|}, if \|x_\alpha\| \geq \frac{1}{2} \\ \frac{y_\alpha}{\|y_\alpha\|}, if \|x_\alpha\| < \frac{1}{2} \end{cases}$$

It is clear that $\{z_{\alpha'}\}$ is a net of norm one and $\lim_\alpha z_{\alpha'} x = 0$. Hence, I consists of JTZD. $\square$

Proof. (Theorem 2.2) Let I consist of JTZD. Then by Lemma 2.3, I_{cxA} consists of JTZD. Hence, there exists a maximal ideal M in cxA consisting of JTZD such that $I_{cxA} \subset M$ [6]. Let $N = M \bigcap A$. Then by Lemma 2.4, N is in A and it consists of JTZD, and $I \subset N$. This N is the required maximal ideal. $\square$

3 Cortex The concept of cortex for a complex Banach algebra has been studied in [5]. The cortex for a complex Banach algebra A is defined as a subset of $Car(A)$. Here, we define the cortex slightly in a different manner.

Definition 3.1. Let A be a real commutative Banach algebra with identity. The set $\{M \in \mathfrak{M}(A) : M \text{ consists of JTZD}\}$ is called the *cortex* of A and is denoted by $Cor(A)$.

Note that for a complex Banach algebra A, $Cor(A)$ can also be looked upon as a subset of $Car(A)$ as $Car(A) \cong \mathfrak{M}(A)$. Here we have considered cortex of a complex Banach algebra A as a subset of $\mathfrak{M}(A)$. The following result for a real Banach algebra A follows immediately from the result of §2.

Theorem 3.2. *$cx^*(Cor(cxA)) = Cor(A)$. Consequently $Cor(A)$ is a nonempty compact subset of $\mathfrak{M}(A)$.*

Corollary 3.3. $\Gamma(A) \subset Cor(A)$.

Proof. $\Gamma(A) = cx^*(\Gamma(cxA))$[3]$\subset cx^*(Cor(cxA))$[5]$= Cor(A)$. $\square$

Lemma 3.4. *Let $\psi \in Car(cxA)$. Then $\ker\psi \in Cor(cxA)$ if and only if $\ker(\bar{\psi}\circ\sigma) \in Cor(cxA)$.*

Proof. Let $(f,g) \in cxA$. Then,$(f,g) \in \ker\psi \Leftrightarrow \psi(f,g) = 0 \Leftrightarrow \bar{\psi}(f,g) = 0$
$\Leftrightarrow (\bar{\psi}\circ\sigma)(f,-g) = 0 \Leftrightarrow (f,-g) \in \ker(\bar{\psi}\circ\sigma)$.

Let $\ker\psi \in Cor(cxA)$. To show that $\ker(\bar{\psi}\circ\sigma) \in Cor(cxA)$, let $(f_i,g_i) \in \ker(\bar{\psi}\circ\sigma)$ for $i = 1,...,n$. Therefore, $(f_i,-g_i) \in \ker\psi$ for $i = 1,...,n$. But $\ker\psi$ consists of JTZD. Hence, for given $\varepsilon > 0$ there exists $(x,y) \in cxA$ with $\|(x,y)\| = 1$ such that

$$\sum_{k=1}^{n} \|(f_k,-g_k)(x,y)\| < \varepsilon.$$

Now, $\|(f_k,-g_k)(x,y)\| = \|(f_k,g_k)(x,-y)\|$ as $\sigma(f,g) = (f,-g)$ is an isometry. So, $\sum_{k=1}^{n} \|(f_k,g_k)(x,-y)\| < \varepsilon$. Hence, $\ker(\bar{\psi}\circ\sigma) \in Cor(cxA)$.

The converse follows from the fact $\overline{\bar{\psi}\circ\sigma}\circ\sigma = \psi$. $\square$

Remark 3.5. *If we consider $F = \ker^{-1}(\Gamma(A))$ and $K = \ker^{-1}(Cor(A))$, then it is clear from the definition of $\Gamma(A)$ that $\ker|_F$ is also two to one onto $\Gamma(A)$. The following result shows that $\ker|_K$ is also two to one onto $Cor(A)$.*

Proposition 3.6. $R\left(\ker^{-1}(Cor\,(cxA))\right) = \ker^{-1}(Cor\,(A))$

Proof. Let $\psi \in \ker^{-1}(Cor\,(cxA))$. Then $\ker\psi \in Cor\,(cxA)$. Now, $R(\psi) = \psi_{|A} = \phi$. To prove $\phi \in \ker^{-1}(Cor\,(A))$, we have to show that $\ker\phi \in Cor\,(A)$. Now, $\ker\phi = \ker\psi \bigcap A$. Therefore, by Lemma 2.4, $\ker\phi$ consists of JTZD. Hence, $\phi \in \ker^{-1}(Cor\,(A))$.

Conversely, let $\phi \in \ker^{-1}(Cor\,(A))$. Then $\ker\phi = N \in Cor\,(A)$. Then, by Lemma 2.3, N_{cxA} consists of JTZD. Hence, there exists a maximal ideal $M \in Cor\,(cxA)$ such that $N_{cxA} \subset M$. But N_{cxA} is contained in only two maximal ideals, $\ker\psi$ and $\ker\left(\bar{\psi}\circ\sigma\right)$. Therefore, either $\ker\psi$ or $\ker\left(\bar{\psi}\circ\sigma\right)$ consists of JTZD. So, by Lemma 3.4 in any case, $\ker\psi \in Cor\,(cxA)$. Therefore, $R\left(\ker^{-1}(Cor\,(cxA))\right) = \ker^{-1}(Cor\,(A))$. ☐

4 Extension and Non-removable ideals

In this section, we characterize the cortex of a real Banach algebra. For this, we define the concepts of extensions and non-removable ideals for a real Banach algebra. Also, we have shown that the smallest complex extension for a real Banach algebra is its complexification.

Definition 4.1. Let A and B be Banach algebras. We say that B is an *extension* of A if there exists an isometrical into isomorphism $\rho : A \to B$. In this case, we write $A \subset B$.

Theorem 4.2. *Let A be a real commutative Banach algebra.*
(i) If B is a real extension of A, then cxB is an extension of cxA with an equivalent norm.
(ii) If B is a complex extension of A, then B is also an extension of cxA, i.e., cxA is the smallest complex extension of A.

Proof. (i) Let B be a real extension of A. Then there exists a real into isometrical isomorphism $\rho : A \to B$. Define $\rho' : cxA \to cxB$ by $\rho'(a,b) = (\rho(a), \rho(b))$. Then it is easy to check that ρ' is an algebra homomorphism. Further,$\|\rho'(a,b)\| = \|(\rho(a), \rho(b))\|$
$\leq 2\max(\|\rho(a)\|, \|\rho(b)\|) = 2\max(\|a\|, \|b\|) \leq 2(\|(a,b)\|)$ and $\|(a,b)\| \leq 2\max(\|a\|, \|b\|)$
$= 2\max(\|\rho(a)\|, \|\rho(b)\|) \leq 2(\|(\rho(a), \rho(b))\|) = 2\|\rho'(a,b)\|$.

Hence, $\frac{1}{2}\|(a,b)\| \leq \|\rho'(a,b)\| \leq 2(\|(a,b)\|)$. Therefore, there exists an algebra norm $|||\cdot|||$ on cxB equivalent to the above norm on cxB such that $|||\rho'(a,b)||| = \|(a,b)\|$ for every $(a,b) \in cxA$ [5]. Hence, cxB is an extension of cxA.

(ii) Let B be a complex extension of A. Then $cxB \cong B$. So, as in part (1), we get B is an extension of cxA. $\square$

Definition 4.3. An ideal I in a commutative Banach algebra A is called *non-removable*, if in every commutative Banach algebra $B \supset A$, there exists a proper ideal J of B such that $J \supset I$.

We shall need the following lemma.

Lemma 4.4. *If I is non-removable in A, then I_{cxA} is non-removable in cxA.*

Proof. Let I be a non-removable ideal in A. To show that I_{cxA} is non-removable in cxA, let B be an extension of cxA. Then B is also an extension of A. Therefore, there exists a proper ideal $J \subset B$ such that $I \subset J$. So, $I_{cxA} \subset J$. Hence, I_{cxA} is non-removable in cxA. $\square$

Theorem 4.5. *An ideal in a real commutative Banach algebra is non-removable if and only if it consists of JTZD.*

Proof. Let A be a real commutative Banach algebra and I be an ideal consisting of JTZD. Then there exists a net (z_α) in A with $\|z_\alpha\| = 1$ and $\lim\limits_{\alpha} xz_\alpha = 0$ for every $x \in I$.

Let $B \supset A$ be a commutative extension of A and let

$$J = \{x_1b_1 + ... + x_nb_n : x_1, ..., x_n \in I, b_1, ..., b_n \in B, n \in \mathbb{N}\}$$

be the smallest ideal in B containing I. Suppose J is not proper. Then $1 \in J$. Therefore, there exists $x_1, ..., x_n \in I$ and $b_1, ..., b_n \in B$ such that $\sum_{k=1}^{n} x_kb_k = 1$.

Then, $1 = \|z_\alpha\| = \|\sum_{k=1}^{n} z_\alpha x_kb_k\| \leq \sum_{k=1}^{n} \|z_\alpha x_k\| \, \|b_k\| \to 0$ a contradiction. Hence, J is proper and so I is non-removable.

Conversely, let I be a non-removable ideal in A. Then I_{cxA} is non-removable ideal in cxA by the above Lemma. Therefore, I_{cxA} consists of JTZD [5]. Hence, $I = I_{cxA} \bigcap A$ also consists of JTZD by Lemma 2.4. $\square$

The next theorem gives characterization of $Cor(A)$.

Theorem 4.6. *Let A be a real commutative Banach algebra and $\phi \in Car\,(A)$. Then the following statements are equivalent:*

(i) $\ker\phi \in Cor\,(A)$.

(ii) For every commutative Banach algebra $B \supset A$, there exists a multiplicative linear functional $\psi \in Car\,(B)$ such that $\phi = \psi_{|A}$.

(iii) For every commutative Banach algebra $B \supset A$, there exists a multiplicative linear functional ψ such that $\ker\psi \in Cor\,(B)$ *and $\phi = \psi_{|A}$.*

Proof. First we prove $(i) \Rightarrow (iii)$. Let $\ker\phi \in Cor\,(A)$. Then, there exists a net (z_α) in A with $\|z_\alpha\| = 1$ and $\lim_\alpha xz_\alpha = 0$ for every $x \in \ker\phi$. Let B be a commutative Banach algebra with $B \supset A$ and $I = \{y \in B : yz_\alpha \to 0\}$. Then $I \supset \ker\phi$ and I consists of JTZD in B, so by Theorem 2.2, there exists a maximal ideal J consists of JTZD in B such that $I \subset J$. Let $J = \ker\psi$. Then $\psi_{|A} = \phi$.

$(iii) \Rightarrow (ii)$ is clear.

Finally, we prove $(ii) \Rightarrow (i)$ If $B \supset A$ and $\psi \in Car\,(B)$ extends ϕ, then $\ker\phi \subset \ker\psi$. Hence, $\ker\phi$ is a non-removable ideal in A. Hence, by Theorem 4.5, $\ker\phi$ consists of JTZD. Therefore, $\ker\phi \in Cor\,(A)$. ☐

Acknowledgements: This research is supported by the SAP programme to the Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar by UGC.

References

[1] R. Arens, *Extensions of Banach Algebras*, Pacific J. Math., **10**, (1960), 1-16.

[2] S. H. Kulkarni and B.V. Limaye, Real function algebras, Monographs and Textbooks in Pure and Applied Mathematics, Dekker, New York, 1992.

[3] B. V. Limaye, *Boundaries for Real Banach algebras*, Can. J.Math., **28**, No. 1, (1976), 42-49

[4] H. S. Mehta, R. D. Mehta and A. N. Roghelia, *Joint topological zero divisors for a real Banach algebra*, Mathematics Today, **30**, (2014), 54-58.

[5] V. Muller, Spectral theory of linear operators and spectral systems in Banach algebras, Birkhasuser Verlag, Basel–Boston–Berlin, 2007.

[6] Z. Slodkowski, *On ideals consisting of joint topological divisors of zero*, Studia Math., **48**, (1973), 83-88.

[7] A. Wawrzynczyk, *On ideals consisting of topological zero divisors*, Studia Math., **142** (8), (2000), 245-251.

[8] W. Zelazko, *On a certain class of non-removable ideals in Banach Algebras*, Studia Math., **44**, (1972), 87-92.

Communicated by *Anthony To-Ming Lau*

Aakar N. Roghelia
BVM Engineering College, Vallabh Vidyanagar, India,388120
E-mail address: aakarkhyati@gmail.com

H. S. Mehta
Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, India, 388120
E-mail address: hs_mehta@spuvvn.edu

R. D. Mehta
Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, India, 388120
E-mail address: vvnspu@yahoo.co.in