CONTRA-γ-IRRESOLUTE MAPPINGS AND RELATED GROUPS *

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Abstract.The aim of the present paper is devoted to discuss some more properties of γ-irresolute mappings and contra-γ-irresolute mappings. Also, we introduce and study two new weak homeomorphisms such as contra-γr-homeomorphisms and contra-γ-homeomorphisms. Further, we investigate some groups related to the mappings above and some examples of them on digital lines.

1 Introduction and preliminaries D.Andrijević [6] (resp. A.A. EL-Atik [15] and J. Dontchev and M. Przemski [13]) introduced independently the concept of b-open sets [6] (resp. γ-open sets [15] and sp-open sets [13]). A subset A of a topological space (X,τ) is called a *γ-open set* [15] (or *b-open set* [6], *sp-open set* [13]), if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ holds in (X,τ); and the complement of a γ-open (or b-open, sp-open) set is called *γ-closed* (or *b-closed, sp-closed*). Throughout the present paper, we use the terminology due to [15] for the naming of the above set, i.e., γ-open sets, γ-closed sets. The *γ-closure* of a subset E of (X,τ) is defined by $\gamma Cl(E) := \bigcap\{F | E \subseteq F, F$ is γ-closed in $(X,\tau)\}$; and it is the smallest γ-closed set containing E (cf. Theorem 4.4(iii)); we recall some importante properties of γ-open sets in Section 4 (Theorem 4.4).

In the present paper, we use the following notations (cf. [28] [19, p.2]):
$\gamma O(X,\tau) := \{U | U$ is γ-open in $(X,\tau)\}$;
$\gamma C(X,\tau) := \{F | F$ is γ-closed in (X,τ), i.e., $Int(Cl(F)) \cap Cl(Int(F)) \subseteq F\}$.
$SO(X,\tau) := \{U | U$ is semi-open in (X,τ), i.e., $U \subseteq Cl(Int(U))\}$ [25];
$SC(X,\tau) := \{F | F$ is semi-closed in (X,τ), i.e., $Int(Cl(F)) \subseteq F\}$.
$\tau^\alpha := \{V | V$ is α-open in (X,τ), i.e., $V \subseteq Int(Cl(Int(V)))\}$ [27].
$\beta O(X,\tau) = SPO(X,\tau) := \{W | W$ is β-open (or semi-preopen) in (X,τ), i.e., $W \subseteq Cl(Int(Cl(W)))\}$ [2],[5]. It is well known that:
$\tau^\alpha \subseteq SO(X,\tau) \subseteq \gamma O(X,\tau) \subseteq \beta O(X,\tau)$ hold in general.

In Section 2, we mention some relations among γ-irresoluteness [12], pre-γ-closedness [15], contra-γ-irresoluteness ([16] [28]) and some mappings (cf. Definitions 2.1, 2.2).

In Section 3, after the work due to A.Keskin and T.Noiri [20], we study a new group, say γr-$h(X;\tau) \cup$ *contra*-γr-$h(X;\tau)$ (Theorem 3.4(i), Corollary 3.6(i)). By the article [20, Definition 4.13, Theorem 4.14(ii)], the concept of the family γr-$h(X;\tau)$ is introduced and it is proved that γr-$h(X;\tau)$ forms a group. The family *contra*-γr-$h(X;\tau)$ is one of all *contra-γ-homeomorhpisms* on (X,τ) (cf. Definition 3.2). The group γr-$h(X;\tau) \cup$ *contra*-γr-$h(X;\tau)$ is one of the group invariants of a topological space (X,τ) under a γr-homeomorphism between topological spaces (Theorem 3.5(i)). By Theorem 3.4(iii)(cf. (iv)), it is shown that the group $h(X;\tau)$ of all homeomorphisms on (X,τ) is a subgroup of the group γr-$h(X;\tau) \cup$ *contra*-γr-$h(X;\tau)$.

In Section 4, we introduce two subgroups of γr-$h(X;\tau)$ (Definition 4.1) and so we can investigate some group structure of γr-$h(H;\tau|H)$ for the subspace $(H,\tau|H)$ of (X,τ) (Theorems 4.2 and 4.9(iii)).

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In Section 5, we study some topological properties on related topics of transformations on the digital line $(\mathbb{Z}, \kappa)$ (so-called Khalimsky lines [21], [22, p.7, line -6], [23, p.905, p.908]), and for a specific subset H of the digital line $(\mathbb{Z}, \kappa)$, we determine the group structure (Example 5.13) of $\gamma r\text{-}h(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa)$, $\gamma r\text{-}h_0(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa)$ and $\gamma r\text{-}h(H; \kappa|H)$.

Throughout the present paper, $(X,\tau), (Y,\sigma)$ and (Z,η) (or simply X, Y and Z) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned.

2 Contra-γ-irresolute mappings and γ-irresolute mappings This section is devoted to discuss the relation among γ-irresolute mappings [15], contra-γ-irresolute mappings [16][28], perfectly contra-γ-irresolute mappings [16] and some mappings (cf. Definitions 2.1, 2.2).

Definition 2.1 A mapping $f : (X,\tau) \to (Y,\sigma)$ is said to be:

(i) *b-continuous* [12] (or *γ-continuous* [15]), if $f^{-1}(V)$ is a b-closed (or γ-closed) set of (X,τ) for each closed set V of (Y,σ);

(ii) *perfectly continuous* [31], if $f^{-1}(V)$ is clopen in (X,τ) for each open set V of (Y,σ);

(iii) *contra-continuous* [11], if $f^{-1}(V)$ is closed in (X,τ) for each open set V of (Y,σ);

(iv) *contra-γ-continuous* [16] (or *contra-b-continuous* [28]) if $f^{-1}(V) \in \gamma C(X,\tau)$ for each open set V of (Y,σ);

(iv)$'$ *strongly contra-γ-continuous* (cf. (iv)), if f is a contra-γ-continuous mapping such that the inverse image of each open set of (Y,σ) has an interior point;

(v) *B-continuous* [34], if $f^{-1}(V)$ is a B-set of (X,τ) for each nonempty open set V of (Y,σ), where the B-set is the intersection of an open set and a semi-closed set of (X,τ) (this is defined by [34], cf. [10, Theorem 2.3]).

(v)$'$ *B^*-continuous* (cf. (v)), if $f^{-1}(V)$ contains a nonempty B-set of (X,τ) for each nonempty open set V of (Y,σ);

(vi) *pre-b-closed* [15] (or *pre-γ-closed*), if $f(G)$ is b-closed (or γ-closed) in (Y,σ) for each b-closed (or γ-closed) set G of (X,τ).

Definition 2.2 A mapping $f : (X,\tau) \to (Y,\sigma)$ is said to be:

(i) *γ-irresolute* (or *b-irresolute* [15]) (resp. *irresolute* [8, Definition 1.1]), if $f^{-1}(U) \in \gamma O(X,\tau)$ (resp. $f^{-1}(U) \in SO(X,\tau)$) for every set $U \in \gamma O(Y,\sigma)$ (resp. $U \in SO(Y,\sigma)$);

(ii) *contra-γ-irresolute* [16] (or *contra-b-irresolute* [28]) (resp.-
contra-irresolute), if $f^{-1}(U) \in \gamma C(X,\tau)$ (resp. $f^{-1}(U) \in SC(X,\tau)$) for every set $U \in \gamma O(Y,\sigma)$ (resp. $U \in SO(Y,\sigma)$);

(iii) *perfectly contra-γ-irresolute* [29] (resp. *perfectly contra-irresolute*), if $f^{-1}(V)$ is γ-clopen (resp. semi-open and semi-closed) in (X,τ) for each set $V \in \gamma O(Y,\sigma)$ (resp. $V \in SO(Y,\sigma)$).

Theorem 2.3 *A mapping $f : (X,\tau) \to (Y,\sigma)$ is B^*-continuous, if one of the following conditions is satisfied:*

(1) f is a strongly contra-γ-continuous mapping,

(2) f is an onto and B-continuous mapping. □

We have the following diagram among several mappings defined above, where p $\to$ q (resp. p' $\not\leftrightarrow$ q') means that p implies q (resp. p' and q' are independent). The implications are not reversible and the independence is explained (cf. Remark 2.4 below).

contra-continuous
$\swarrow$ $\searrow$
contra-γ-continuous $\not\leftrightarrow$ B-continuous $\leftarrow$ onto and B-continuous
$\uparrow$ $\downarrow$
strongly contra-γ-continuous $\rightarrow$ B^*-contiuous.

Remark 2.4 (i) Let $(\mathbb{R}, \epsilon)$ be the real line with the Euclidean topology ϵ. The following functions $f, 1_{\mathbb{R}} : (\mathbb{R}, \epsilon) \to (\mathbb{R}, \epsilon)$ of (i) below are seen in [12].

(i) (i-1) Let $f : (\mathbb{R}, \epsilon) \to (\mathbb{R}, \epsilon)$ be a mapping defined by $f(x) = [x]$, where $[x]$ is the Gaussian symbol. Then, f is contra-γ-continuous (cf. Definition 2.1(iv)). However, f is not contra-continuous, because for an open interval $(1/2, 3/2)$, $f^{-1}((1/2, 3/2)) = [1, 2)$ is not closed in $(\mathbb{R}, \epsilon)$.

(i-2) The identity mapping $1_{\mathbb{R}} : (\mathbb{R}, \epsilon) \to (\mathbb{R}, \epsilon)$ is B-continuous (cf. Definition 2.1(v)) but not contra-γ-continuous, since the inverse image of each singleton is not γ-open. Moreover, $1_{\mathbb{R}}$ is not contra-continuous.

(ii) The following mapping $f : (X, \tau) \to (X, \tau)$ is contra-γ-continuous; but f is not B-continuous. Let $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a, b\}, X\}$. Then, we have $\gamma C(X, \tau) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$ and $SC(X, \tau) = \{\emptyset, \{c\}, X\}$. We define the mapping f by $f(a) := a, f(b) := c, f(c) := b$.

(iii) The converse of Theorem 2.3 under the assumption (1) is not reversible. Let $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) := b, f(b) := c, f(c) := a$. Then, since $\gamma C(X, \tau) = SC(X, \tau) = P(X) \setminus \{\{a, b\}\}$, we show f is B-continuous and onto. By Theorem 2.3 under the assumption (2), it is obtained that f is B^*-continuous. This mapping f is contra-γ-continuous; but $Int(f^{-1}(\{a\})) = Int(\{c\}) = \emptyset$ hold; and so f is not strongly contra-γ-continuous.

(iv) The converse of Theorem 2.3 under the assumption (2) is not reversible. The mapping $f : (X, \tau) \to (X, \tau)$ defined in (ii) above is not B-continuous (cf. (ii)). But, f is B^*-continuous, because $\{c\}$ and X are the nonempty B-sets.

(v) The contra-γ-continuous mapping $f : (X, \tau) \to (X, \tau)$ of (ii) above is not strongly contra-γ-continuous (cf. Definition 2.1(iv)$'$), because
$Int(f^{-1}(\{a, b\})) = \emptyset$.

Remark 2.5 (i) Let $X = \{a, b\}, \tau = \{\emptyset, X, \{a\}\}$ and $\sigma = \{\emptyset, X, \{b\}\}$. Then the identity mapping $1_X : (X, \tau) \to (X, \sigma)$ is a contra-γ-continuous mapping but it is not γ-continuous.

(ii) The identity mapping $1_{\mathbb{R}} : (\mathbb{R}, \epsilon) \to (\mathbb{R}, \epsilon)$ of Remark 2.4(i)(i-2) is γ-continuous but it is not contra-γ-continuous.

Remark 2.6 The following properties are well known. (i) [4, Theorem 3.7(i)] if $f : (X, \tau) \to (Y, \sigma)$ is contra-γ-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is γ-continuous, then $g \circ f$ is contra-γ-continuous.

(ii) Every homeomorphism is γ-irresolute.

Remark 2.7 (i) By the following examples (i-1) and (i-2), it is shown that the contra-γ-irresoluteness and γ-irresoluteness are independent notions: let $X := \{a, b, c\}$ and $\tau := \{X, \emptyset, \{a\}, \{a, b\}\}$.

(i-1) The identity mapping on (X, τ) above is γ-irresolute; but it is not contra-γ-irresolute.

(i-2) Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) := b, f(b) := b, f(c) := a$. Then, f is contra-γ-irresolute; but f is not γ-irresolute.

(ii) In general, for any topological space (X, τ), the identity mapping $1_X : (X, \tau) \to (X, \tau)$ is contra-γ-irresolute if and only if $\gamma O(X, \tau) = \gamma C(X, \tau)$ holds. And, 1_X on any topological space (X, τ) is γ-irresolute.

Remark 2.8 (i) Every contra-γ-irresolute mapping is contra-γ-continuous, but it is shown that its converse is not true, by the following example. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) := c, f(b) := a, f(c) := b$.

(ii) For a mapping $f : (X, \tau) \to (Y, \sigma)$, f is contra-γ-irresolute if and only if the inverse image $f^{-1}(F)$ of each γ-closed set F of (Y, σ) is γ-open in (X, τ).

Remark 2.9 (i) The following diagram of implications is well known:
· contra-irresolute $\longleftarrow$ perfectly contra-irresolute $\longrightarrow$ irresolute.
We have the following diagram of implications:
· contra-γ-irresolute $\longleftarrow$ perfectly contra-γ-irresolute $\longrightarrow$ γ-irresolute;
and they are not reversible (cf. Remark 2.7(i) above and Remark 2.10 below):

(ii) In the implications above, the irresoluteness (resp. contra-irresoluteness, perfectly contra-irresoluteness) and the γ-irresoluteness (resp. contra-γ-irresoluteness, perfectly contra-γ-irresoluteness) are independent (cf. (a), (b), (c) below).

Let $X = \{a, b, c\}$. We consider the following topologies on $X : \tau := \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\tau_1 := \{X, \emptyset, \{a\}, \{a, b\}\}$, $\tau_2 := \{X, \emptyset, \{c\}, \{a, b\}\}$ and $\tau_3 := \{X, \emptyset\}$. We have the following dates: $SO(X, \tau) = \gamma O(X, \tau) = P(X) \setminus \{\{c\}\}$; $SO(X, \tau_1) = \gamma O(X, \tau_1)=\{\emptyset, \{a\}, \{a, b\},\{a, c\}, X\}$; $SO(X, \tau_2) = \tau_2, \gamma O(X, \tau_2) = P(X)$; $SO(X, \tau_3) = \{\emptyset, X\}, \gamma O(X, \tau_3) = P(X)$.

(a) (a-1) Define a mapping $f : (X, \tau) \to (X, \tau_2)$ as follows: $f(a) = a, f(b) = c$ and $f(c) = b$. Then f is irresolute; f is not γ-irresolute.
(a-2) Let $f : (X, \tau_3) \to (X, \tau)$ be the identity mapping. Then f is γ-irresolute; f is not irresolute.

(b) (b-1) Let $f : (X, \tau_2) \to (X, \tau_1)$ be the identity mapping. Then f is contra-γ-irresolute; f is not contra-irresolute.
(b-2) Define a mapping $f : (X, \tau_1) \to (X, \tau_2)$ as follows: $f(a) := a, f(b) := a, f(c) := b$. Then f is contra-irresolute; f is not contra-γ-irresolute.

(c) (c-1) Let $f : (X, \tau_3) \to (X, \tau_2)$ be the identity mapping. Then f is perfectly contra-γ-irresolute; f is not perfectly contra-irresolute.
(c-2) Define a mapping $f : (X, \tau) \to (X, \tau_2)$ as follows: $f(a) := c, f(b) := a, f(c) := b$. Then f is perfectly contra-irresolute; f is not perfectly contra-γ-irresolute.

Remark 2.10 We have a decomposition of perfectly contra-γ-irresolute mappings. The following conditions (1) and (2) are equivalent: (1) $f : (X, \tau) \to (Y, \sigma)$ is perfectly contra-γ-irresolute; (2) $f : (X, \tau) \to (Y, \sigma)$ is contra-γ-irresolute and γ-irresolute.

3 Groups γr-$h(X; \tau) \cup$ *contra*-γr-$h(X; \tau)$ and $h(X; \tau) \cup$ *contra*-$h(X; \tau)$ We have a new homeomorphism invariant for topological spaces (cf. Theorems 3.4, 3.5, Corollary 3.6).

Definition 3.1 (i) A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(i-1) ([20, Definiton 4.12]) a *γr-homeomorphism* if f is a γ-irresolute bijection and f^{-1} is γ-irresolute;

(i-2) a *contra-γr-homeomorphism* if f is a contra-γ-irresolute bijection and f^{-1} is contra-γ-irresolute;

(ii) (ii-1) ([20, Definition 4.12]) a *γ-homeomorphism* if f is a γ-continuous bijection and f^{-1} is γ-continuous;

(ii-2) a *contra-γ-homeomorphism* (resp. *contra-homeomorphism*) if f is a contra-γ-continuous (resp. contra-continuous) bijection and f^{-1} is contra-γ-continuous (resp. contra-continuous).

Definition 3.2 We recall and define the following families of mappings from (X, τ) onto itself.
· ([20, Definition 4.13]) γr-$h(X; \tau) := \{f | f : (X, \tau) \to (X, \tau)$ is a γr-homeomorphism$\}$ (by [20, Theorem 4.14(ii)], it is proved that γr-$h(X; \tau)$ forms a group under the composition of mappings);
· *contra*-γr-$h(X; \tau) := \{f | f : (X, \tau) \to (X, \tau)$ is a contra-γr-homeomorphism$\}$;
· $h(X; \tau) := \{f | f : (X, \tau) \to (X, \tau)$ is a homeomorphism$\}$;
· *contra*-$h(X; \tau) := \{f | f : (X, \tau) \to (X, \tau)$ is a contra-homeomorphism$\}$;
· $G_{(X,\tau)} := \gamma r$-$h(X; \tau) \cup$ *contra*-γr-$h(X; \tau)$;
· $H_{(X,\tau)} := h(X; \tau) \cup$ *contra*-$h(X; \tau)$.

Proposition 3.3 *Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two mappings between topological spaces.*

(i) (i-1) ([20, Theorem 4.14(ii)]) *If f and g are γ-irresolute, then $g\circ f$ is γ-irresolute.*

(i-2) ([20, Theorem 4.14(ii)]) *The identity mapping $1_X:(X,\tau)\to(X,\tau)$ is γ-irresolute.*

(i-3) *If f and g are contra-γ-irresolute, then $g\circ f$ is γ-irresolute.*

(ii) (ii-1) *If f is contra-γ-irresolute and g is γ-irresolute, then $g\circ f$ is contra-γ-irresolute.*

(ii-2) *If f is γ-irresolute and g is contra-γ-irresolute, then $g\circ f$ is contra-γ-irresolute.* □

Theorem 3.4 *Let $G_{(X,\tau)}$ and $H_{(X,\tau)}$ be the families of mappings defned in* Definition 3.2.

(i) *$G_{(X,\tau)}$ forms a group under the composition of mappings.*

(ii) *γr-$h(X;\tau)$ forms a subgroup of $G_{(X,\tau)}$* (cf. [20, Theorem 4.14(ii)]).

(iii) *The group $h(X;\tau)$ is a subgroup of γr-$h(X;\tau)$* ([20, Theorem 4.14(iii)]) *and $h(X;\tau)$ is also a subgroup of $G_{(X,\tau)}$.*

(iv) *$H_{(X,\tau)}$ forms a group under the composition of mappings. The group $h(X;\tau)$ is a subgroup of $H_{(X,\tau)}$.*

(v) *If $\tau=\gamma O(X,\tau)$ holds, then $G_{(X,\tau)}=H_{(X,\tau)}$.* □

We note that the binary operation $\omega_{G(X,\tau)}:G_{(X,\tau)}\times G_{(X,\tau)}\to G_{(X,\tau)}$ is well defined by $\omega_{G(X,\tau)}(a,b):=b\circ a$, where $a,b\in G_{(X,\tau)}$ and $b\circ a$ denotes the composition of two mappings a, b defined by $(b\circ a)(x)=b(a(x))$ for any $x\in X$ (cf. Proposition 3.3). And, the restriction $\omega_{G(X,\tau)}|\gamma r\text{-}h(X;\tau)\times\gamma r\text{-}h(X;\tau)$ is denoted shortly by ω_X.

Theorem 3.5 (i) *If $f:(X,\tau)\to(Y,\sigma)$ is a γr-homeomorphism* (resp. *contra-γr-homeomorphism*), *then the mapping f induces an isomorphism $f_*:G_{(X,\tau)}\to G_{(Y,\sigma)}$, where f_* is defined by $f_*(a):=f\circ a\circ f^{-1}$ for any $a\in G_{(X,\tau)}$. Moreover,*

(a) *$(g\circ f)_*=g_*\circ f_*:G_{(X,\tau)}\to G_{(Z,\eta)}$, where $g:(Y,\sigma)\to(Z,\eta)$ is a γr-homeomorphism* (resp. *contra-γr-homeomorphism*),

(b) *$(1_X)_*=1:G_{(X,\tau)}\to G_{(X,\tau)}$ is the identity isomorphism,*

(c) *$f_*(\gamma r\text{-}h(X;\tau))=\gamma r\text{-}h(Y;\sigma)$, $f_*(h(X;\tau))\subseteq\gamma r\text{-}h(Y;\sigma)$ and $f_*(contra\text{-}\gamma r\text{-}h(X;\tau))=contra\text{-}\gamma r\text{-}h(Y;\sigma)$ hold.*

(ii) *Especially, if $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ are homeomorphisms, then the induced mappings $f_*:G_{(X,\tau)}\to G_{(Y,\sigma)}$ and $g_*:G_{(Y,\sigma)}\to G_{(Z,\eta)}$ are isomorphisms* (cf. (i)). *Moreover, they have the same property of* (a),(b) *and* (c) *in* (i). *We note that, in* (c), *$f_*(h(X;\tau))=h(Y;\sigma)$ holds.* □

Corollary 3.6 (cf. Definition 3.2, Theorem 3.5) (i) *If $G_{(X,\tau)}\not\cong G_{(Y,\sigma)}$* (i.e. *$G_{(X,\tau)}$ is not isomorphic to $G_{(Y,\sigma)}$ as groups), then there does not exist any γr-homeomorphism between two topological spaces (X,τ) and (Y,σ); and hence $(X,\tau)\not\cong(Y,\sigma)$* (i.e., *$(X,\tau)$ is not homeomorphic to (Y,σ)*).

(ii) *If γr-$h(X;\tau)\not\cong\gamma r$-$h(Y;\sigma)$* (i.e., *$\gamma r$-$h(X;\tau)$ is not isomorphic to γr-$h(Y;\sigma)$ as groups), then there does not exist any γr-homeomorphism between (X,τ) and (Y,σ).* □

Example 3.7 (i) In Section 5, we give a special example of group γr-$h(H,\kappa|H)$, where $(H,\kappa|H)$ is a subspace of the digital line $(\mathbb{Z},\kappa)$(cf. Example 5.13).

(ii) Let (X,τ) and (Y,σ) be two topological spaces, where $X=Y:=\{a,b,c\}$, $\tau:=\{\emptyset,\{a\},\{b,c\},X\}$ and $\sigma:=\{\emptyset,\{a\},\{b\},\{a,b\},Y\}$. Then, it is shown that $G_{(X,\tau)}=\gamma r$-$h(X;\tau)\cong S_3$(=the symetric group of degree 3) and $G_{(Y,\sigma)}=\gamma r$-$h(Y;\sigma)=\{1_Y,h_c\}$, where $h_c:(Y,\sigma)\to(Y,\sigma)$ is a bijection defined by $h_c(a):=b,h_c(b):=a,h_c(c):=c$; and hence $G_{(X,\tau)}\not\cong G_{(Y,\sigma)}$. Thus, using Corollary 3.6(i), we can assure that there is never exists any γr-homeomorphism between (X,τ) and (Y,σ). We note that $h(X;\tau)=\{1_X,h_a\}$ and $h(Y;\sigma)=\{1_Y,h_c\}$ hold, where $h_a:(X,\tau)\to(X,\tau)$ is a bijection defined by $h_a(a):=a,h_a(b):=c,h_a(c):=b$; and so $h(X;\tau)\cong h(Y;\sigma)$ holds.

(iii) Let (X,τ) be the topological space of (ii) above and let (Y_1,σ_1) be a topological space such that $Y_1:=\{a,b,c\}$ and $\sigma_1:=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\}$

$,Y_1\}$. Then, we have that $G_{(X,\tau)} \ncong G_{(Y_1,\sigma_1)}$ and $h(X;\tau) \ncong h(Y_1;\sigma_1)$. Using Corollary 3.6, there is never exist any γr-homeomorphism between (X,τ) and (Y_1,σ_1).

(iv) Let (Y_1,σ_1) be the topological space of (iii) above and let (Y_2,σ_2) be a topological space such that $Y_2 := \{a,b,c\}$ and $\sigma_2 := \{\emptyset, \{a\}, Y_2\}$. Then, we have that $G_{(Y_1,\sigma_1)} \cong G_{(Y_2,\sigma_2)}$, $\gamma r\text{-}h(Y_1,\sigma_1) \ncong \gamma r\text{-}h(Y_2,\sigma_2)$ and $h(Y_1,\sigma_1) \ncong h(Y_2,\sigma_2)$ hold. We can apply Corollary 3.6(ii) for this example (iii).

(v) For the digital line $(\mathbb{Z},\kappa)$, we have an example of a subgroup of $H_{(\mathbb{Z},\kappa)}$ (cf. Example 5.10(iv)).

4 Two subgroups of γr-$h(X;\tau)$ and their properties The purpose of the present section is to prove Theorem 4.9.

Definition 4.1 For a subset G of X, we define the following families of mappings:

(i) $\gamma r\text{-}h(X,G;\tau) := \{a|\ a \in \gamma r\text{-}h(X;\tau) \text{ and } a(G) = G\}$;

(ii) $\gamma r\text{-}h_0(X,G;\tau) := \{a|\ a \in \gamma r\text{-}h(X;\tau) \text{ and } a(x) = x \text{ for every point } x \in G\}$.

Theorem 4.2 *Let H be a subset of a topological space (X,τ). The families $\gamma r\text{-}h(X,X\setminus H;\tau)$ and $\gamma r\text{-}h_0(X,X\setminus H;\tau)$ form two subgroups of $\gamma r\text{-}h(X,\tau)$ and $\gamma r\text{-}h(X,X\setminus H;\tau) = \gamma r\text{-}h(X,H;\tau)$ holds.* □

For the group $\gamma r\text{-}h(X,X\setminus H;\tau)$, say A, (resp. $\gamma r\text{-}h_0(X,X\setminus H;\tau)$, say A_0,) of Theorem 4.2, we define the binary operation $\omega_{X,H} : A\times A \to A$ (resp. $\omega_{X,H_0} : A_0\times A_0 \to A_0$) by $\omega_{X,H}(a,b) := (\omega_{G(X,\tau)}|A\times A)(a,b) = b\circ a$ (resp. $\omega_{X,H_0}(a,b) := (\omega_{G(X,\tau)}|A_0\times A_0)(a,b) = b\circ a$) (cf. a few lines after Theorem 3.4).

In order to investigate precisely some structures of $\gamma r\text{-}h(H,X\setminus H;\tau|H)$ (cf. Theorem 4.9), we need the following definitions and properties.

Definition 4.3 Let H, K be subsets of X and Y, respectively. For a mapping $f : X \to Y$ satisfying a property $K = f(H)$, we define the following mapping $r_{H,K}(f) : H \to K$ by $r_{H,K}(f)(x) = f(x)$ for every $x \in H$.

Then, we have the following properties:

(4.a) $j_K \circ r_{H,K}(f) = f|H : H \to Y$, where $j_K : K \to Y$ be the inclusion defined by $j_K(y) = y$ for every $y \in K$ and $f|H : H \to Y$ is the restriction of f to H defined by $(f|H)(x) = f(x)$ for every $x \in H$.

(4.b) Especially, we consider the following case where $X = Y, H = K \subseteq X$. If $a(H) = H$ and $b(H) = H$, then $r_{H,H}(b\circ a) = r_{H,H}(b)\circ r_{H,H}(a)$ holds, where $a, b : X \to X$ are mappings.

(4.c) If a mapping $a : X \to X$ is a bijection such that $a(H) = H$, then $r_{H,H}(a) : H \to H$ is bijective and $r_{H,H}(a^{-1}) = (r_{H,H}(a))^{-1}$.

In Theorem 4.4 below, we recall well known properties on γ-open sets and they are needed later. For a subset H of (X,τ) and a subset $U \subseteq H$, $Int_H(U)$ (resp. $Cl_H(U)$) is the interior (resp. closure) of the set U in a subspace $(H,\tau|H)$. The γ-interior of a subset A of (X,τ) is defined by

· $\gamma Int(A) := \bigcup\{V|V \subseteq A, V \in \gamma O(X,\tau)\}$. It is well known that: for a set $A \subseteq X$,

· ([6, Proposition 2.5]) $\gamma Int(A) = A \cap (Int(Cl(A)) \cup Cl(Int(A)))$ and

· $\gamma Cl(A) = A \cup (Int(Cl(A)) \cap Cl(Int(A)))$ hold (e.g., [19, Lemma 2.6(iii)], [3, Lemma 3.2]).

And, by [6, Proposition 2.3(a)] (cf. Theorem 4.4(iii)), it is shown that

· $\gamma Cl(A) \in \gamma C(X,\tau)$ and $\gamma Int(A) \in \gamma O(X,\tau)$, where A is a subset of (X,τ).

· $\gamma O(H,\tau|H) := \{U \subseteq H|\ U \text{ is } \gamma\text{-open in } (H,\tau|H)\}$;

· $\gamma C(H,\tau|H) := \{F \subseteq H|\ F \text{ is } \gamma\text{-closed in } (H,\tau|H)\}$;

· $\gamma Cl_H(U) := \bigcap\{F|\ U \subseteq F, F \in \gamma C(H,\tau|H)\}$, where $U \subseteq H \subseteq X$.

Theorem 4.4 (i) ([15],e.g.,[14, Lemma 2.2];[1, Proof of Theorem 2.3(3)]). *Let $H \subseteq X$ and $A_1 \subseteq X$. If H is α-open in (X,τ) and A_1 is γ-open in (X,τ), then $A_1 \cap H$ is γ-open in $(H,\tau|H)$.*

(ii) ([15];e.g.,[14, Lemma 2.4]) *Let $A \subseteq H \subseteq X$. If A is γ-open in $(H,\tau|H)$ and H is α-open in (X,τ), then A is γ-open in (X,τ).*

(iii) ([6, Proposition 2.3(a)]) *Arbitrary union of γ-open sets of (X,τ) is γ-open in (X,τ).*

(iv) ([6, Proposition 2.4(2)]) *Let $H \subseteq X$ and $A_1 \subseteq X$. If H is α-open in (X,τ) and A_1 is γ-open in (X,τ), then $A_1 \cap H$ is γ-open in (X,τ).*

(v) *If $B \subseteq H \subseteq X$ and H is α-open in (X,τ), then $\gamma Cl(B) \cap H = \gamma Cl_H(B)$ holds.*

(vi) *Let $F \subseteq H \subseteq X$. If H is α-open and γ-closed in (X,τ) and F is γ-closed in $(H,\tau|H)$, then F is γ-closed in (X,τ).* □

Remark 4.5 It follows from the following example that one of the assumptions of Theorem 4.4(vi) is not removed. Let $X := \{a,b,c\}$ and $\tau := \{\emptyset,\{a\},X\}$ (cf. the space (Y_2,σ_2) of Example 3.7(iv)). For a subset $H := \{a,c\}$, the set H is γ-closed in $(H,\tau|H)$ and it is α-open in (X,τ), but H is not γ-closed in (X,τ).

Proposition 4.6 (i) *If $f : (X,\tau) \to (Y,\sigma)$ is γ-irresolute and a subset H is α-open in (X,τ), then $f|H : (H,\tau|H) \to (Y,\sigma)$ is γ-irresolute.*

(ii) *Let $k : (X,\tau) \to (K,\sigma|K)$ be a mapping and $j_K : (K,\sigma|K) \to (Y,\sigma)$ be the inclusion, where $K \subseteq Y$. Then, the following properties* (1), (2) *are equivalent, under the assumption that K is α-open in (Y,σ):*
(1) *$k : (X,\tau) \to (K,\sigma|K)$ is γ-irresolute;*
(2) *$j_K \circ k : (X,\tau) \to (Y,\sigma)$ is γ-irresolute.*

(iii) *If $f : (X,\tau) \to (Y,\sigma)$ is γ-irresolute, H is α-open in (X,τ) and $f(H)$ is α-open in (Y,σ), then $r_{H,f(H)}(f) : (H,\tau|H) \to (f(H),\sigma|f(H))$ is γ-irresolute* (cf. Definition 4.3).

Proof. The properties (i) and (ii)(1)⇒(2) (resp. (ii)(2)⇒(1)) are proved by using Theorem 4.4(i) (resp. Theorem 4.4(ii)). The property (iii) is proved by (i),(ii) above and (4.a) after Definition 4.3. □

Definition 4.7 For an α-open subset H of (X,τ), the following mappings $(r_H)_* : \gamma r\text{-}h(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ and $(r_H)_{*,0} : \gamma r\text{-}h_0(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ are well defined as follows (cf. Proposition 4.6(iii)), respectively:

$(r_H)_*(f) := r_{H,H}(f)$ for every $f \in \gamma r\text{-}h(X,X\setminus H;\tau)$;

$(r_H)_{*,0}(g) := r_{H,H}(g)$ for every $g \in \gamma r\text{-}h_0(X,X\setminus H;\tau)$.

Lemma 4.8 (A pasting lemma for γ-irresolute mappings) *Let $X = U_1 \cup U_2$, where U_1 and U_2 are α-open sets in (X,τ), and $f_1 : (U_1,\tau|U_1) \to (Y,\sigma)$ and $f_2 : (U_2,\tau|U_2) \to (Y,\sigma)$ are γ-irresolute mappings such that $f_1(x) = f_2(x)$ for every point $x \in U_1 \cap U_2$. Then its combination $f_1\nabla f_2 : (X,\tau) \to (Y,\sigma)$ is γ-irresolute, where $(f_1\nabla f_2)(x) := f_j(x)$ for every $x \in U_j (j \in \{1,2\})$.*

Proof. Let $V \in \gamma O(Y,\sigma)$. By Theorem 4.4 (ii) and (iii), it is proved that $(f_1\nabla f_2)^{-1}(V) \in \gamma O(X,\tau)$, because $f_i^{-1}(V) \in \gamma O(U_i,\tau|U_i), f_i^{-1}(V) \in$-
$\gamma O(X,\tau)$ for each $i \in \{1,2\}$ and $(f_1\nabla f_2)^{-1}(V) = f_1^{-1}(V) \cup f_2^{-1}(V)$ hold. □

Theorem 4.9 *Let H be a subset of a topological space (X,τ).*

(i) (i-1) *If H is α-open in (X,τ), then the mappings $(r_H)_* : \gamma r\text{-}h(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ and $(r_H)_{*,0} : \gamma r\text{-}h_0(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ are homomorphisms of groups* (cf. Definition 4.7). *Moreover, $(r_H)_*|B_0 = (r_H)_{*,0}$ holds, where $B_0 := \gamma r\text{-}h_0(X,X\setminus H;\tau)$.*

(i-2) *If H is α-open and α-closed in (X,τ), then the mappings $(r_H)_* : \gamma r\text{-}h(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ and $(r_H)_{*,0} : \gamma r\text{-}h_0(X,X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ are onto homomorphisms of groups.*

(ii) *For an α-open subset H of (X,τ), we have the following isomorphisms of groups:*

(ii-1) $\gamma r\text{-}h(X, X\setminus H;\tau)/Ker(r_H)_* \cong Im(r_H)_*$;

(ii-2) $\gamma r\text{-}h_0(X, X\setminus H;\tau) \cong Im(r_H)_{*,0}$, *where* $Ker(r_H)_*{:=}\ \{a \in \gamma r\text{-}h(X, X\setminus H;\tau)|\ (r_H)_*(a) = 1_X\}$ *is a normal subgroup of* $\gamma r\text{-}h(X, X\setminus H;\tau)$; $Im(r_H)_*{:=}\{(r_H)_*(a)|\ a \in \gamma r\text{-}h(X, X\setminus H;\tau)\}$ *and* $Im(r_H)_{*,0}{:=}\{(r_H)_{*,0}(b)|\ b \in \gamma r\text{-}h_0(X, X\setminus H;\tau)\}$ *are subgroups of* $\gamma r\text{-}h(H;\tau)$.

(iii) *For an α-open and α-closed subset H of (X,τ), we have the following isomorphisms of groups:*

(iii-1) $\gamma r\text{-}h(H;\tau|H) \cong \gamma r\text{-}h(X, X\setminus H;\tau)/Ker(r_H)_*$;

(iii-2) $\gamma r\text{-}h(H;\tau|H) \cong \gamma r\text{-}h_0(X, X\setminus H;\tau)$.

Proof. (i) (i-1) Since H is α-open in (X,τ), the mappings $(r_H)_*$ and $(r_H)_{*,0}$ are well defined (cf. Definition 4.7). Let $a, b \in \gamma r\text{-}h(X, X\setminus H;\tau)$ and $\omega_{X,H} : \gamma r\text{-}h(X, X\setminus H;\tau)\times \gamma r\text{-}h(X, X\setminus H;\tau)) \to \gamma r\text{-}h(X, X\setminus H;\tau)$ be the binary operation of the group $\gamma r\text{-}h(X, X\setminus H;\tau)$ (cf. a few lines after Theorem 4.2). Then, $(r_H)_*(\omega_{X,H}(a, b)) = (r_H)_*(b\circ a) = r_{H,H}(b\circ a) = (r_{H,H}(b))\circ(r_{H,H}(a)) = \omega_H((r_H)_*(a),(r_H)_*(b))$ hold, where ω_H is the binary operation of the group $\gamma r\text{-}h(H;\tau|H)$ (cf. a few lines after Theorem 3.4). Thus, $(r_H)_*$ is a homomorphism of group. Similarly, the mapping $(r_H)_{*,0} : \gamma r\text{-}h_0(X, X\setminus H;\tau) \to \gamma r\text{-}h(H;\tau|H)$ is also a homomorphism of groups. It is obviously shown that $(r_H)_*|\gamma r\text{-}h_0(X, X\setminus H;\tau) = (r_H)_{*,0}$ holds (cf. Definition 4.1, Definition 4.7).

(i-2) Let $h \in \gamma r\text{-}h(H;\tau|H)$. We consider the combination $h_1 := (j_H\circ h)\nabla(j_{X\setminus H}\circ 1_{X\setminus H}) : (X,\tau)\to(X,\tau)$. By Proposition 4.6 (ii) and the assumption of α-openness of H, it is shown that the two mappings $j_H\circ h : (H,\tau|H) \to (X,\tau)$ and $j_H\circ h^{-1} : (H,\tau|H) \to (X,\tau)$ are γ-irresolute. Moreover, under the assumption of α-openness of $X\setminus H$, $j_{X\setminus H}\circ 1_{X\setminus H} : (X\setminus H,\tau|(X\setminus H)) \to (X,\tau)$ is γ-irresolute. By using Lemma 4.8 for an α-open cover $\{H, X\setminus H\}$ of X, the combination above $h_1 : (X,\tau)\to(X,\tau)$ is γ-irresolute and h_1 is bijective and its inverse mapping $h_1^{-1} = (j_H\circ h^{-1})\nabla(j_{X\setminus H}\circ 1_{X\setminus H})$ is also γ-irresolute. Thus, we have that $h_1 \in \gamma r\text{-}h(X,\tau)$. Since $h_1(x) = x$ for every point $x\in X\setminus H$, we conclude that $h_1 \in \gamma r\text{-}h_0(X, X\setminus H;\tau)$ and so $h_1 \in \gamma r\text{-}h(X, X\setminus H;\tau)$; moreover, $(r_H)_{*,0}(h_1) = (r_H)_*(h_1) = r_{H,H}(h_1) = h$.

(ii) By (i-1) above and the first isomorphism theorem of group theory, it is shown that there are group isomorphisms below, under the assumption of the α-openness of H in (X,τ):
(4.d) $\gamma r\text{-}h(X, X\setminus H;\tau)/Ker(r_H)_* \cong Im(r_H)_*$ and
(4.e) $\gamma r\text{-}h_0(X, X\setminus H;\tau)/Ker(r_H)_{*,0} \cong Im(r_H)_{*,0}$, where $Ker(r_H)_{*,0} := \{a \in \gamma r\text{-}h_0(X, X\setminus H;\tau)|\ (r_H)_{*,0}(a) = 1_X\}$.

It is shown that $Ker(r_H)_{*,0} = \{1_X\}$. Indeed, let $u_0 \in Ker(r_H)_{*,0} \subset \gamma r\text{-}h_0(X, X\setminus H;\tau)$; then $(r_H)_{*,0}(u_0) = 1_H$, where 1_H is the identity element of $\gamma r\text{-}h(H;\tau|H)$. By Definitions 4.7 and 4.3, we have that, for any point $x \in H$, $((r_H)_{*,0}(u_0))(x) = (r_{H,H}(u_0))(x) = u_0(x)$ and so, $u_0(x) = 1_H(x)$; and, for any point $x\in X\setminus H$, $u_0(x) = x$ (cf. Definition 4.1(ii)). Thus, we conclude that $u_0 = 1_X$; and hence $Ker(r_H)_{*,0} = \{1_X\}$. Therefore, by using the isomorphism (4.e) above, we have the isomorphism (ii-2).

(iii) By (i-2) and (ii), the isomorphisms (iii-1) and (iii-2) are obtained. □

Example 4.10 (i) In Example 5.13 of Section 5, the groups in Theorem 4.9 above are given for a special subspace $(H,\kappa|H)$ of the digital line $(\mathbb{Z},\kappa)$.

(ii) Let (X,τ) be the topological space of Example 3.7(ii) throughout the present Example 4.10(ii).

(ii-1) Let $H := \{a\}$. Since $H = \{a\}$ is α-open and α-closed in the topologica space (X,τ), then we apply Theorem 4.9(iii) to the present case; and so, we have the following result:
$\gamma r\text{-}h(H;\tau|H) \cong \gamma r\text{-}h(X, X\setminus H;\tau)/Ker(r_H)_* \cong \gamma r\text{-}h_0(X, X\setminus H;\tau)$.
We can check directly the group isomorphisms as follows: we have the date: $\gamma r\text{-}h(X, X\setminus H;\tau) = \{1_X, h_a\}$, $Ker(r_H)_* = \{1_X, h_a\}$, $\gamma O(H,\tau|H) = \{\emptyset, H\}$, $\gamma r\text{-}h(H;\tau|H) = \{1_H\}$ and $\gamma r\text{-}h_0(X, X\setminus H;\tau) = \{1_X\}$, where $\tau|H = \{\emptyset, H\}$.

(ii-2) Let $H := \{b, c\}$. Then H is α-open and α-closed in (X,τ). Now, we apply Theorem 4.9

(iii) to the present case; and we can also check directly the group isomorphisms: we have the date as follows: $\gamma r\text{-}h(X, X \setminus H; \tau) = \{1_X, h_a\}$, $Ker(r_H)_* = \{1_X\}$, $\gamma O(H, \tau|H) = P(H)$, $\gamma r\text{-}h(H; \tau|H) = \{1_H, h_a|H\}$ and $\gamma r\text{-}h_0(X, X \setminus H; \tau) = \{1_X, h_a\}$, where $\tau|H = \{\emptyset, H\}$.

Example 4.11 Even if a subset H of a topological space (X, τ) is not α-closed and it is α-open (cf. Theorem 4.9(i)(i-2)), we have some examples such that the homomorphisms $(r_H)_*$ and $(r_H)_{*,0}$ are onto.

(i) For example, let (X, τ) be a topological space and $(H, \tau|H)$ a subspace of (X, τ), where $X := \{a, b, c\}$, $H =: \{a, b\}$ and $\tau := \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$; and so, $\tau|H = \{\emptyset, \{a\}, \{b\}, H\}$. Then, we see that $\gamma O(X, \tau) = P(X) \setminus \{\{c\}\}$ and $\tau^\alpha = \tau$. The subset H is α-open and it is not α-closed in (X, τ). Hence by Theorem 4.9(i)(i-1), the mappings $(r_H)_*$ and $(r_H)_{*,0}$ are homomorphisms of groups. Because of $X \setminus H = \{c\}$, we see that $\gamma r\text{-}h_0(X, X \setminus H; \tau) = \gamma r\text{-}h(X, X \setminus H; \tau)$ and $(r_{H,0})_* = (r_H)_*$. And it is shown directly that $\gamma r\text{-}h(X, X \setminus H; \tau) = \{1_X, h_c\} \cong \mathbb{Z}_2, (h_c)^2 = 1_X$, and $\gamma r\text{-}h(H; \tau|H) = \{1_H, t_{a,b}\}$, where $h_c : (X, \tau) \to (X, \tau)$ and $t_{a,b} : (H, \tau|H) \to (H, \tau|H)$ are the bijections defined by $h_c(a) = b, h_c(b) = a, h_c(c) = c$ and $t_{a,b}(a) = b, t_{a,b}(b) = a$, respectively. Then, we prove that : $(r_H)_* : \gamma r\text{-}h(X, X \setminus H; \kappa) \to \gamma r\text{-}h(H; \tau|H)$ is onto; $Ker(r_H)_* = \{1_X\}$. By using Theorem 4.9(i)(i-1) and (ii), we have that $\gamma r\text{-}h(H; \tau|H) \cong \gamma r\text{-}h(X, X \setminus H; \tau)/Ker(r_H)_* = \gamma r\text{-}h(X, X \setminus H; \tau)$ hold.

(ii) In Section 5, we give an example of an onto homomorphism $(r_H)_*$, where $H := \{-1, 0, 1\}$ of the digital line $(\mathbb{Z}, \kappa)$ (cf. Example 5.13(iv)).

5 Examples on the digital line $(\mathbb{Z}, \kappa)$ We recall that *the digital line* is the set of the integers, $\mathbb{Z}$, equipped with the topology κ having $\{\{2s-1, 2s, 2s+1\}|\ s \in \mathbb{Z}\}$, say $\mathbf{G}$, as a subbase (e.g., [24, p.175], [26, Section 3(I)], [23, p.905,p.908]). This topological space is denoted by $(\mathbb{Z}, \kappa)$. By the definition of topology κ, every singleton $\{2u+1\}$ is open in $(\mathbb{Z}, \kappa)$ and it is not closed in $(\mathbb{Z}, \kappa)$, where $u \in \mathbb{Z}$. Every singleton $\{2s\}$ is closed in $(\mathbb{Z}, \kappa)$ and it is not open in $(\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$. In the present paper, we denote: $U(2s) := \{2s-1, 2s, 2s+1\}$ and $U(2u+1) := \{2u+1\}$ for each point $2s$ and $2u+1$ of $(\mathbb{Z}, \kappa)$, respectively; and $U(2s)$ and $U(2u+1)$ are two typical open sets of $(\mathbb{Z}, \kappa)$. And, $U(x)$ above is called *the smallest open set containing the point* x of $(\mathbb{Z}, \kappa)$, where $x \in \mathbb{Z}$. It is well known that: for a nonempty open set U and a point x of $(\mathbb{Z}, \kappa)$, if $x \in U$, then $U(x) \subseteq U$ holds (e.g., [26, Section 3]).

(I) Characterizations of γ-open sets in the digital line $(\mathbb{Z}, \kappa)$ (cf. Theorems 5.1 and 5.5 below). First, we recall some properties on the digital line $(\mathbb{Z}, \kappa)$: $\kappa = PO(\mathbb{Z}, \kappa)$ and $PO(\mathbb{Z}, \kappa) \subseteq SO(\mathbb{Z}, \kappa) = \gamma O(\mathbb{Z}, \kappa) = \beta O(\mathbb{Z}, \kappa)$ (cf. [9], [17], [33]). Secondly, we need some notations and properties (e.g., [18, Sections 1, 2], [26, Sections 2, 3]): let A be a nonempty subset of $(\mathbb{Z}, \kappa)$, $A_\kappa := \{x \in A|\ \{x\}$ is open in $(\mathbb{Z}, \kappa)\}$; $A_{\mathbf{F}} := \{x \in A|\ \{x\}$ is closed in $(\mathbb{Z}, \kappa)\}$. It is easily shown that:

(i) $A_\kappa = \{2s+1 \in A|\ s \in \mathbb{Z}\}$; $A_{\mathbf{F}} = \{2m \in A|\ m \in \mathbb{Z}\}$; and

(ii) $A = A_\kappa \cup A_{\mathbf{F}}$ ($A_\kappa \cap A_{\mathbf{F}} = \emptyset$), where A is any subset of $(\mathbb{Z}, \kappa)$.

By Takigawa [32, Theorems 1, 2 and 3], some characterizations of any preopen sets, semi-open sets and semi-preopen sets in the digital n-space $(\mathbb{Z}^n, \kappa^n)$ are investigated, where $n \geq 1$. The following property is obtained by a special case of [32, Theorem 2 or Theorem 3] for the digital line (i.e., $n = 1$).

Theorem 5.1 (A special case of Takigawa [32, Theorem 2 or Theorem 3]) *A subset E is γ-open in $(\mathbb{Z}, \kappa)$ if and only if $E \subseteq Cl(E_\kappa)$ holds in $(\mathbb{Z}, \kappa)$.*

Remark 5.2 (i) If $A_\kappa = \emptyset$ for a subset A of $(\mathbb{Z}, \kappa)$, then A is closed in $(\mathbb{Z}, \kappa)$. The converse of above implication is not true; a subset $\{2s, 2s+1, 2s+2\}$ is closed in $(\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$; and $(\{2s, 2s+1, 2s+2\})_\kappa = \{2s+1\} \neq \emptyset$.

(ii) $Cl(A) = Cl(A_\kappa) \cup A$ holds for a subset A of $(\mathbb{Z}, \kappa)$.

Definition 5.3 ([7, Definition 5.3]) Let A be a subset of $(\mathbb{Z}, \kappa)$.

(i) For a point $x \in \mathbb{Z}$, the following set $V_A(x)$ is defined: if $x+1 \in A$, then $V_A(x) := \{x, x+1\}$ (sometimes it is denoted by ${V_A}^+(x)$, or shortly $V^+(x)$); if $x+1 \notin A$, then $V_A(x) := \{x-1, x\}$ (sometimes it is denoted by ${V_A}^-(x)$, or shortly $V^-(x)$). Thus, we have that $V_A(x) = {V_A}^+(x)$ or ${V_A}^-(x)$.

(ii) $V_A := \bigcup\{V_A(x) |\ x \in A_{\mathbf{F}}\}$ if $A_{\mathbf{F}} \neq \emptyset$; $V_A := \emptyset$ if $A_{\mathbf{F}} = \emptyset$.

Example 5.4 (i) A subset $\{x, x+1\}$ of $\mathbb{Z}$ is γ-open and γ-closed in $(\mathbb{Z}, \kappa)$ for any point $x \in \mathbb{Z}$.

(ii) (cf. [7, Example 5.5]) For a point $x \in \mathbb{Z}$ and a subset $A \subseteq \mathbb{Z}$, the set $V_A(x)$ is both γ-open and γ-closed in $(\mathbb{Z}, \kappa)$ (cf. Definition 5.3).

Finally, the following characterization (Theorem 5.5) is obtained by using the equality $\gamma O(\mathbb{Z}, \kappa) = \beta O(\mathbb{Z}, \kappa)$ and [7, Theorem 5.7]. We note that we are able to have directly an alternative proof of Theorem 5.5 using the characterization of Theorem 5.1 above.

Theorem 5.5 ([7, Theorem 5.7]) *Let B be a nonempty subset of $(\mathbb{Z}, \kappa)$. Then the following statements hold.*

(i) *Assume that $B_{\mathbf{F}} \neq \emptyset$.*

(i-1) *If B is γ-open in $(\mathbb{Z}, \kappa)$, then B is expressible as the union: $B = V_B \cup B_\kappa$, where* $V_B := \bigcup\{V_B(x) | x \in B_{\mathbf{F}}\}$ (cf. Definiton 5.3).

(i-2) *If B satisfies a property that $B = V_B \cup B_\kappa$, then B is γ-open in $(\mathbb{Z}, \kappa)$.*

(ii) *Assume that $B_{\mathbf{F}} = \emptyset$. Then, $V_B = \emptyset$ and $B = B_\kappa$ hold and B is open in $(\mathbb{Z}, \kappa)$; and so B is γ-open in $(\mathbb{Z}, \kappa)$.* □

Example 5.6 Suppose that a singleton $\{x\}$ is closed in $(\mathbb{Z}, \kappa)$ (i.e., x is even in $\mathbb{Z}$) and y is any point with $y \neq x$. Then,
(i) $\{x, y\}$ is γ-closed in $(\mathbb{Z}, \kappa)$;
(ii) $\{x, y\}$ is γ-open if and only if $y = x+1$ or $y = x-1$.

(II) Some transformations on $(\mathbb{Z}, \kappa)$.

Definition 5.7 Let $t_{e+,o-} : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa), t_- : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$ and $f_s : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$, where $s \in \mathbb{Z}$, be the transformations defined by the following form, respectively: for every point $x \in \mathbb{Z}$,
(i) $t_{e+,o-}(x) := x+1$ if x is even and $t_{e+,o-}(x) := x-1$ if x is odd;
(ii) $t_-(x) := -x$; (iii) $f_s(x) := x+s$.

Theorem 5.8 *For any γ-open set A of $(\mathbb{Z}, \kappa)$, we have the following properties:*

(i) ${t_{e+,o-}}^{-1}(A)$ *is explessible as the union of arbitrary collection of γ-closed sets of $(\mathbb{Z}, \kappa)$;*

(ii) ${t_-}^{-1}(A)$ *is explessible as the union of arbitrary collection of γ-closed sets of $(\mathbb{Z}, \kappa)$;*

(iii)([7, Lemma 5.8(vii), Theorem 5.10(iii)]) ${f_{2m+1}}^{-1}(A)$ *and* $f_{2m+1}(A)$ *are explessible as the union of arbitrary collection of γ-closed sets of $(\mathbb{Z}, \kappa)$, where $m \in \mathbb{Z}$.*

Proof. (i) By using Definition 5.3, Example 5.6(i) and Definition 5.7, it is shown that, for any set B and any point $x \in \mathbb{Z}$, ${t_{e+,o-}}^{-1}(V_B(x))$ is γ-closed in $(\mathbb{Z}, \kappa)$ (cf. Definition 5.3(i), Example 5.6(i), Definition 5.7); ${t_{e+,o-}}^{-1}(B_\kappa) = \bigcup\{\{2s\} |\ 2s+1 \in B\}$ holds, because $B_\kappa = \bigcup\{\{2s+1\} |\ 2s+1 \in B\})$. And, so ${t_{e+,o-}}^{-1}(B_\kappa)$ is the union of the collection $\{\{2s\} |\ 2s+1 \in B\}$ of γ-closed sets. Let $A \in \gamma O(\mathbb{Z}, \kappa)$. By Theorem 5.5(i-1) and (ii), it is shown that ${t_{e+,o-}}^{-1}(A) = (\bigcup\{{t_{e+,o-}}^{-1}(V_A(x)) |\ x \in A_{\mathbf{F}}\}) \cup {t_{e+,o-}}^{-1}(A_\kappa)$ (if $A_{\mathbf{F}} \neq \emptyset$) and ${t_{e+,o-}}^{-1}(A) = {t_{e+,o-}}^{-1}(A_\kappa)$ (if $A_{\mathbf{F}} = \emptyset$); and so, by the properties above respectively, ${t_{e+,o-}}^{-1}(A)$ is the union of a collection of γ-closed sets.

(ii) By an argument similar to that in (i), the statement (ii) is proved (cf. Definition 5.3, Example 5.4).

(iii) This is shown by the property that $\gamma O(\mathbb{Z}, \kappa) = \beta O(\mathbb{Z}, \kappa)$ (cf. (I) above) and the corresponding property on β-openess version [7, Lemma 5.8(vii), Theorem 5.10(iii)]. □

Remark 5.9 Let $A_{2k} := \{2k, 2k+1\} \cup \{2(k+1)+1, 2(k+1)+2\}$. Since $Int(Cl(A_{2k})) \cap Cl(Int(A_{2k})) = \{2k+1, 2(k+1), 2(k+1)+1\} \not\subseteq A_{2k}$ hold, A_{2k} is not γ-closed. But, A_{2k} is the union of two γ-closed sets $\{2k, 2k+1\}$ and $\{2(k+1)+1, 2(k+2)\}$ of $(\mathbb{Z}, \kappa)$ (cf. Example 5.4 (i)).

Example 5.10 (i) $t_{e+,o-} \notin \gamma r\text{-}h(\mathbb{Z};\kappa)$ and $t_{e+,o-} \notin contra\text{-}\gamma r\text{-}h(\mathbb{Z};\kappa)$ hold.

(ii) $t_- \in h(\mathbb{Z},\kappa)$ holds and so $t_- \in \gamma r\text{-}h(\mathbb{Z};\kappa)$.

(iii) (iii-1) $f_{2m+1} \notin \gamma r\text{-}h(\mathbb{Z};\kappa)$ and $f_{2m+1} \notin contra\text{-}\gamma r\text{-}h(\mathbb{Z};\kappa)$;
(iii-2) $f_{2m+1} \notin h(\mathbb{Z};\kappa)$.

(iv) $f_{2m} \in h(\mathbb{Z};\kappa)$ and $f_{2m+1} \in contra\text{-}h(\mathbb{Z};\kappa)$ hold; and hence $\{f_s | s \in \mathbb{Z}\}$ forms a subgroup of $H_{(\mathbb{Z},\kappa)}$.

(III) A group structure of γr-$h(H;\kappa|H)$, where $H := \{-1, 0, 1\}$.

Lemma 5.11 *Let $s, u \in \mathbb{Z}$. If $f : (\mathbb{Z},\kappa) \to (\mathbb{Z},\kappa)$ is a γr-homeomorphism* (i.e., $f \in \gamma r$-$h(\mathbb{Z},\kappa)$), *then*

(i) *$f(U(2s)) = U(2a)$ holds for some point $2a \in \mathbb{Z}$;*

(ii) *$f(U(2u+1)) = U(2v+1)$ holds for some point $2v+1 \in \mathbb{Z}$.* □

Notation Let H be the smallest open set containing 0, $U(0) := \{-1, 0, +1\}$, which is used in Example 5.13 below. A family of subsets of $(\mathbb{Z},\kappa)$, say $\{H_j | j \in \mathbb{Z}$ with $j \geq 1\}$, is defined by : $H_1 := H = U(0)$ and $H_i := U(-(2i-2)) \cup H_{i-1} \cup U(2i-2)$ for each integer $i \geq 2$, where $U(2s) := \{2s-1, 2s, 2s+1\} (s \in \mathbb{Z})$.

It is easily shown that $H_i = \bigcup\{U(-(2j-2)) \cup U(2j-2) |\ j \in \mathbb{Z}$ with $1 \leq j \leq i\}$ holds for each integer $i \geq 2$; and if $i \leq j$, then $H_i \subseteq H_j$ and $\bigcup\{H_j |\ j \in \mathbb{Z}$ with $j \geq 1\} = \mathbb{Z}$.

Lemma 5.12 below is proved by an argument similar to that in [30, Claim in Proof of Proposition 6.1]; we use induction on $m \in \mathbb{Z}$ and Lemma 5.11; and so we omite the proof.

Lemma 5.12 *Let $f \in \gamma r$-$h(\mathbb{Z}, \mathbb{Z} \setminus H;\kappa)$ and $\{H_j | j \in \mathbb{Z}$ with $j \geq 1\}$ be the family of subsets defined by Notation above, where $H = H_1 = \{-1, 0, 1\}$,* i.e., $H = U(0)$.

(i) *If $f|H = t_-|H$, then $f|H_m = t_-|H_m$ for any interger m with $m \geq 2$.*

(ii) *If $f|H = 1_H$, then $f|H_m = 1_{H_m}$ for any integer m with $m \geq 2$.* □

Using Lemma 5.11 and Lemma 5.12, we can examine the isomorphisms of Theorem 4.9(ii) for the following α-open set $H := U(0)$ which is not α-closed in $(\mathbb{Z},\kappa)$.

Example 5.13 Let $(H, \kappa|H)$ be a subspace of $(\mathbb{Z},\kappa)$, where $H := \{-1, 0, +1\}$ is the smallest open set containing $0 \in \mathbb{Z}$, i.e., $H = U(0)$. Then, we have the following properties: (i) γr-$h(\mathbb{Z}, \mathbb{Z} \setminus H;\kappa) = \{1_{\mathbb{Z}}, t_-\}$; (ii) γr-$h_0(\mathbb{Z}, \mathbb{Z} \setminus H;\kappa) = \{1_{\mathbb{Z}}\}$; (iii) γr-$h(H;\kappa|H) = \{1_H, t_-|H\}$; (iv) $Im(r_H)_* = \{1_H, t_-|H\}$ and $(r_H)_* : \gamma r$-$h(\mathbb{Z}, \mathbb{Z} \setminus H;\kappa) \to \gamma r$-$h(H, \kappa|H)$ is onto; (v) $Ker(r_H)_* = \{1_{\mathbb{Z}}\}$.

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