Scientiae Mathematicae Japonicae

ON DUCCI MATRIX SEQUENCES

Takanori Hida

Received December 1, 2015

Abstract. For each irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$, there is a unique Ducci matrix sequence $M_{j_\alpha(1)}, M_{j_\alpha(2)}, \ldots$ associated with it. We first consider the function j that maps each $\alpha \in (0,1) \setminus \mathbb{Q}$ to the sequence $j(\alpha) := \langle j_\alpha(1), j_\alpha(2), \ldots \rangle$ of indexes of its Ducci matrix sequence expansion. While continuity of j and j^{-1} is easily checked, we show that j^{-1} is moreover uniformly continuous. We then study the distribution of Ducci matrices in the Ducci matrix sequence expansion of a given irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$ by considering the following three conditions on the sequence $j(\alpha)$:

$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$$
$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6} \text{ for every } \jmath \in \{1,2,\ldots,6\};$$
$$\lim_{n\to\infty} \sqrt[p]{\frac{\Sigma_{i=1}^n j_\alpha(i)^p}{n}} = \sqrt[p]{\frac{1^p + 2^p + \cdots + 6^p}{6}}.$$

We prove that the top implies the middle and the middle implies the bottom. We also give examples witnessing that the converse to these two implications are not true in general. In addition, various equivalent statements to the first condition will be presented. Furthermore, we shall give measure theoretic treatment of the subject: We prove that for almost every α, each Ducci matrix appears in the Ducci matrix sequence expansion of α infinitely often. We then ask if the second (and also the third) condition above holds almost everywhere. Related questions as well as several partial results will be presented.

1 Introduction.

A Ducci sequence is a sequence of vectors generated by iterating the following *Ducci map* D to a starting vector:

$$(v_1, v_2, \ldots, v_n) \stackrel{D}{\longmapsto} (|v_1 - v_2|, |v_2 - v_3|, \ldots, |v_n - v_1|)$$

Ciamberlini and Marengoni attributed a question about the limiting behavior of such sequences to E. Ducci in their paper [4]. Since then, a substantial amount of literature on various generalizations as well as the dynamics of the Ducci map has appeared ([2] provides a large list of references.)

Due to the simplicity of the definition, one can consider the Ducci map on various domains. While more works can be found on the Ducci map on $\mathbb{Z}^n$, there are several important results in the real setting, i.e. $\mathbb{R}^n$. For $n = 4$, though every vector in $\mathbb{Z}^4$ is known to converge to the zero vector in finite time [1, 4], Lotan [8] constructed vectors in $\mathbb{R}^4$ whose Ducci sequence never reach the zero vector. However, not many vectors exhibit such asymptotic behavior — A vector does not reach the zero vector if and only if it reaches a trivial transformation of the vector $(1, q, q^2, q^3)$ after finite time, where $1 < q < 2$ is the unique positive solution of the equation $x^3 - x^2 - x - 1 = 0$ [8]. For $n = 3$, Brockman and Zerr [2] proved that if a starting vector $\boldsymbol{v}$ is *heterogeneous*, i.e. $\lambda(\boldsymbol{v} + (x,x,x)) \notin \mathbb{Q}^3$ holds for all $\lambda, x \in \mathbb{R}$ with $\lambda \neq 0$, then

2010 *Mathematics Subject Classification.* 39A05, 28E99, 26E99.

Key words and phrases. Ducci map; Continued fraction; Ducci matrix sequence; Distribution; Measure; Almost everywhere.

its Ducci sequence will never become periodic and approaches the zero vector asymptotically. For a non-heterogeneous starting vector, it was also proved in [2] that its Ducci sequence is eventually periodic. (An alternative proof can be found in [5].) For a general n, it is known [3] that any starting vector in $\mathbb{R}^n$ converges asymptotically to a periodic sequence, but not necessarily to the sequence of zero vectors. Indeed, a vector which converges asymptotically to a non-trivial periodic sequence is constructed in [3] for $n = 7$.

Hogenson et al. [6] made a new approach to the subject by introducing the concept *Ducci matrix sequences*. For each vector in $\mathbb{R}^n$, one can find an $n \times n$ matrix whose application to the vector is equivalent to the application of the Ducci map. This matrix depends, of course, on the chosen vector. Thus, one may associate with a vector $\boldsymbol{v}$ not a single matrix but a sequence $M_{j_1}, M_{j_2}, \dots$ of matrices such that the matrix M_{j_n} implements the n-th application of the Ducci map to $\boldsymbol{v}$. By considering those starting vectors in $\mathbb{R}^3$ that lead to unique Ducci matrix sequences, Hogenson et al. [6] established a connection between the Ducci map, the process of forming *mediants* of rational numbers and the *Stern-Brocot tree*.

In this paper, we focus on the Ducci map on $\mathbb{R}^3$. After presenting necessary concepts and their properties in Section 2, we consider in Section 3 the function j that maps each $\alpha \in (0,1) \setminus \mathbb{Q}$ to the sequence $j(\alpha) := \langle j_\alpha(1), j_\alpha(2), \dots \rangle$ of indexes of its Ducci matrix sequence expansion. While continuity of j and j^{-1} is easily checked, we show that j^{-1} is moreover uniformly continuous. We then study the distribution of Ducci matrices in the Ducci matrix sequence expansion of a given irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$ by considering the following three conditions on the sequence $j(\alpha)$:

$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$$
$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6} \text{ for every } \jmath \in \{1, 2, \dots, 6\};$$
$$\lim_{n\to\infty} \sqrt[p]{\frac{\Sigma_{i=1}^n j_\alpha(i)^p}{n}} = \sqrt[p]{\frac{1^p + 2^p + \cdots + 6^p}{6}}.$$

In Section 4, we prove that the top implies the middle and the middle implies the bottom. We also give examples witnessing that the converse to these two implications are not true in general. In addition, various equivalent statements to the first condition will be presented. In the final section, we shall provide measure theoretic treatment of the subject: We prove that for almost every α, each Ducci matrix appears in the Ducci matrix sequence expansion of α infinitely often. We then ask if the second (and the third) condition above holds almost everywhere. We have not succeeded in solving these questions; We will however see the following partial result:

$$\limsup_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) = \jmath\}|}{n} \ge \frac{1}{6} \text{ for every } \jmath \in \{1, 2, \dots, 6\} \text{ holds a.e.}$$

Related questions as well as some other partial results will be presented.

2 Preliminaries.

In this preparatory section, we recapitulate materials presented in [5]. For more details, we refer the reader to [5].

2.1 Ducci map and continued fractions.

Let us start by fixing certain terminology on continued fractions (as taken from Khinchin's book [7]). We write $[a_0; a_1, a_2, \dots]$ and $[a_0; a_1, \dots, a_l]$ for the following infinite and finite continued fraction, respectively:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \qquad \text{and} \qquad a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_l}}}.$$

We assume that a_0 is an integer and $a_1, a_2, \ldots$ are positive integers. We call $a_0, a_1, \ldots$ the *elements* of a continued fraction. For an infinite continued fraction $\alpha = [a_0; a_1, a_2, \ldots]$, we call $s_k := [a_0; a_1, \ldots, a_k]$ and $r_k := [a_k; a_{k+1}, \ldots]$ a *segment* and a *remainder* of α, respectively. Obviously, remainders satisfy the relation $r_k = r_{k+1}^{-1} + a_k$. For finite continued fractions, segments and remainders are defined analogously.

Another important concept in the theory of continued fractions is that of *convergent.* For a given $\alpha = [a_0; a_1, a_2, \ldots]$, we write p_k/q_k for the k-th order convergent, i.e. p_k and q_k are non-negative relatively prime integers such that $p_k/q_k = s_k$. It is customary to set $p_{-1} := 1$ and $q_{-1} := 0$. A folklore theorem gives us the rule for the formation of the convergents: For any $k \geq 1$, it holds that $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$.

It is well-known that continued fraction can be used as an apparatus for representing real numbers (A proof of the next theorem can be found in, e.g. [7, Theorem 14]):

Theorem 1. *Assume that the last element of any finite continued fraction is greater than* 1*. Then, to every real number α, there corresponds a unique continued fraction with value equal to α. This fraction is finite if α is rational, and is infinite if α is irrational.* □

Using continued fraction expansion, one can completely describe the orbit of $(0, \alpha, 1)$ under the Ducci map D for irrational $\alpha > 0$ as follows. Observe that $\alpha > 0$ implies that the first element a_0 of α's continued fraction expansion is non-negative.

Theorem 2 ([5])**.** *Let $\alpha = [a_0; a_1, a_2, \ldots] > 0$. For a given positive integer $n \geq 1$, let $k = k(n)$ be the least integer satisfying the relation $n \leq \Sigma_{i=0}^{k} a_i$. Then*

$$D^n(0, \alpha, 1) = \frac{\alpha}{r_0 \cdots r_k} \tau_{n,k} \cdot \begin{pmatrix} 1 \\ r_{k+1}^{-1} + \Sigma_{i=0}^{k} a_i - n \\ r_{k+1}^{-1} + \Sigma_{i=0}^{k} a_i - n + 1 \end{pmatrix}^{\mathrm{T}},$$

where $\tau_{n,k} \in \mathfrak{S}_3$ is a permutation that depends only on n if $k = 0$, and n and a segment s_{k-1} if $k > 0$. □

(We put $\tau \cdot \boldsymbol{v} := (v_{\tau(1)}, v_{\tau(2)}, v_{\tau(3)})$ for a permutation $\tau \in \mathfrak{S}_3$ and a vector $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$.) It is not hard to check the validity of this theorem also for finite continued fractions. More precisely, for a finite continued fraction $\alpha = [a_0; a_1, \ldots, a_l]$, the formula is correct for $n = 1, 2, \ldots, \Sigma_{i=0}^{l-1} a_i$. For n with $\Sigma_{i=0}^{l-1} a_i < n \leq \Sigma_{i=0}^{l} a_i$, we obtain a correct formula by deleting all the occurrences of the term r_{l+1}^{-1} in the entries of the vector. Specifically, we have $D^n(0, \alpha, 1) = (\alpha/r_0 \cdots r_l)\, \tau_{n,l} \cdot (1, \Sigma_{i=0}^{l} a_i - n, \Sigma_{i=0}^{l} a_i - n + 1)$ in this case.

For convenience, let us introduce one more concept here:

Definition 1. *We say that a real vector $\boldsymbol{v} \in \mathbb{R}^3$ is of*

- type 1 *if it is of the form $\boldsymbol{v}_1\langle c; x; n\rangle := c(1, x+n, x+n+1)$ for some $c > 0, 0 < x < 1$ and a natural number $n \geq 1$;*
- type 2 *if it is of the form $\boldsymbol{v}_2\langle c; x; n\rangle := c(x+n, 1, x+n+1)$ for some $c > 0, 0 < x < 1$ and a natural number $n \geq 1$;*

- type 3 *if it is of the form* $\boldsymbol{v}_3\langle c;x;n\rangle := c(x+n, x+n+1, 1)$ *for some* $c>0, 0<x<1$ *and a natural number* $n \geq 1$;
- type 4 *if it is of the form* $\boldsymbol{v}_4\langle c;x;n\rangle := c(1, x+n+1, x+n)$ *for some* $c>0, 0<x<1$ *and a natural number* $n \geq 1$;
- type 5 *if it is of the form* $\boldsymbol{v}_5\langle c;x;n\rangle := c(x+n+1, 1, x+n)$ *for some* $c>0, 0<x<1$ *and a natural number* $n \geq 1$;
- type 6 *if it is of the form* $\boldsymbol{v}_6\langle c;x;n\rangle := c(x+n+1, x+n, 1)$ *for some* $c>0, 0<x<1$ *and a natural number* $n \geq 1$.

In any of these cases, we call n the *integer part* of the vector $\boldsymbol{v}_i\langle c;x;n\rangle$.

Let an irrational number α be given. Observe that its reminders r_i satisfy $0 < r_i^{-1} < 1$, and that we have

$$\frac{\alpha}{r_0 \cdots r_k}\tau_{n,k}\cdot\begin{pmatrix}1\\ r_{k+1}^{-1}\\ r_{k+1}^{-1}+1\end{pmatrix}^{\mathrm{T}} = \frac{\alpha}{r_0\cdots r_k r_{k+1}}\tau_{n,k}\cdot\begin{pmatrix}r_{k+2}^{-1}+a_{k+1}\\ 1\\ r_{k+2}^{-1}+a_{k+1}+1\end{pmatrix}^{\mathrm{T}}$$

with $a_{k+1} \geq 1$. It is then not hard to see from Theorem 2 that for every $n \geq 1$, the vector $D^n(0,\alpha,1)$ is of some type.

An easy computation shows the following

Proposition 1 ([5])**.** *Let* $\alpha = [a_0; a_1, a_2, \dots] > 0$ *be irrational. Then for any positive real number* $c > 0$ *and a natural number* $n > 1$*, it holds that* $D(\boldsymbol{v}_i\langle c; r_k^{-1}; n\rangle) = \boldsymbol{v}_{i+1}\langle c; r_k^{-1}; n-1\rangle$ *for every* $k \geq 0$ *and* $i = 1, 2, \dots, 6$*, where any subscript greater than* 6 *is to be understood by modulo* 6*.*

If the integer part of $\boldsymbol{v}_i$ *is* 1*, then we have the following:*

- $D(\boldsymbol{v}_1\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_1\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_2\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_4\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_3\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_3\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_4\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_6\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_5\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_5\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_6\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_2\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$. □

Therefore, an application of the Ducci map D to a vector of the form $\boldsymbol{v}_i\langle c; r_k^{-1}; n\rangle$ with $n \geq 1$ yields the increment of the type by 1 (modulo 6) if and only if the integer part n is greater than 1. This property will play a key role later on.

The above proposition will bring the reader clearer understanding of the computation of the permutation $\tau_{n,k(n)}$.

2.2 Ducci matrix sequence.

Definition 2. *The regions* $\mathcal{R}_1, \dots, \mathcal{R}_6 \subset \mathbb{R}^3$ *are defined as follows:*

- $\mathcal{R}_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \leq x_2 \leq x_3\}$;
- $\mathcal{R}_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \leq x_1 \leq x_3\}$;

- $\mathcal{R}_3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \le x_1 \le x_2\}$;
- $\mathcal{R}_4 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \le x_3 \le x_2\}$;
- $\mathcal{R}_5 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \le x_3 \le x_1\}$;
- $\mathcal{R}_6 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \le x_2 \le x_1\}$.

We say that a matrix M *implements* the action of the Ducci map D on $\boldsymbol{v} \in \mathbb{R}^3$ if $D\boldsymbol{v} = \boldsymbol{v}M$ holds. Matrices $M_1, \dots, M_6$ are defined so that M_i implements the application of the Ducci map to any vector in the region $\mathcal{R}_i$ uniformly, i.e. $D\boldsymbol{v} = \boldsymbol{v}M_i$ holds for every $\boldsymbol{v} \in \mathcal{R}_i$. For instance,

$$M_1 = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$

Observe that two distinct regions can overlap each other. For example, $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(x_1, x_2, x_3) \mid x_1 = x_2 \le x_3\} \ne \emptyset$. Consequently, either M_1 or M_2 serves as an implementation of an application of the Ducci map to any vector $\boldsymbol{v} \in \mathcal{R}_1 \cap \mathcal{R}_2$. It is also easy to observe that if all entries of a vector $\boldsymbol{v}$ are pairwise distinct, then $\boldsymbol{v}$ belongs to a unique region, and hence has only one implementation.

It would be interesting to consider a sequence of implementations of applications of the Ducci map to a given starting vector. To make this precise, let us introduce one more piece of terminology.

Definition 3 ([6]). *For a given vector $\boldsymbol{v} \in \mathbb{R}^3$, a* Ducci matrix sequence associated with *$\boldsymbol{v}$ is a sequence $M_{j_1}, M_{j_2}, \dots$ of matrices with $j_1, j_2, \dots \in \{1, 2, \dots, 6\}$ such that $D^n\boldsymbol{v} = \boldsymbol{v}M_{j_1} \cdots M_{j_n}$ holds for all $n \ge 1$.*

For a real number $\alpha \in \mathbb{R}$, we define a Ducci matrix sequence associated with α *to be a Ducci matrix sequence associated with the vector* $(0, \alpha, 1)$.

One may naturally ask which α have a unique Ducci matrix sequence. This question has been answered in [6] as follows (A different proof can be found in [5].):

Theorem 3 ([6]). *α is irrational if and only if there is only one Ducci matrix sequence associated with α.* □

Thus, for a given α, we call the unique Ducci matrix sequence associated with it the *Ducci matrix sequence expansion* of α.

3 Uniform continuity. At the end of the last section, we mentioned the result that α is irrational if and only if there is only one Ducci matrix sequence associated with α. This gives us a function j that sends an irrational number $\alpha \in (0, 1)$ to the sequence $j(\alpha) = \langle j_\alpha(1), j_\alpha(2), \dots \rangle \in \{1, 2, \dots, 6\}^\omega$ of indexes of the Ducci matrix sequence expansion $M_{j_\alpha(1)}, M_{j_\alpha(2)}, \dots$ of α. In this section, we shall study (uniform) continuity of j and its inverse j^{-1}.

Before proceeding any further, it will be useful to summarize the relationship among relevant concepts defined so far:

Proposition 2. *For irrational $\alpha > 0$ and $n \ge 1$, we have the following relations:*

$$D^n(0, \alpha, 1) \text{ is of type } t \iff D^n(0, \alpha, 1) \in \mathcal{R}_t \iff j_\alpha(n + 1) = t. \qquad \square$$

Remark 1. *Actually, the second equivalence in the above proposition holds for $n = 0$. We have formulated the above proposition in this way because the type of vector $D^0(0, \alpha, 1)$ is undefined.*

The uniqueness of the continued fraction expansion (Theorem 1) and Theorem 2 entail the injectivity of j. Thus, considered as a function from $(0, 1) \setminus \mathbb{Q}$ to $j((0, 1) \setminus \mathbb{Q})$, j is bijective.

In order to see that j is continuous, the following result plays a key role:

Theorem 4 ([5]). *Let two distinct positive irrational numbers $\alpha, \alpha' > 0$ be given, and consider their infinite continued fraction expansions: $[a_0; a_1, a_2, \dots]$ and $[a'_0; a'_1, a'_2, \dots]$. If we have $a_l < a'_l$ for $l = \min\{\, l \mid a_l \neq a'_l \,\}$, then the length of the maximal common initial segment of Ducci matrix sequence expansions of α and α' is $\Sigma_{i=0}^{l} a_i$.* □

For a given irrational $\alpha \in (0, 1)$, take an irrational α' sufficiently close to α so that the first n elements of their continued fraction expansions coincide. Then, by virtue of this theorem, the sequences $j(\alpha)$ and $j(\alpha')$ are identical up to the first $\min\{\Sigma_{i=0}^{n} a_i, \Sigma_{i=0}^{n} a'_i\} \geq n$ segments. Given the definition of the standard metric on $\{1, 2, \dots, 6\}^\omega$, i.e. the distance of two sequences is set to be 2^{-m}, where m is the first place at which the sequences differ, it is easily observed that continuity of j follows from this argument.

Uniform continuity however does not hold for j, as witnessed by the example below:

Example 1. *Take a positive irrational number $\varepsilon < 1/6$ and consider two irrational numbers $1/2 - \varepsilon$ and $1/2 + \varepsilon$. The first element of the continued fraction expansion of the former is 2, while the latter is 1. This means that, no matter how small ε is, and consequently no matter how close the numbers $1/2 - \varepsilon$ and $1/2 + \varepsilon$ are, the second coordinates of $j(1/2 - \varepsilon)$ and $j(1/2 + \varepsilon)$ are different.*

When it comes to the inverse $j^{-1} : j((0, 1) \setminus \mathbb{Q}) \to (0, 1) \setminus \mathbb{Q}$, not only continuity but also uniform continuity holds. Let us prove this assertion.

Theorem 5. $j^{-1} : j((0, 1) \setminus \mathbb{Q}) \to (0, 1) \setminus \mathbb{Q}$ *is uniformly continuous.*

Proof. Let $\varepsilon > 0$ be given. Since $j((0, 1) \setminus \mathbb{Q})$ is considered as a subspace of $\{1, 2, \dots, 6\}^\omega$, in view of the definition of the standard metric on $\{1, 2, \dots, 6\}^\omega$, it suffices to show that there exists an n such that $|\alpha - \alpha'| < \varepsilon$ holds for any $\alpha, \alpha' \in (0, 1) \setminus \mathbb{Q}$ whenever $j(\alpha) = \langle j_\alpha(1), j_\alpha(2), \dots \rangle$ and $j(\alpha') = \langle j_{\alpha'}(1), j_{\alpha'}(2), \dots \rangle$ agree up to first n elements.

We claim that any natural number n greater than $1/\varepsilon$ has the desired property. To see this, take $\alpha, \alpha' \in (0, 1) \setminus \mathbb{Q}$ so that the initial segments of $j(\alpha)$ and $j(\alpha')$ are identical up to n. Let $k = k(n)$ and $k' = k'(n)$ be as in the statement of Theorem 2, i.e. $k \geq 1$ (resp. $k' \geq 1$) is the least integer satisfying that $n \leq \Sigma_{i=0}^{k} a_i$ (resp. $n \leq \Sigma_{i=0}^{k} a'_i$). It is not difficult to see from Theorem 4 that k and k' are equal and that we have $a_0 = a'_0, \dots, a_{k-1} = a'_{k-1}$.

Now let p_l/q_l and p'_l/q'_l denote the k-th order convergent of α and α', respectively: $p_l/q_l = s_l$ and $p'_l/q'_l = s'_l$. It is known [7, pp. 8] that α and α' can be expressed in terms of their convergents and remainders as follows:

$$\alpha = \frac{p_{l-1} r_l + p_{l-2}}{q_{l-1} r_l + q_{l-2}} \quad \text{and} \quad \alpha' = \frac{p'_{l-1} r'_l + p'_{l-2}}{q'_{l-1} r'_l + q'_{l-2}} \quad \text{for every } l \geq 1.$$

Note that $a_0 = a'_0, \dots, a_{k-1} = a'_{k-1}$ imply $p_l = p'_l$ and $q_l = q'_l$ for $l \leq k$. Therefore, by writing $f(x) = (p_{k-1}x + p_{k-2})/(q_{k-1}x + q_{k-2})$, we have $\alpha = f(r_k)$ and $\alpha' = f(r'_k)$.

Since $q_{k-1} \geq 1$ and $q_{k-2} \geq 0$, the function $f(x)$ is monotone on $(0, \infty)$. As we have $r_k \geq \lfloor r_k \rfloor = a_k \geq n - \Sigma_{i=0}^{k-1} a_i > 0$ and similarly $r'_k \geq n - \Sigma_{i=0}^{k-1} a_i > 0$, this monotonicity of

$f(x)$ proves that $\alpha = f(r_k)$ and $\alpha' = f(r'_k)$ lie between $f(n - \Sigma_{i=0}^{k-1} a_i)$ and $\lim_{x\to\infty} f(x) = p_{k-1}/q_{k-1}$. Hence it holds that

$$\begin{aligned}|\alpha - \alpha'| &\leq \left| f(n - \Sigma_{i=0}^{k-1} a_i) - \lim_{x\to\infty} f(x) \right| \\ &= \left| \frac{p_{k-1}(n - \Sigma_{i=0}^{k-1} a_i) + p_{k-2}}{q_{k-1}(n - \Sigma_{i=0}^{k-1} a_i) + q_{k-2}} - \frac{p_{k-1}}{q_{k-1}} \right| \\ &= \frac{|p_{k-2}q_{k-1} - p_{k-1}q_{k-2}|}{q_{k-1}\{q_{k-1}(n - \Sigma_{i=0}^{k-1} a_i) + q_{k-2}\}} \\ &= \frac{1}{q_{k-1}\{q_{k-1}(n - \Sigma_{i=0}^{k-1} a_i) + q_{k-2}\}}.\end{aligned}$$

(The well-known identity $p_{k-2}q_{k-1} - p_{k-1}q_{k-2} = (-1)^{k-1}$ was used in the last step.)

Since n is greater than $1/\varepsilon$, if $k = 1$, then $q_0 = 1$ and $q_{-1} = 0$ proves that $|\alpha - \alpha'| \leq 1/n < \varepsilon$. In order to deal with the case that $k \geq 2$, we need the following lemma, which is easily proved via induction.

Lemma 1. *$q_{k-1} \geq \Sigma_{i=0}^{k-1} a_i$.* □

Using this lemma, we resume the evaluation of the difference $|\alpha - \alpha'|$:

$$\begin{aligned}|\alpha - \alpha'| &\leq \frac{1}{q_{k-1}\{q_{k-1}(n - \Sigma_{i=0}^{k-1} a_i) + q_{k-2}\}} \\ &\leq \frac{1}{\Sigma_{i=0}^{k-1} a_i\{\Sigma_{i=0}^{k-1} a_i(n - \Sigma_{i=0}^{k-1} a_i) + q_{k-2}\}} \\ &\leq \frac{1}{\Sigma_{i=0}^{k-1} a_i(n - \Sigma_{i=0}^{k-1} a_i) + 1} \\ &\leq \frac{1}{n} < \varepsilon.\end{aligned}$$

This makes the proof of the theorem complete. □

Remark 2. *If the domain of a continuous function is compact, then uniform continuity follows automatically from continuity. This time, however, we had to prove the above theorem directly because the domain $j((0,1)\setminus\mathbb{Q}) \subset \{1,2,\ldots,6\}^\omega$ of the continuous function j^{-1} is not compact. Indeed, there is a Cauchy sequence $\{j([0;n,1,1,1,\ldots])\}_{n\geq 1}$ in $j((0,1)\setminus\mathbb{Q})$ with its limit $\langle 1\rangle^\frown\langle 1,2,\ldots,6\rangle^\omega \in \{1,2,\ldots,6\}^\omega$ outside $j((0,1)\setminus\mathbb{Q})$. (Here and in what follows, we use the symbol $^\frown$ for the concatenation of two sequences: $\langle x_1,x_2,\ldots,x_n\rangle^\frown\langle y_1,y_2,\ldots\rangle := \langle x_1,x_2,\ldots,x_n,y_1,y_2,\ldots\rangle$.)*

4 Distribution of Ducci matrices. For a given irrational number $\alpha \in (0,1)\setminus\mathbb{Q}$, we are interested in the distribution of indexes in the sequence $j(\alpha) = \langle j_\alpha(1), j_\alpha(2), \ldots\rangle$. In this section, we consider several statements regarding the distribution of Ducci matrices in a given sequence $j(\alpha)$ and examine their relationships. Note that, since we shall deal with irrational numbers only from $(0,1)\setminus\mathbb{Q}$, we always have $a_0 = 0$ from now on. Also, the index i of the sequence $\{j_\alpha(i)\}_i$ starts from 1. For notational convenience, we thus make the following convention: Throughout this and the next section, any index and subscript start from 1 unless otherwise stated.

To begin with, let us prove the following

Lemma 2. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$ and $l \geq 1$, we have*

$$\begin{aligned} &|\,\{i \leq n \mid j_\alpha(i) + l \equiv j_\alpha(i+1) + l - 1 \equiv \cdots \equiv j_\alpha(i+l) \pmod 6\}\,| \\ &\qquad\qquad \geq n - l \cdot |\,\{m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n\}\,| - \frac{l(l-1)}{2}. \end{aligned}$$

Equality holds if $l = 1$.

Proof. It is not hard to check that for any $i \geq 2$, we have

$$\begin{aligned} j_\alpha(i) + 1 \not\equiv j_\alpha(i+1) \pmod 6 &\overset{(1)}{\iff} i \text{ is of the form } \Sigma_{s=1}^m a_s \text{ for some } m \geq 1 \\ &\overset{(2)}{\iff} \text{The integer part of } D^{i-1}(0,\alpha,1) \text{ is } 1. \end{aligned}$$

(Actually, Equivalence (1) holds for $i = 1$; We wrote $i \geq 2$ above because the integer part of $D^n(0,\alpha,1)$ for $n = 0$ is undefined. See Remark 1.)

The assertion for $l = 1$ follows at once from Equivalence (1), which holds for any $i \geq 1$.

In order to deal with a general $l > 1$, let us put

$$\begin{aligned} \hat{I}_p^n &:= \{i \leq n \mid j_\alpha(i) + l \equiv j_\alpha(i+1) + l - 1 \equiv \cdots \\ &\qquad \equiv j_\alpha(i+p-1) + l - p + 1 \not\equiv j_\alpha(i+p) + l - p \pmod 6\} \text{ and} \\ I_p^n &:= \{i \leq n \mid \text{The integer part of } D^i(0,\alpha,1) \text{ is } p\} \end{aligned}$$

for $1 \leq p \leq l$.

If $p = 1$, then (1) implies that $|\,\hat{I}_1^n\,| = |\,\{m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n\}\,|$.

If $p > 1$, then (1), (2) and Proposition 1 tell us that $\hat{I}_p^n = I_{p-1}^n$. Now we evaluate the size $|\,I_{p-1}^n\,|$ of I_{p-1}^n. If $p > 2$, then $i \in I_{p-1}^n$ implies $i + 1 \in I_{p-2}^n$ for any $i < n$. Note however that the converse is in general not true. (For example, it can happen that an $i \in I_1^n$ satisfies $i + 1 \in I_{p-2}^n$.) Therefore, we have $|\,I_{p-1}^n\,| \leq |\,I_{p-2}^n\,| + 1$. Applying this inequality repeatedly, we obtain $|\,I_{p-1}^n\,| \leq |\,I_1^n\,| + p - 2$ for any $p > 1$. Using (2), we thus see that

$$\begin{aligned} |\,\hat{I}_p^n\,| &= |\,I_{p-1}^n\,| \\ &\leq |\,\{i \leq n \mid i + 1 \text{ is of the form } \Sigma_{s=1}^m a_s \text{ for some } m \geq 1\}\,| + p - 2 \\ &\leq |\,\{i \leq n \mid i \text{ is of the form } \Sigma_{s=1}^m a_s \text{ for some } m \geq 1\}\,| + p - 1 \\ &= |\,\{m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n\}\,| + p - 1, \end{aligned}$$

for any $p > 1$.

Putting these arguments together, we get

$$\begin{aligned} &|\,\{i \leq n \mid j_\alpha(i) + l \equiv j_\alpha(i+1) + l - 1 \equiv \cdots \equiv j_\alpha(i+l) \pmod 6\}\,| \\ &= n - \Sigma_{p=1}^l |\,\hat{I}_p^n\,| \\ &\geq n - |\,\{m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n\}\,| - \Sigma_{p=2}^l \left(|\,\{m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n\}\,| + p - 1\right) \\ &= n - l \cdot |\,\{m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \leq n\}\,| - \frac{l(l-1)}{2}, \end{aligned}$$

as desired. □

We are interested in the relation $j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6$, especially in the frequency that this happens in a given sequence $j(\alpha)$. In some situations, this relation between $j_\alpha(i)$ and $j_\alpha(i+1)$ can be equivalently expressed using only elements a_i of α. Specifically, we have

Theorem 6. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, the following are equivalent:*

1. $\displaystyle\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$

2. $\displaystyle\lim_{n\to\infty} \frac{|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \le n\}|}{n} = 0;$

3. $\displaystyle\frac{\Sigma_{i=1}^n a_i}{n}$ *diverges.*

Proof. $1 \Leftrightarrow 2$: This follows easily from Lemma 2 (for the case $l = 1$).
$2 \Rightarrow 3$: Let us temporarily put $A(n) := |\{m \in \mathbb{Z}_{>0} \mid \Sigma_{s=1}^m a_s \le n\}|$. It is then clear that $n = A(\Sigma_{i=1}^n a_i)$. Since $\Sigma_{i=1}^n a_i \to \infty$ as $n \to \infty$, 2 implies $A(\Sigma_{i=1}^n a_i)/\Sigma_{i=1}^n a_i$ converges to 0. Hence $\Sigma_{i=1}^n a_i/n = \Sigma_{i=1}^n a_i/A(\Sigma_{i=1}^n a_i)$ diverges.
$3 \Rightarrow 2$: From the definition of $A(n)$, it follows that $\Sigma_{s=1}^{A(n)} a_s \le n$. Hence we have $0 \le A(n)/n \le A(n)/\Sigma_{s=1}^{A(n)} a_s$. Since $A(n)$ diverges as $n \to \infty$, 3 implies $A(n)/\Sigma_{s=1}^{A(n)} a_s$ converges to 0. Therefore $A(n)/n$ also converges to 0. ◻

Corollary 1. *No $\alpha \in (0,1) \setminus \mathbb{Q}$ with bounded elements satisfies* $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$.

Proof. Let M be such that $a_i \le M$ holds for all $i \in \mathbb{Z}_{>0}$. This implies that $\Sigma_{i=1}^n a_i/n \le M$, in particular, $\Sigma_{i=1}^n a_i/n$ is not divergent. Theorem 6 now proves our assertion. ◻

Corollary 2. *The set of all α satisfying* $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ *is dense.*

Proof. The set of all α with $\Sigma_{i=1}^n a_i/n$ divergent is dense. The assertion again follows from Theorem 6. ◻

Our condition above concerns the relationship between only two indexes. Here, the following question arises naturally: does it give rise to any difference if we require more than two indexes, say $j_\alpha(i), j_\alpha(i+1), \ldots, j_\alpha(i+l)$ with $l > 1$, to be consecutive by modulo 6? The next theorem answers this question negatively.

Theorem 7. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$ and $l \ge 1$, we have*

$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1 \Longleftrightarrow$$
$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) + l \equiv j_\alpha(i+1) + l - 1 \equiv \cdots \equiv j_\alpha(i+l) \pmod 6\}|}{n} = 1.$$

Proof. Evidently, the first condition follows from the second. So let us prove the other direction:

Assume that the first condition is true for a given α. From Theorem 6, it then follows that $\Sigma_{i=1}^m a_i/m$ is divergent. For a given $\varepsilon > 0$, let $M \in \mathbb{Z}_{>0}$ be such that every $M' \ge M$ satisfies $\Sigma_{i=1}^{M'} a_i/M' > 2l/\varepsilon$. Take an $n \ge \max\{\Sigma_{i=1}^M a_i, l(l-1)/\varepsilon\}$. Then, since we have $|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \le n\}| \ge M$, it holds that

$$\frac{n}{|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \le n\}|} \ge \frac{\Sigma_{i=1}^{|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \le n\}|} a_i}{|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \le n\}|} > \frac{2l}{\varepsilon}.$$

Using Lemma 2, we thus obtain

$$\begin{aligned} 1 &\geq \frac{|\{i \leq n \mid j_\alpha(i) + l \equiv j_\alpha(i+1) + l - 1 \equiv \cdots \equiv j_\alpha(i+l) \pmod 6\}|}{n} \\ &\geq 1 - \frac{l \cdot |\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \leq n\}|}{n} - \frac{l(l-1)}{2n} \\ &> 1 - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this completes the proof. □

Now we turn to another condition on the distribution of indexes in the sequence $j(\alpha) = \langle j_\alpha(1), j_\alpha(2), \dots \rangle$. If the sequence $j(\alpha)$ is distributed uniformly, any $\jmath \in \{1, 2, \dots, 6\}$ will occur with the same probability. As there are only six possible values of $\jmath$, the probability should then be $1/6$. Hence the following statement is to be seen as a necessary condition for a sequence $j(\alpha)$ to be uniformly distributed:

$$\lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6} \text{ holds for every } \jmath \in \{1, 2, \dots, 6\}.$$

The next theorem shows that this condition follows from our first condition.

Theorem 8. *Let an $\alpha \in (0,1) \setminus \mathbb{Q}$ be given. If $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ holds, then $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ holds for every $\jmath \in \{1, 2, \dots, 6\}$.*

Proof. In view of Theorem 6, it suffices to prove the consequent assuming that $\Sigma_{i=1}^n a_i/n$ is divergent.

Fix a $\jmath \in \{1, 2, \dots, 6\}$ and choose an $\varepsilon > 0$ arbitrarily. Since $\Sigma_{i=1}^n a_i/n$ diverges, there exists a positive integer $K > 1$ such that $\Sigma_{i=1}^l a_i/l > 3/\varepsilon$ holds for every $l \geq K$. For this K, we claim that the difference between $|\{i \leq n \mid j_\alpha(i) = \jmath\}|/n$ and $1/6$ is less than ε whenever n is greater than $\Sigma_{i=1}^K a_i$. For this purpose, fix an $n > \Sigma_{i=1}^K a_i$ and let $k = k(n)$ be as in the statement of Theorem 2.

Using the sequence $j_\alpha(1), j_\alpha(2), \dots, j_\alpha(n)$, we shall construct a new sequence w (of finite length) as follows: Consider two numbers $j_\alpha(1+\Sigma_{i=1}^p a_i)$ and $j_\alpha(\Sigma_{i=1}^p a_i)$ for $p = 1, \dots, k-1$, and iterate the next process from $p = 1$ to $p = k-1$. If $j_\alpha(1 + \Sigma_{i=1}^p a_i) = j_\alpha(\Sigma_{i=1}^p a_i)$, then remove $j_\alpha(\Sigma_{i=1}^p a_i)$ from the initial sequence. If, on the other hand, we have $j_\alpha(1+\Sigma_{i=1}^p a_i) \equiv j_\alpha(\Sigma_{i=1}^p a_i) + 2 \pmod 6$, then insert the number $j_\alpha(\Sigma_{i=1}^p a_i) + 1 \pmod 6$ between these two. Call the resulting sequence w. It is then immediate from Propositions 1 and 2 that the new finite sequence w is eventually periodic: $w = \langle 1, 1, 2, \dots, 6, 1, 2, \dots, 6, \dots \rangle$. (Periodic part starts from the second coordinate.)

Since we have removed or inserted $k-1$ numbers, the length $\mathrm{lh}(w)$ of w is at most $n+k-1$ and at least $n-k+1$. Eventual periodicity of w implies that at most $\lfloor (\mathrm{lh}(w)-1)/6 \rfloor + 2$ and at least $\lfloor (\mathrm{lh}(w)-1)/6 \rfloor$ coordinates of w are equal to $\jmath$. It might be the case that all removed numbers are equal to $\jmath$; it might be the case that all inserted numbers are equal to $\jmath$. Taking these worst case scenarios into account, one obtains the following estimate:

$$\frac{\lfloor \frac{n-k}{6} \rfloor - (k-1)}{n} \leq \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} \leq \frac{\lfloor \frac{n+k-2}{6} \rfloor + k + 1}{n}.$$

As we have $\Sigma_{i=1}^k a_i \geq n > \Sigma_{i=1}^K a_i$ for $k = k(n)$, it follows that $k - 1 \geq K > 1$. In view of the definition of K, we thus see that

$$\frac{1}{n} < \frac{k-1}{n} < \frac{k-1}{\Sigma_{i=1}^{k-1} a_i} < \frac{\varepsilon}{3}.$$

Putting these arguments together, we compute the difference as follows:

$$\begin{aligned}\left|\frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} - \frac{1}{6}\right| &\leq \frac{7k+4}{6n} \\ &= \frac{7(k-1)}{6n} + \frac{11}{6n} \\ &< \frac{7\varepsilon}{18} + \frac{11\varepsilon}{18} = \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ and $\jmath \in \{1, 2, \ldots, 6\}$ were chosen arbitrarily, this proves that the limit exists and is equal to $1/6$, as desired. $\square$

We note that the converse to this theorem is not true. Here is a witness:

Example 2. *Consider the following eventually periodic sequence of Ducci matrices:*

$$\langle M_1\rangle^\frown\langle M_1, M_2, M_3, M_4, M_5, M_6, M_1, M_1, M_2, M_3, M_4, M_5, M_6, M_2, M_3, M_4, M_5, M_6\rangle^\omega.$$

It is not hard to check that this is the Ducci matrix sequence expansion of an irrational number $\alpha := [0; 8, 6, 12, 6, 12, 6, 12, 6, \ldots] \in (0,1) \setminus \mathbb{Q}$.

Since the elements of α *is bounded by* 12, *it is clear that* $\Sigma_{i=1}^n a_i/n$ *is not divergent, and accordingly, we do not have* $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$.

We need to check that this α *satisfies* $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ *for every* $\jmath \in \{1, 2, \ldots, 6\}$. *To see this, choose an* $\varepsilon > 0$ *and a* $\jmath \in \{1, 2, \ldots, 6\}$ *arbitrarily and pick a natural number* $N > 4/\varepsilon$. *Take an arbitrary natural number* $n \geq N$ *and express it as* $n = 1 + 18a + b$ *with non-negative integers* $a \geq 0$ *and* $0 \leq b < 18$. *Note that in the Ducci matrix sequence expansion of* α, *the number of occurrences of the Ducci matrix* $M_\jmath$ *from the* $18m + 2$*nd matrix to the* $18(m+1) + 1$*st matrix is three for every* $m \geq 0$. *Therefore,*

$$\frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} \leq \frac{3a+4}{n} \leq \frac{1}{6} + \frac{4}{n} < \frac{1}{6} + \varepsilon$$

and

$$\frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} \geq \frac{3a}{n} = \frac{1}{6} - \frac{1+b}{6n} > \frac{1}{6} - \frac{4}{n} > \frac{1}{6} - \varepsilon.$$

Since $\varepsilon > 0$ *was chosen arbitrarily, this proves that* $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$. *As we took* $\jmath$ *also arbitrarily, this proves our claim.*

In studying distribution, one natural attempt is to take the average. The uniformity of the distribution of indexes in the sequence $\langle j_\alpha(1), j_\alpha(2), \ldots\rangle$ can also be captured using the notion of average. Specifically, we consider the following formula to be a plausible formulation of uniformity: $\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$. Let us investigate the relationship of this condition to the preceding one.

Proposition 3. *Let* $p \geq 1$ *be a positive integer. For any* $\alpha \in (0,1) \setminus \mathbb{Q}$, *if* $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ *holds for every* $\jmath \in \{1, 2, \ldots, 6\}$, *then we have*

$$\lim_{n\to\infty} \sqrt[p]{\frac{\Sigma_{i=1}^n j_\alpha(i)^p}{n}} = \sqrt[p]{\frac{1^p + 2^p + \cdots + 6^p}{6}}.$$

Proof. Let $\varepsilon > 0$ be given. Then there exists an $N \in \mathbb{Z}_{>0}$ such that $|\,|\{i \leq n \mid j_\alpha(i) = \jmath\}|/n - 1/6\,| < \varepsilon$ holds for every $\jmath$ and $n \geq N$. This means that

$$\left(\frac{1}{6} - \varepsilon\right) n < |\{i \leq n \mid j_\alpha(i) = \jmath\}| < \left(\frac{1}{6} + \varepsilon\right) n$$

holds for every $\jmath$. By multiplying by $\jmath^p$ and taking the sum over all $\jmath$, we get

$$(1^p+2^p+\cdots+6^p)\left(\frac{1}{6}-\varepsilon\right)n < \Sigma_\jmath\left(\jmath^p|\{i\le n \mid j_\alpha(i)=\jmath\}|\right) < (1^p+2^p+\cdots+6^p)\left(\frac{1}{6}+\varepsilon\right)n.$$

Since $\Sigma_\jmath\left(\jmath^p|\{i\le n \mid j_\alpha(i)=\jmath\}|\right)$ is simply $\Sigma_{i=1}^n j_\alpha(i)^p$, this easily entails the assertion. $\square$

The reader may wonder if the converse to the above implication is true. In order to answer this question, let us introduce the following example:

Example 3. *For any given $p\ge 1$, we define an infinite sequence $\mathbf{M}_p$ of Ducci matrices by putting $\mathbf{M}_p := \langle M_1\rangle^\frown(\vec{M}_1{}^\frown\vec{M}_2{}^\frown\cdots{}^\frown\vec{M}_{5^p-1})^\omega$, where a finite sequence $\vec{M}_i$ $(1\le i\le 5^p-1)$ is given by*

$$\vec{M}_i := \begin{cases} \langle M_1,M_1,M_2,M_4,M_5,M_5,M_6\rangle & (i\le 3^p-1)\\ \langle M_1,M_1,M_2,M_4,M_5,M_6\rangle & (3^p-1<i\le 5^p-3^p)\ .\\ \langle M_1,M_2,M_4,M_5,M_6\rangle & (5^p-3^p<i\le 5^p-1)\end{cases}$$

With the help of Proposition 1, one can check that for each $p\ge 1$, the (eventually periodic) sequence $\mathbf{M}_p$ is realized as the Ducci matrix sequence expansion of some $\alpha_p\in(0,1)\setminus\mathbb{Q}$. For example, we have

$$\begin{aligned}\mathbf{M}_1 = \langle M_1\rangle^\frown\langle &M_1,M_1,M_2,M_4,M_5,M_5,M_6,M_1,M_1,M_2,M_4,M_5,M_5,M_6,\\ &M_1,M_2,M_4,M_5,M_6,M_1,M_2,M_4,M_5,M_6\rangle^\omega,\\ \alpha_1 = [0;2,&\underline{2,2,3,2,2,4,5,4},\underline{2,2,3,2,2,4,5,4},\underline{2,2,3,2,2,4,5,4},\dots].\end{aligned}$$

By construction, M_3 does not appear in the Ducci matrix sequence expansion $\mathbf{M}_p$ of α_p. Therefore, we have $\lim_{n\to\infty}|\{i\le n \mid j_{\alpha_p}(i)=3\}|/n = 0\ne 1/6$.

One can easily check that the number of occurrences of M_{2i} $(i=1,2,3)$ in the finite sequence $\vec{M}_1{}^\frown\vec{M}_2{}^\frown\cdots{}^\frown\vec{M}_{5^p-1}$ is 5^p-1. In this finite sequence, M_1 appears $2\cdot 5^p-3^p-1$ times and M_5 appears 5^p+3^p-2 times. By the definition of $\mathbf{M}_p$, it is thus clear that

$$\begin{aligned}\Sigma_{i=6m(5^p-1)+2}^{6(m+1)(5^p-1)+1} j_{\alpha_p}(i)^p &= (2\cdot 5^p-3^p-1)\cdot 1^p+(5^p-1)\cdot 2^p+(5^p-1)\cdot 4^p\\ &\quad +(5^p+3^p-2)\cdot 5^p+(5^p-1)\cdot 6^p\\ &= (5^p-1)\cdot(1^p+2^p+3^p+4^p+5^p+6^p)\\ &= \Sigma_{i=6m(5^p-1)+2}^{6(m+1)(5^p-1)+1} i^p,\end{aligned}$$

for every $m\in\mathbb{Z}_{\ge 0}$. From this, it is not hard to conclude $\lim_{n\to\infty}\sqrt[p]{\Sigma_{i=1}^n j_{\alpha_p}(i)^p/n} = \sqrt[p]{(1^p+2^p+\cdots+6^p)/6}$ for this α_p.

For any given positive integer q, the above argument also proves that

$$\begin{aligned}&\lim_{n\to\infty}\sqrt[q]{\frac{\Sigma_{i=1}^n j_{\alpha_p}(i)^q}{n}}\\ &=\sqrt[q]{\frac{(2\cdot 5^p-3^p-1)\cdot 1^q+(5^p-1)\cdot 2^q+(5^p-1)\cdot 4^q+(5^p+3^p-2)\cdot 5^q+(5^p-1)\cdot 6^q}{6(5^p-1)}}\\ &=\sqrt[q]{\frac{(5^p-1)\cdot(1^q+2^q+4^q+5^q+6^q)+5^p-3^p+3^p\cdot 5^q-5^q}{6(5^p-1)}}.\end{aligned}$$

In order for this value to be equal to

$$\sqrt[q]{\frac{1^q + 2^q + \cdots + 6^q}{6}} = \sqrt[q]{\frac{(5^p - 1)\cdot(1^q + 2^q + 3^q + 4^q + 5^q + 6^q)}{6(5^p - 1)}},$$

$q \geq 1$ has to satisfy $5^p - 3^p + 3^p \cdot 5^q - 5^q = 5^p \cdot 3^q - 3^q$. An elementary computation shows that this happens only when q is equal to p. Hence for any q different from p, we have $\lim_{n\to\infty} \sqrt[q]{\Sigma_{i=1}^n j_{\alpha_p}(i)^q/n} \neq \sqrt[q]{(1^q + 2^q + \cdots + 6^q)/6}$.

From this example, we can conclude as follows:

Theorem 9. *For every positive integer $p \geq 1$, the condition "$\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$" is weaker than "$\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ for every $\jmath$".*

Moreover, the statements "$\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$" for $p = p_1$ and for $p = p_2$ are independent from each other whenever p_1 and p_2 are distinct. □

Before ending this section, let us present variants to the preceding results. As in the proof of Theorem 6, one can prove

Theorem 10. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, the following are equivalent:*

1. $\limsup_{n\to\infty} \dfrac{|\{i \leq n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$
2. $\liminf_{n\to\infty} \dfrac{|\{m \in \mathbb{Z}_{>0} \mid \Sigma_{i=1}^m a_i \leq n\}|}{n} = 0;$
3. $\dfrac{\Sigma_{i=1}^n a_i}{n}$ *is unbounded.* □

Also, one can show as in the proof of Theorem 8 that

Theorem 11. *If $\limsup_{n\to\infty} |\{i \leq n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ holds for a given $\alpha \in (0,1) \setminus \mathbb{Q}$, then this α satisfies $\limsup_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n \geq 1/6$ for every $\jmath \in \{1, 2, \ldots, 6\}$.* □

Observe that Example 2 witnesses the failure of the converse to Theorem 8 but also to Theorem 11.

5 Measure theory. Given an irrational number $\alpha > 0$, how often for a fixed index $\jmath$, does the matrix $M_\jmath$ appear in its Ducci matrix sequence expansion $M_{j_\alpha(1)}, M_{j_\alpha(2)}, \ldots$? We shall present several measure theoretic approaches around this problem. In this section, measure refers to the Lebesgue measure on $\mathbb{R}$.

Our first result is the next

Theorem 12. *The following set is of measure zero:*

$$\{\alpha \in (0,1) \setminus \mathbb{Q} \mid \exists \jmath\, (j_\alpha(n) = \jmath \text{ holds for only finitely many } n)\}.$$

Before proving this theorem, let us remind the reader of the following result (For a proof, see, e.g. [7, Theorem 29]):

Theorem 13. *The set of all numbers in the interval $(0,1)$ with bounded elements is of measure zero.* □

Therefore, it is sufficient to prove that every element α from the set that we are concerning has bounded elements.

Proof of Theorem 12. Let α and $\jmath$ be such that only finitely many n satisfy $j_\alpha(n) = \jmath$. Then there exists an N such that $j_\alpha(n) \neq \jmath$ holds for all $n \geq \Sigma_{i=1}^N a_i$. In particular, we have $j_\alpha(1 + \Sigma_{i=1}^N a_i) \neq \jmath$. Since the integer part of the vector $D^{\Sigma_{i=1}^N a_i}(0, \alpha, 1)$ is a_{N+1}, if $a_{N+1} \geq 2$, then Proposition 1 implies that the type of $D^{1+\Sigma_{i=1}^N a_i}(0, \alpha, 1)$ is 1 plus the type of $D^{\Sigma_{i=1}^N a_i}(0, \alpha, 1)$ modulo 6. In view of Proposition 2, this yields that $j_\alpha(2 + \Sigma_{i=1}^N a_i) \equiv j_\alpha(1 + \Sigma_{i=1}^N a_i) + 1 \pmod 6$. Now, since the integer part of $D^{1+\Sigma_{i=1}^N a_i}(0, \alpha, 1)$ is $a_{N+1} - 1$, if $a_{N+1} - 1 \geq 2$, the same reasoning proves $j_\alpha(3 + \Sigma_{i=1}^N a_i) \equiv j_\alpha(2 + \Sigma_{i=1}^N a_i) + 1 \pmod 6$. Repeating in this manner, we see that, for $m = 1, 2, \ldots, a_{N+1} - 1$

$$j_\alpha(1 + m + \Sigma_{i=1}^N a_i) \equiv j_\alpha(1 + \Sigma_{i=1}^N a_i) + m \pmod 6.$$

These arguments, together with the assumption that $j_\alpha(n) \neq \jmath$ holds for all $n \geq \Sigma_{i=1}^N a_i$, proves that $j_\alpha(1 + \Sigma_{i=1}^N a_i) + m \not\equiv \jmath \pmod 6$ for $0 \leq m \leq a_{N+1} - 1$. This clearly entails that $a_{N+1} < 6$.

Continuing this way, we reach the conclusion that $a_{N+l} < 6$ holds for all $l \geq 1$. As every element a_i satisfies $a_i \leq \max\{a_0, a_1, \ldots, a_N, 6\}$, this finishes the proof. □

In the last section, we considered the condition "$\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ for every $\jmath \in \{1, 2, \ldots, 6\}$". One can of course study this condition from the viewpoint of measure theory:

Question 1. *Does the following property hold for a.e.* $\alpha \in (0,1) \setminus \mathbb{Q}$*?*

$$\lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6} \text{ for every } \jmath = 1, 2, \ldots, 6.$$

We do not know the answer to this question. Note however that Theorems 10 and 11, together with the fact that $\Sigma_{i=1}^n a_i/n$ is unbounded almost everywhere [7, pp. 94], shows that

Theorem 14. *Almost every* $\alpha \in (0,1) \setminus \mathbb{Q}$ *has the following property:*

$$\limsup_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} \geq \frac{1}{6} \text{ for every } \jmath \in \{1, 2, \ldots, 6\}.$$ □

Similarly, one will be able to conclude that $\liminf_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n \leq 1/6$ for every $\jmath \in \{1, 2, \ldots, 6\}$.

Here is another partial result:

Theorem 15. *For almost every* $\alpha \in (0,1) \setminus \mathbb{Q}$*, the sequence* $|\{i \leq n \mid j_\alpha(i) = \jmath\}|/n$ *has* $1/6$ *as an accumulation point for every* $\jmath \in \{1, 2, \ldots, 6\}$*.*

Proof. The following theorem plays an important role in our proof:

Theorem 16 ([7, Theorem 30])**.** *Let* $\varphi(i)$ *be an arbitrary positive function with natural argument* n*. If the series* $\Sigma_{i=1}^\infty 1/\varphi(i)$ *diverges, then for almost every* α*, infinitely many* i *satisfy the inequality* $a_i \geq \varphi(i)$*.* □

Applying this theorem for $\varphi(i) := Ki$ ($K \in \mathbb{Z}_{>0}$) and taking the countable intersection for all positive integers K, one sees that almost every $\alpha \in (0,1) \setminus \mathbb{Q}$ has the following property: For any positive real number x, infinitely many i satisfy the inequality $a_i \geq xi$. We claim that $1/6$ is an accumulation point of the sequence $|\{i \leq n \mid j_\alpha(i) = \jmath\}|/n$ for every $\jmath \in \{1, 2, \ldots, 6\}$ whenever $\alpha \in (0,1) \setminus \mathbb{Q}$ has the above property. To prove our claim, choose an α having the above property and take $\jmath$ from $\{1, 2, \ldots, 6\}$ arbitrarily.

Let $\varepsilon > 0$ and $N \in \mathbb{Z}_{>0}$ be given. What we have to show is that there exists an $n' \geq N$ such that $||\{i \leq n' \mid j_\alpha(i) = \jmath\}|/n' - 1/6| < \varepsilon$ holds. By the assumption on α, there exists an $n > \max\{N, 5/3\varepsilon\}$ satisfying the inequality $a_n > 2n/\varepsilon$. We claim that $a_1 + \cdots + a_n$ ($\geq n > N$) works as n'.

For $m \geq 2$, Proposition 1 tells us that the sequence $j_\alpha((\Sigma_{s=1}^{m-1} a_s) + 1), j_\alpha((\Sigma_{s=1}^{m-1} a_s) + 2), \ldots, j_\alpha((\Sigma_{s=1}^{m-1} a_s) + a_m) = j_\alpha(\Sigma_{s=1}^{m} a_s)$ is periodic. This periodicity enables us to estimate the number of occurrences of $\jmath$ in this sequence:

$$\left\lfloor \frac{a_m}{6} \right\rfloor < |\{\Sigma_{s=1}^{m-1} a_s < i \leq \Sigma_{s=1}^{m} a_s \mid j_\alpha(i) = \jmath\}| < \left\lfloor \frac{a_m}{6} \right\rfloor + 1.$$

When $m = 1$, periodic part is $j_\alpha(2), j_\alpha(3), \ldots, j_\alpha(a_1)$. Hence we have

$$\left\lfloor \frac{a_1 - 1}{6} \right\rfloor < |\{i \leq a_1 \mid j_\alpha(i) = \jmath\}| < \left\lfloor \frac{a_1 - 1}{6} \right\rfloor + 2.$$

These arguments, together with inequalities $2n/\varepsilon < a_n \leq \Sigma_{s=0}^{n} a_s$ and $5/3\varepsilon < n \leq \Sigma_{s=1}^{n} a_s$, proves that

$$\begin{aligned} \frac{|\{i \leq \Sigma_{s=1}^{n} a_s \mid j_\alpha(i) = \jmath\}|}{\Sigma_{s=1}^{n} a_s} &\leq \frac{\frac{\Sigma_{s=1}^{n} a_s}{6} + n + \frac{5}{6}}{\Sigma_{s=1}^{n} a_s} \\ &< \frac{1}{6} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \frac{1}{6} + \varepsilon. \end{aligned}$$

Similarly, one can prove

$$\frac{|\{i \leq \Sigma_{s=1}^{n} a_s \mid j_\alpha(i) = \jmath\}|}{\Sigma_{s=1}^{n} a_s} > \frac{1}{6} - \varepsilon.$$

This completes the proof. □

Note that Theorem 14 follows also from this result as a corollary.

The rest of this paper studies another possible question concerning Question 1. Specifically, we ask if it gives rise to any difference to replace "for *every* $\jmath$" in Question 1 with "for *some* $\jmath$". The next results shows that the notion of parity plays a crucial part in this investigation.

Theorem 17. *Let $\jmath_1, \jmath_2 \in \{1, 2, \ldots, 6\}$ be two distinct numbers with the same parity. If $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath_1\}|/n = 1/6$ holds a.e., then so does $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath_2\}|/n = 1/6$.*

Proof. We give the proof only for the case $\jmath_1 = 1$ and $\jmath_2 = 3$; the other cases are left to the reader.

For $\jmath \in \{1, 2, \ldots, 6\}$, put

$$N_\jmath := \left\{ \alpha \in (0,1) \setminus \mathbb{Q} \;\middle|\; \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} \text{ does not converge to } \frac{1}{6} \right\}.$$

What we need to prove is that if N_1 is of measure zero, then so is N_3.

Define a function $g : (0,1) \to (0,1/3)$ by $g(x) = x/(2x+1)$. Clearly, g is a homeomorphism on $(0,1)$. An elementary computation shows that g is invertible and that both g and g^{-1} are *bi-Lipschitz*, in particular both send measure zero sets to measure zero sets. Since we trivially have $N_3 = \{\alpha \in N_3 \mid a_1 \geq 3\} \cup \{\alpha \in N_3 \mid a_1 = 1 \text{ or } 2\}$, in order to prove that N_3 is of measure zero, it suffices to show the next two identities:

$$\{\alpha \in N_3 \mid a_1 \geq 3\} = g(N_1);$$
$$\{\alpha \in N_3 \mid a_1 = 1 \text{ or } 2\} = g^{-2}(\{\alpha \in N_1 \mid a_1 = 5 \text{ or } 6\}).$$

Observe that g maps $[0; a_1, a_2, a_3, \dots]$ to $[0; a_1 + 2, a_2, a_3, \dots]$, i.e. adds 2 to the first element of the continued fraction expansion. This observation, together with Proposition 1, leads to the next

Lemma 3. *For every $\alpha \in (0,1)$ and $n \geq 1$, the type of $D^{n+2}(0, g(\alpha), 1)$ is the type of $D^n(0, \alpha, 1)$ plus 2 (modulo 6).* □

This lemma implies that α satisfies $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = 1\}|/n = 1/6$ if and only if $g(\alpha)$ satisfies $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = 3\}|/n = 1/6$. In other words, $\alpha \in N_1 \iff g(\alpha) \in N_3$. The desired two identities follows from this easily. □

Remark 3. *In the above proof, the value $1/6$ did not play any role. Indeed, the statement remains valid even when $1/6$ is replaced by any other real number in $(0,1)$.*

In the proof of the preceding theorem, Lemma 3 played an important role. A similar statement when the parity of $\jmath_1$ is different from that of $\jmath_2$ is no longer true. This makes it difficult to prove an analogous statement to Theorem 17 for a pair of different parity.

Question 2. *Let two distinct numbers $\jmath_1, \jmath_2 \in \{1, 2, \dots, 6\}$ with different parity be given. If $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath_1\}|/n = 1/6$ holds a.e., does $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath_2\}|/n = 1/6$ also hold a.e. ?*

If this (technical) question has a positive answer, then we can actually replace "for *every* $\jmath$" with (seemingly weaker) "for *some* $\jmath$" in the statement of Question 1. Indeed, suppose there exists some $\jmath \in \{1, 2, \dots, 6\}$ such that $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ holds almost everywhere. Then, we know from Theorem 17 and the positive answer to Question 2 that $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath'\}|/n = 1/6$ holds almost everywhere for every $\jmath'$. Hence we see that $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath'\}|/n = 1/6$ for every $\jmath' = 1, 2, \dots, 6$ at almost everywhere.

Note that an affirmative answer to the following question solves Question 2 positively:

Question 3. *Are the following two conditions equivalent for every $\jmath \in \{1, 2, \dots, 6\}$?*

1. $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) \text{ and } \jmath \text{ have the same parity}\}|/n = 1/2$ *holds a.e.;*

2. $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ *holds a.e.*

One can easily deduce the first condition from the second one. Indeed, 2 and Theorem 17 imply that we have $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath'\}|/n = 1/6$ a.e. for every $\jmath'$ having the same parity as $\jmath$. As the conjunction of finitely many properties that hold almost everywhere again holds almost everywhere, 1 now follows.

We now deduce Question 2 assuming that 2 follows from 1: Let us suppose $\lim_{n\to\infty} |\{i \leq n \mid j_\alpha(i) = \jmath_1\}|/n = 1/6$ holds a.e. for a given $\jmath_1 \in \{1, 2, \dots, 6\}$. Take a $\jmath_2 \in \{1, 2, \dots, 6\}$

having different parity from that of $\jmath_1$. We wish to prove that $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) = \jmath_2\}|/n = 1/6$ holds a.e.

Since 2 implies 1, $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i)$ and $\jmath_1$ have the same parity$\}|/n = 1/2$ holds almost everywhere. Now observe that for an arbitrary $\alpha \in (0,1) \setminus \mathbb{Q}$, $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i)$ is even$\}|/n = 1/2$ holds if and only if $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i)$ is odd$\}|/n = 1/2$ holds. Therefore, we have

$$\lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) \text{ is even}\}|}{n} = \frac{1}{2} \ a.e. \Longleftrightarrow \lim_{n\to\infty} \frac{|\{i \le n \mid j_\alpha(i) \text{ is odd}\}|}{n} = \frac{1}{2} \ a.e.$$

As $\jmath_1$ and $\jmath_2$ have different parity, it follows that $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i)$ and $\jmath_2$ have the same parity$\}|/n = 1/2$ holds a.e. By applying our assumption that the condition 2 follows from 1, we get the desired result.

One can also ask a

Question 4. *Do we have* $\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$ *almost everywhere?*

We also do not know the answer to this question. What we can certainly say is that, in view of Proposition 3, Question 4 is at least as likely to be true as Question 1. Although we know from Theorem 9 that the condition "$\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$" for $p = p_1$ and for $p = p_2$ are independent from each other whenever p_1 and p_2 are distinct, there might be some relationship between statements "$\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$ holds almost everywhere" for $p = p_1$ and for $p = p_2$ even when $p_1 \ne p_2$; It is interesting to see how the strength of the above statement changes as the value of p increases.

Our final remark is on "mod 2". Instead of modulo 6, one can consider indexes of Ducci matrices by modulo 2 and formulate statements for them, e.g. $j_\alpha(i) \equiv \jmath \pmod 2$ in place of $j_\alpha(i) = \jmath$. Even if we do so, the results in this paper remain valid (with trivial modifications).

Acknowledgement. The author wishes to thank the referee for his useful comments that improved the quality of the presentation of this work.

Note Added in Proof. At the time of submission of the manuscript, the author was not aware of the following fact: Almost every $\alpha = [0; a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$ satisfies $\lim_{n\to\infty} \Sigma_{i=1}^n a_i/n = \infty$. This appears to be a standard result in ergodic theory.

On the other hand, Theorem 6 states that for each $\alpha \in (0,1) \setminus \mathbb{Q}$, $\Sigma_{i=1}^n a_i/n$ diverges if and only if $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ holds. Combining these two results, we thus get a

Corollary 3. $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ *holds at almost every* $\alpha \in (0,1) \setminus \mathbb{Q}$. □

In view of Theorem 8 and Proposition 3, this in turn gives us a

Corollary 4.
1. $\lim_{n\to\infty} |\{i \le n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ *for every* $\jmath = 1, 2, \dots, 6$ *holds at almost every* $\alpha \in (0,1) \setminus \mathbb{Q}$;
2. *For every positive integer* p, $\lim_{n\to\infty} \sqrt[p]{\Sigma_{i=1}^n j_\alpha(i)^p/n} = \sqrt[p]{(1^p + 2^p + \cdots + 6^p)/6}$ *holds at almost every* $\alpha \in (0,1) \setminus \mathbb{Q}$. □

1 and 2 positively answer Questions 1 and 4, respectively. Questions 2 and 3 are vacuously true.

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Communicated by *Jarkko Kari*