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EXISTENCE OF UNBOUNDED SOLUTIONS TO A ONE DIMENSIONAL ISENTROPIC PERIODIC FLOW OF A COMPRESSIBLE VISCOUS FLUID WITH SELF-GRAVITATION

Masahiro Sawada and Yoshitaka Yamamoto

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Abstract. We consider a one dimensional isentropic periodic flow of a compressible viscous fluid driven by a self-gravitation of the fluid. We show the existence of an unbounded solution of a system describing the flow. A sufficient condition for the unboundedness is given in terms of the initial values of an energy form.

1 Introduction Let us consider a one dimensional isentropic flow of a compressible viscous fluid in the Lagrangian mass coordinates:

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x (a v^{-\gamma}) - \mu \partial_x \left(\dfrac{\partial_x u}{v} \right) = \mathcal{G}, \end{cases} \tag{1.1}$$

where specific volume v, assumed to take positive values, and velocity u of the fluid are unknown functions of the time and space variables $t \geq 0$ and $x \in \mathbf{R}$, pressure $av^{-\gamma}$ a function of v with constants $a > 0$ and $\gamma \geq 1$, and $\mu > 0$ the viscosity constant. The second member $\mathcal{G}$ is an external force specified below. We are mainly concerned with the so-called isentropic flow, i.e., $\gamma > 1$, though, we occasionally refer to the isothermal flow, i.e. $\gamma = 1$ for the sake of comparison.

The initial or initial-boundary value problem for (1.1) with a prescribed forcing term $\mathcal{G}$ has been studied by several authors. Since the pioneering work of Kanel' [3], showing the existence of global bounded solutions to the system (1.1) on the whole line with $\mathcal{G} \equiv 0$, the boundedness is one of the crucial keys to study the asymptotic behavior of the solutions. Closely related with the present paper are the works of Matsumura and Nishida [4], and Matsumura and Yanagi [5]. In [4] it is shown that the isothermal system on a finite interval with a general bounded forcing term $\mathcal{G}$ has a unique global bounded solution for any smooth initial data. In [5] a similar result was obtained for the isentropic system but on the assumption of smallness of $\gamma - 1$ depending on the data. Both the results fail to mention whether an unbounded solution exists or not for the isentropic system with a bounded forcing term.

This paper handles the system (1.1) under the L-periodic condition:

$$v(t, x + L) = v(t, x), \quad u(t, x + L) = u(t, x) \tag{1.2}$$

with a rather special forcing term depending on the unknowns:

$$\mathcal{G}(t, x) = -\frac{4\pi G}{\bar{v}} \partial_x \int_0^L K_L(x, y)(v(t, y) - \bar{v}) dy, \tag{1.3}$$

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where $K_L(x,y)$ is the Green kernel of the operator $-\partial_x^2$ on the L-periodic functions with average 0:

$$K_L(x,y) = \sum_{n=1}^{\infty} \frac{L}{2\pi^2 n^2} \cos \frac{2\pi n}{L}(x-y),$$

or

$$K_L(x,y) = -\frac{|x-y|}{2} + \frac{(x-y)^2}{2L} + \frac{L}{12}, \quad 0 \leq x, y \leq L, \tag{1.4}$$

$\bar{v}$ the average of the specific volume:

$$\bar{v} = \frac{1}{L}\int_0^L v(t,x)dx,$$

and $G > 0$ the gravitational constant. This is the representation in the Lagrangian mass coordinates of a force field that takes into account only the part of Newton's gravitation corresponding to the disturbance in an infinite homogeneous fluid, and is often adopted in the classical theory of gravitational instability for the fluid. See Weinberg [7], Chapter 15. Notice that the field is consistent with static equilibria of the fluid.

Since the average $\bar{v}$ as well as that of u is a constant of motion in view of (1.1), the forcing term (1.3) is a bounded function of the variables t and x. This enables us to show the boundedness of any smooth solutions to the isothermal system just in the same manner as in [4]. As for the isentropic system, however, the situation proves to be quite different. Indeed, on the assumption $1 < \gamma < 2$ we show the existence of unbounded solutions in the sense that

$$\sup_{t,x} v(t,x) = \infty.$$

To be precise we present an initial condition for unbounded solutions in terms of the form for a state (v,u) given by

$$\mathcal{E}(v,u) = \int_0^L \frac{1}{2}u(x)^2 dx + \mathcal{E}(v) \tag{1.5}$$

with

$$\mathcal{E}(v) = \int_0^L a\left(\frac{v(x)-\bar{v}}{\bar{v}^\gamma} - \frac{v(x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}\right)dx - \frac{2\pi G}{\bar{v}}\int_0^L\int_0^L K_L(x,y)(v(x)-\bar{v})(v(y)-\bar{v})dxdy. \tag{1.6}$$

This form, called the energy form associated with the system (1.1)–(1.3), is decreasing and bounded along the orbit of a solution to the system, which turns out to be the key to find out the unboundedness condition.

The paper is organized as follows. In Section 2, after a brief comment on the class of solutions concerned, we present two theorems. One refers to the structure of the whole stationary solutions. The other constitutes the main part of the paper showing an initial condition for unbounded solutions to the Cauchy problem. In Section 3 we study the large time behavior of bounded solutions and show a reason why the stationary problem is inevitably related to the unboundedness of solutions to the Cauchy problem. In Section 4 we study the structure of the whole stationary solutions with the comparison of the values of energy form at the stationary solutions. From this together with the decreasing property of energy form we formulate an initial condition for unbounded solutions. Finally in Section 5, focusing on the behavior of the energy form near the stationary solutions, we supplement the condition for unboundedness to make it meaningful.

This paper completes the preceding one [6] with details of the unboundedness of solutions to the isentropic system. By replacing (1.6) with

$$\begin{aligned}\mathcal{E}(v) = &\int_0^L a\left(\frac{v(x)-\bar{v}}{\bar{v}} - \log\frac{v(x)}{\bar{v}}\right)dx \\ &- \frac{2\pi G}{\bar{v}}\int_0^L\int_0^L K_L(x,y)(v(x)-\bar{v})(v(y)-\bar{v})dxdy\end{aligned}$$

some results of the present paper are, with natural modifications, valid also for the isothermal system. See [6].

2 Notation and main results For a nonnegative integer m and a positive number L let C^m be the space of m times continuously differentiable periodic real-valued functions on $\mathbf{R}$ with period L, and H^m the Sobolev space of locally square integrable L-periodic real-valued functions on $\mathbf{R}$ equipped with scalar product

$$(h_1,h_2)_{H^m} = \sum_{j=0}^m \int_0^L \partial_x^j h_1(x)\partial_x^j h_2(x)dx$$

and norm $\|h\|_{H^m} = \sqrt{(h,h)_{H^m}}$. We write $H^0 = L^2$ as usual. Let s be a nonnegative integer and X a Banach space with norm $\|\cdot\|$. The space of s-times continuously differentiable functions on $[0,\infty)$ with values in X is denoted by $C^s([0,\infty);X)$. $H^s_{\rm loc}(0,\infty;X)$ denotes the space of X-valued strongly measurable functions on $[0,\infty)$ whose distributional derivatives up to order s are locally square integrable, i.e.,

$$\int_0^T \|\partial_t^k u(t)\|^2 dt < \infty \quad \text{for any } k=0,\ldots,s \text{ and } T>0.$$

Noting that the forcing term (1.3) is a bounded function of the variables t and x on the time interval of existence for a solution, we are allowed to consider a unique global solution for the Cauchy problem of (1.1)–(1.3) having initial value $(v_0,u_0)\in H^1\times H^1$ with $v_0>0$ arbitrarily given at $t=0$, as for the initial-boundary value problem on a finite interval supplemented by solid boundary condition with a general bounded forcing term. In what follows the solution of the Cauchy problem is meant by a unique global solution having the property

$$\begin{cases} v\in C^1([0,\infty);L^2)\cap C^0([0,\infty);H^1), \quad v(t,\,\cdot\,)>0,\\ u\in H^1_{\rm loc}(0,\infty;L^2)\cap L^2_{\rm loc}(0,\infty;H^2).\end{cases}$$

Without loss of generality we may assume that the average of u vanishes, taking $(v,u-\bar{u})$ as new unknown functions, if necessary.

In order to present an initial condition for unbounded solutions we first refer to the structure of the stationary solutions to (1.1)–(1.3). Noting that the average $\bar{v}$ of v is a constant of motion, we consider the stationary solutions on the following manifold in $H^1\times H^1$ parametrized by a positive number V:

$$M_V = \{(v,u)\in H^1\times H^1;\, v>0,\ \ \bar{v}=V,\ \ \bar{u}=0\}.$$

Clearly, the trivial solution $(V, 0)$ lies in M_V. A non-trivial stationary solution, if exists, has the least period L/j for some positive integer j. Let us now introduce a function I_γ on the interval $(0, (\gamma-1)^{-1/2})$ expressed as

$$I_\gamma(\theta) = \theta \int_0^1 \frac{1}{\sqrt{1-y}} \frac{1}{f_+(F_+{}^{-1}(\theta^2 y))}\, dy + \theta \int_0^1 \frac{1}{\sqrt{1-y}} \frac{1}{f_-(F_-{}^{-1}(\theta^2 y))}\, dy, \tag{2.1}$$

where the functions $f_+(r)$, $F_+(r)$ on $r \geq 0$, and $f_-(r)$, $F_-(r)$ on $0 \leq r < 1$ are given by

$$f_+(r) = 1 - (1+r)^{-1/\gamma}, \qquad F_+(r) = \int_0^r f_+(s)\, ds,$$
$$f_-(r) = -\{1 - (1-r)^{-1/\gamma}\}, \quad F_-(r) = \int_0^r f_-(s)\, ds.$$

As shown by **Lemma 4** below in Section 4, I_γ is a monotone increasing function with $I_\gamma(\theta) > \sqrt{2\gamma}\pi$ provided that $1 < \gamma < 2$. Moreover, $I_\gamma(\theta)$ has a finite limit as $\theta \to (\gamma-1)^{-1/2} - 0$.

Theorem 1 Assume $1 < \gamma < 2$. For $V > 0$ let $k_{\min}$ and $k_{\max}$, respectively, be the smallest and the largest integers j satisfying

$$\left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2} < \frac{L}{j} < \frac{I_\gamma((\gamma-1)^{-1/2} - 0)}{\sqrt{2\gamma}\pi} \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}. \tag{2.2}$$

Then, for $j = k_{\min}, \ldots, k_{\max}$ there exists on M_V a stationary solution of (1.1)–(1.3) with least period L/j. The whole stationary solutions lying in M_V except for the trivial one are given by $(\tilde{v}^{(j)}(\,\cdot\, - \alpha), 0)$, $0 \leq \alpha < L/j$, $j = k_{\min}, \ldots, k_{\max}$, where $(\tilde{v}^{(j)}, 0)$ is one of the stationary solutions with least period L/j.

Remark 1 When $V \leq \left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma}$, no integer satisfies the condition (2.2), and hence the stationary problem admits on M_V only the trivial solution. When $\left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma} < V < \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$, (2.2) holds with $j = 1$, and hence $k_{\min} = 1$, while when $V \geq \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$, $k_{\min}$, if it makes sense, must be greater than or equal to 2.

Let us recall the energy form (1.5) with (1.6). By L-periodicity of v we have $\mathcal{E}(v(\,\cdot\, - \alpha)) = \mathcal{E}(v)$ for any $\alpha \in \mathbf{R}$. The following theorem gives an initial condition for unbounded solutions to the isentropic system (1.1)–(1.3) with $1 < \gamma < 2$.

Theorem 2 Assume $1 < \gamma < 2$. Let $V \geq \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$ and $\tilde{v}^{(k_{\min})}$ be as in **Theorem 1**.
(i) The subset of $H^1 \times H^1$ given by

$$A_V = \left\{ (v,u) \in M_V \;\middle|\; \mathcal{E}(v,u) < \begin{cases} \mathcal{E}(\tilde{v}^{(k_{\min})}), & \text{if integers } j \text{ with (2.2) exist,} \\ 0, & \text{otherwise} \end{cases} \right\} \tag{2.3}$$

is nonempty.
(ii) Any solution of (1.1)–(1.3) with initial value from A_V is unbounded, i.e.,

$$\sup_{t,x} v(t,x) = \infty.$$

Remark 2 In view of the decreasing property of the energy form shown by **Lemma 1** in the following section the statement of **Theorem 2** suggests that $\mathcal{E}(\tilde{v}^{(k_{\min})})$ if it makes sense or else $\mathcal{E}(V) = 0$ is minimal amongst the values of the energy form evaluated at the stationary solutions on M_V. This itself is true also for the case $V < \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$ as shown by **Proposition 1** in Section 4, however, in this case we fail to refer to the existence or nonexistence of unbounded solutions for some technical reasons. See **Remark 3** in the final section.

3 Large time behavior of bounded solutions As a preliminary but vital step, we devote this section to the study of the global behavior of a solution of (1.1)–(1.3) subject to

$$\sup_{t,x} v(t,x) < \infty. \tag{3.1}$$

The results in the present section have already been given in [6] with rather detailed proofs, however, we give them for the sake of completeness.

We first show that the energy form (1.5) with (1.6) is non-increasing along the orbits of solutions. This is true for any $\gamma > 1$ regardless of the boundedness of solutions.

Lemma 1 For a solution (v, u) of (1.1)–(1.3) put

$$E(t) = \mathcal{E}(v(t,\,\cdot\,), u(t,\,\cdot\,)), \quad t \geq 0. \tag{3.2}$$

Then we have

$$\frac{dE}{dt}(t) = -\mu \int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx \leq 0, \quad \inf_t E(t) = E(\infty) > -\infty. \tag{3.3}$$

Proof: Taking the derivative of E and then using the symmetry of the integral kernel K_L, we obtain an expression for the derivative dE/dt as

$$\int_0^L \Bigg\{ u(t,x)\partial_t u(t,x) + \left(a\bar{v}^{-\gamma} - av(t,x)^{-\gamma}\right)\partial_t v(t,x) \\ - \frac{4\pi G}{\bar{v}} \int_0^L K_L(x,y)(v(t,y)-\bar{v})dy\,\partial_t v(t,x) \Bigg\} dx.$$

After substituting $\partial_x u$ for $\partial_t v$, by integration by parts we get

$$\frac{dE}{dt}(t) = \int_0^L u(t,x) \left\{ \partial_t u(t,x) + \partial_x \left(av(t,x)^{-\gamma}\right) + \frac{4\pi G}{\bar{v}} \partial_x \int_0^L K_L(x,y)(v(t,y)-\bar{v})dy \right\} dx.$$

Using the second equation of (1.1), by integration by parts we obtain the desired equality for dE/dt. The boundedness of E from below follows from

$$\frac{v-\bar{v}}{\bar{v}^\gamma} - \frac{v^{1-\gamma} - \bar{v}^{1-\gamma}}{1-\gamma} \geq 0,$$

the positivity of v and the boundedness of the kernel K_L. $\square$

Integrating the equality (3.3) over $(0, \infty)$, we obtain the following.

Corollary of Lemma 1 We have

$$\mu \int_0^\infty \left(\int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx \right) dt = E(0) - E(\infty) < \infty. \tag{3.4}$$

The following lemma shows that if $1 < \gamma \leq 2$, the upper bound of v controls the H^1 norm as well as the lower bound of v of a solution. Notice that the same result holds true of the isothermal system without any assumptions on a priori bounds of a solution. See Matsumura and Nishida [4].

Lemma 2 Assume $1 < \gamma \leq 2$. For a solution (v, u) of (1.1)–(1.3) with $\bar{u} = 0$, if it is bounded in the sense of (3.1), then we have

$$\sup_t \|v(t,\,\cdot\,)\|_{H^1} < \infty, \quad \sup_t \|u(t,\,\cdot\,)\|_{H^1} < \infty, \quad \inf_{t,x} v(t,x) > 0. \tag{3.5}$$

Proof: We consider the forcing term (1.3) as a bounded function of the variables t and x, and make use of the equalities

$$\begin{aligned} &\frac{d}{dt} \int_0^L \left\{ \frac{1}{2} u(t,x)^2 + a \left(\frac{v(t,x) - \bar{v}}{\bar{v}^\gamma} - \frac{v(t,x)^{1-\gamma} - \bar{v}^{1-\gamma}}{1-\gamma} \right) \right\} dx \qquad (3.6)\\ &= -\mu \int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx + \int_0^L \mathcal{G}(t,x) u(t,x) dx, \\ &\frac{d}{dt} \int_0^L \left(\frac{\mu}{2} \frac{\partial_x v(t,x)^2}{v(t,x)^2} - u(t,x) \frac{\partial_x v(t,x)}{v(t,x)} \right) dx \qquad (3.7)\\ &= -a\gamma \int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^{\gamma+2}} dx + \int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx - \int_0^L \mathcal{G}(t,x) \frac{\partial_x v(t,x)}{v(t,x)} dx. \end{aligned}$$

Combining the equalities as (3.6)+$(\mu/2)\times$(3.7), we prove that the quantity

$$\begin{aligned} \int_0^L \bigg\{ &\frac{1}{2} u(t,x)^2 - \frac{\mu}{2} u(t,x) \frac{\partial_x v(t,x)}{v(t,x)} + \frac{\mu^2}{4} \frac{\partial_x v(t,x)^2}{v(t,x)^2} \\ &+ a \left(\frac{v(t,x) - \bar{v}}{\bar{v}^\gamma} - \frac{v(t,x)^{1-\gamma} - \bar{v}^{1-\gamma}}{1-\gamma} \right) \bigg\} dx \end{aligned} \tag{3.8}$$

is bounded with respect to the variable t. For this purpose we need to estimate the bounds of $\int_0^L u(t,x)^2 dx$, $\int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^2} dx$ and $\int_0^L \left(\frac{v(t,x) - \bar{v}}{\bar{v}^\gamma} - \frac{v(t,x)^{1-\gamma} - \bar{v}^{1-\gamma}}{1-\gamma} \right) dx$ in terms of $\int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx$ and $\int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^{\gamma+2}} dx$. Since $\bar{u} = 0$, choosing such an $x_t \in [0, L)$ as $u(t, x_t) = 0$ for every $t \geq 0$, and then using Schwarz' lemma, for $x \in [0, L]$ we have

$$\begin{aligned} |u(t,x)| &= \left| \int_{x_t}^x \partial_y u(t,y) dy \right| \\ &\leq \int_0^L v(t,y)^{1/2} \frac{|\partial_y u(t,y)|}{v(t,y)^{1/2}} dy \\ &\leq \left(\int_0^L v(t,y) dy \right)^{1/2} \left(\int_0^L \frac{\partial_y u(t,y)^2}{v(t,y)} dy \right)^{1/2}, \end{aligned}$$

from which we obtain

$$\int_0^L u(t,x)^2 dx \leq L^2 \bar{v} \int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx. \tag{3.9}$$

As for the estimate of $\int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^2}dx$ and $\int_0^L \left(\frac{v(t,x)-\bar{v}}{\bar{v}^\gamma} - \frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}\right)dx$, we make use of the assumption (3.1) noting that $1<\gamma\le 2$. It is clear that

$$\int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^2}dx \le \left(\sup_{t,x} v(t,x)\right)^\gamma \int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^{\gamma+2}}dx.$$

For every $t\ge 0$ choosing $x_t\in[0,L)$ so that $v(t,x_t)=\bar{v}$ holds, by Schwarz' lemma we have

$$\begin{aligned}\left|\frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}\right| &= \left|\int_{x_t}^x \frac{\partial_y v(t,y)}{v(t,y)^\gamma}dy\right| \\ &\le \left(\int_0^L v(t,y)^{2-\gamma}dy\right)^{1/2}\left(\int_0^L \frac{\partial_y v(t,y)^2}{v(t,y)^{\gamma+2}}dy\right)^{1/2} \\ &\le L^{1/2}\left(\sup_{t,y} v(t,y)\right)^{(2-\gamma)/2}\left(\int_0^L \frac{\partial_y v(t,y)^2}{v(t,y)^{\gamma+2}}dy\right)^{1/2}\end{aligned}$$

for $x\in[0,L]$, and hence,

$$\begin{aligned}&\int_0^L \left(\frac{v(t,x)-\bar{v}}{\bar{v}^\gamma} - \frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}\right)dx \\ &= \left|\int_0^L \frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}dx\right| \\ &\le L^{3/2}\left(\sup_{t,x} v(t,x)\right)^{(2-\gamma)/2}\left(\int_0^L \frac{\partial_x v(t,x)^2}{v(t,x)^{\gamma+2}}dx\right)^{1/2}.\end{aligned}$$

We thus obtain a differential inequality for (3.8) showing its boundedness. Since the first three terms of the integrand of (3.8) constitute a positive quadratic form in two variables u and $\partial_x v/v$, the boundedness of v as in (3.5) follows from that of (3.8) immediately. Once the boundedness of v is obtained, that of u in H^1 follows just in the same manner as in [4]. We thus conclude (3.5). □

Let (v,u) be a solution of (1.1)–(1.3) with initial value (v_0,u_0). If (3.5) holds, then the orbit of the solution is a precompact set of $C^0\times C^0$ by the Ascoli-Arzelá theorem. In particular, the ω-limit set of the orbit defined by

$$\omega(v_0,u_0)=\bigcap_{n=1}^{\infty} \overline{\{(v(t,\cdot),u(t,\cdot));t\ge n\}}^{C^0\times C^0}$$

is nonempty. The following lemma shows that the large time behavior of a bounded solution is under the control of the set of stationary solutions.

Lemma 3 Assume that $1<\gamma\le 2$. Let (v,u) be a bounded solution of (1.1)–(1.3) with initial value (v_0,u_0) and $\bar{u}=0$. Then, for $(v_\omega,u_\omega)\in\omega(v_0,u_0)$ we have $v_\omega\in C^\infty$, $v_\omega>0$, $\overline{v_\omega}=\overline{v_0}$, $u_\omega=0$, and

$$\partial_x\left(av_\omega(x)^{-\gamma}\right) = -\frac{4\pi G}{\overline{v_\omega}}\partial_x\int_0^L K_L(x,y)(v_\omega(y)-\overline{v_\omega})dy, \tag{3.10}$$

that is, (v_ω,u_ω) is a static and stationary solution of (1.1)–(1.3) having the average in common with the initial value.

Proof: By **Lemma 2** we have $\inf_{t,x} v(t,x) > 0$, and hence $v_\omega > 0$. It is clear that $\overline{v_\omega} = \overline{v_0}$.

We show that $u_\omega = 0$. Choose an increasing sequence $\{t_n; n = 1, 2, \ldots, \}$ of positive numbers such that $t_n \geq n$ and $\lim_{n\to\infty}(v(t_n,\,\cdot), u(t_n,\,\cdot)) = (v_\omega, u_\omega)$ in $C^0 \times C^0$. Since E given by (3.2) is decreasing, we have

$$\lim_{t\to\infty} E(t) = \lim_{n\to\infty} E(t_n) = \mathcal{E}(v_\omega, u_\omega)$$

and

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} E(t)dt = \mathcal{E}(v_\omega, u_\omega). \tag{3.11}$$

Representing $\int_{t_n}^{t_n+1} E(t)dt$ as

$$\int_{t_n}^{t_n+1} \left(\int_0^L \frac{1}{2} u(t,x)^2 dx \right) dt + \int_{t_n}^{t_n+1} \left(\mathcal{E}(v(t,\,\cdot)) - \mathcal{E}(v(t_n,\,\cdot)) \right) dt + \mathcal{E}(v(t_n,\,\cdot)),$$

we take the limit in (3.11) term by term. Since

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} \left(\int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx \right) dt = 0 \tag{3.12}$$

by (3.4), it follows from (3.9) that

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} \left(\int_0^L \frac{1}{2} u(t,x)^2 dx \right) dt = 0. \tag{3.13}$$

We next take the limit of the second term using

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} \left(\int_0^L |v(t,x) - v(t_n,x)| dx \right) dt = 0. \tag{3.14}$$

This follows from estimating the integral with respect to the variable x in (3.14) with the use of the equality $v(t,x) - v(t_n,x) = \int_{t_n}^{t} \partial_x u(s,x) ds$, $t \in [t_n, t_n+1]$, due to the first equation of (1.1), as

$$\begin{aligned}
&\int_0^L |v(t,x) - v(t_n,x)| dx \\
&\leq \int_{t_n}^{t_n+1} \left(\int_0^L |\partial_x u(s,x)| dx \right) ds \\
&\leq \left\{ \int_{t_n}^{t_n+1} \left(\int_0^L v(s,x) dx \right) ds \right\}^{1/2} \left\{ \int_{t_n}^{t_n+1} \left(\int_0^L \frac{\partial_x u(s,x)^2}{v(s,x)} dx \right) ds \right\}^{1/2} \\
&= \sqrt{L\bar{v}} \left\{ \int_{t_n}^{t_n+1} \left(\int_0^L \frac{\partial_x u(s,x)^2}{v(s,x)} dx \right) ds \right\}^{1/2},
\end{aligned}$$

and then applying (3.12). From the expression

$$v(t,x)^{1-\gamma} - v(t_n,x)^{1-\gamma} = (1-\gamma) \int_0^1 \{\xi v(t,x) + (1-\xi) v(t_n,x)\}^{-\gamma} d\xi \, (v(t,x) - v(t_n,x))$$

we have

$$\int_{t_n}^{t_n+1}\left(\int_0^L \frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}dx-\int_0^L \frac{v(t_n,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}dx\right)dt$$
$$=\int_0^1\left\{\int_{t_n}^{t_n+1}\left(\int_0^L \{\xi v(t,x)+(1-\xi)v(t_n,x)\}^{-\gamma}(v(t,x)-v(t_n,x))dx\right)dt\right\}d\xi.$$

Since $\{\xi v(t,x)+(1-\xi)v(t_n,x)\}^{-\gamma}\le(\inf_{t,x}v(t,x))^{-\gamma}$, $0\le\xi\le1$, we obtain

$$\lim_{n\to\infty}\int_{t_n}^{t_n+1}\left(\int_0^L \frac{v(t,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}dx-\int_0^L \frac{v(t_n,x)^{1-\gamma}-\bar{v}^{1-\gamma}}{1-\gamma}dx\right)dt=0.$$

Similarly, it follows from

$$(v(t,x)-\bar{v})(v(t,y)-\bar{v})-(v(t_n,x)-\bar{v})(v(t_n,y)-\bar{v})$$
$$=(v(t,x)-v(t_n,x))(v(t,y)-\bar{v})+(v(t_n,x)-\bar{v})(v(t,y)-v(t_n,y))$$

and the boundedness of the kernel K_L that

$$\lim_{n\to\infty}\int_{t_n}^{t_n+1}\left(\int_0^L\int_0^L K_L(x,y)(v(t,x)-\bar{v})(v(t,y)-\bar{v})dxdy\right.$$
$$\left.-\int_0^L\int_0^L K_L(x,y)(v(t_n,x)-\bar{v})(v(t_n,y)-\bar{v})dxdy\right)dt=0.$$

We thus obtain

$$\lim_{n\to\infty}\int_{t_n}^{t_n+1}(\mathcal{E}(v(t,\cdot))-\mathcal{E}(v(t_n,\cdot)))\,dt=0,$$

and hence,

$$\lim_{n\to\infty}\int_{t_n}^{t_n+1}E(t)dt=\lim_{n\to\infty}\mathcal{E}(v(t_n,\cdot))=\mathcal{E}(v_\omega).$$

Comparing this result with (3.11), we have $\int_0^L u_\omega(x)^2dx=0$, that is, $u_\omega=0$.

Finally we prove that v_ω is smooth and subject to (3.10). Let $\{t_n; n=1,2,\dots,\}$ be as above. Take a test function $\phi\in H^1$, and a smooth function θ of the real variable with support contained in the interval $(0,1)$, $\theta\ge0$, and $\int_0^1\theta(t)dt=1$. Multiply the second equation of (1.1) by $\theta(t-t_n)\phi(x)$ and integrate the both sides of the result over $[t_n,t_n+1]\times[0,L]$. Integration by parts yields

$$-\int_{t_n}^{t_n+1}\theta'(t-t_n)\left(\int_0^L\phi(x)u(t,x)dx\right)dt$$
$$-\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L\partial_x\phi(x)av(t,x)^{-\gamma}dx\right)dt$$
$$+\mu\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L\partial_x\phi(x)\frac{\partial_xu(t,x)}{v(t,x)}dx\right)dt$$
$$=\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L\partial_x\phi(x)\frac{4\pi G}{\bar{v}}\int_0^L K_L(x,y)(v(t,y)-\bar{v})dydx\right)dt.$$

With the use of (3.14) we can handle the second term on the left-hand side and the term on the right-hand side in the same manner as shown above:

$$\begin{aligned}
&\lim_{n\to\infty}\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L \partial_x\phi(x)av(t,x)^{-\gamma}dx\right)dt\\
&=\int_0^L \partial_x\phi(x)av_\omega(x)^{-\gamma}dx,\\
&\lim_{n\to\infty}\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L \partial_x\phi(x)\frac{4\pi G}{\bar v}\int_0^L K_L(x,y)(v(t,y)-\bar v)dydx\right)dt\\
&=\int_0^L \partial_x\phi(x)\frac{4\pi G}{\bar v}\int_0^L K_L(x,y)(v_\omega(y)-\bar v)dydx.
\end{aligned}$$

As for the third term on the left-hand side we have the following estimate by Schwarz' lemma:

$$\begin{aligned}
&\left|\int_{t_n}^{t_n+1}\theta(t-t_n)\left(\int_0^L \partial_x\phi(x)\frac{\partial_x u(t,x)}{v(t,x)}dx\right)dt\right|\\
&\le\left\{\int_{t_n}^{t_n+1}\theta(t-t_n)^2\left(\int_0^L\frac{\partial_x\phi(x)^2}{v(t,x)}dx\right)dt\right\}^{1/2}\left\{\int_{t_n}^{t_n+1}\left(\int_0^L\frac{\partial_x u(t,x)^2}{v(t,x)}dx\right)dt\right\}^{1/2}\\
&\le\left(\frac{1}{\inf_{t,x}v(t,x)}\int_0^1\theta(t)^2dt\int_0^L\partial_x\phi(x)^2dx\right)^{1/2}\left\{\int_{t_n}^{t_n+1}\left(\int_0^L\frac{\partial_x u(t,x)^2}{v(t,x)}dx\right)dt\right\}^{1/2},
\end{aligned}$$

which shows that the term tends to 0 as $n\to\infty$ in view of (3.12). Similarly, the first term on the left-hand side tends to 0 as $n\to\infty$ from (3.13). Thus we obtain

$$-\int_0^L\partial_x\phi(x)av_\omega(x)^{-\gamma}dx=\int_0^L\partial_x\phi(x)\frac{4\pi G}{\overline{v_\omega}}\int_0^L K_L(x,y)(v_\omega(y)-\overline{v_\omega})dydx,$$

the equality (3.10) for v_ω in the distribution sense. Using the smoothing property of the integral operator with kernel K_L, by bootstrap argument we can derive the smoothness of v_ω from $v_\omega\in C^0$. □

4 Structure of stationary solutions From the observation of the large time behavior of bounded solutions to (1.1)–(1.3) we see that a solution is necessarily unbounded if it fails to approach the set of stationary solutions. This together with **Lemma 1**, which claims that the energy form is decreasing along the orbit of any solution, implies that, if there exists a state on M_V at which the energy form takes a value smaller than those of the energy form evaluated at the stationary solutions on M_V, then the orbit passing such a state is apart from the set of the stationary solutions and must be unbounded. This gives us the idea of providing, in terms of the energy form, an initial condition for unbounded solutions in reference to the structure of stationary solutions. Based on this idea, we first prove **Theorem 1**, and then examine at which stationary solution on M_V the energy form takes the minimal value.

Let us consider the stationary problem for (1.1)–(1.3):

$$\begin{cases}\partial_x u(x)=0,\\ \partial_x(av(x)^{-\gamma})=-\dfrac{4\pi G}{\bar v}\partial_x\displaystyle\int_0^L K_L(x,y)(v(y)-\bar v)dy.\end{cases}\tag{4.1}$$

Our first task in the present section is to seek all the solutions of (4.1) lying in M_V for every $V > 0$. Clearly we have $u = 0$. By the change of unknown functions $r(x) = (v(x)/V)^{-\gamma} - 1$, we transform the problem into an equivalent one of finding L-periodic solutions to the following differential equation:

$$\partial_x^2 r(x) + \lambda f(r(x)) = 0, \quad r(x) > -1, \tag{4.2}$$

with

$$f(r) = 1 - (1+r)^{-1/\gamma}, \quad \lambda = \frac{4\pi G V^\gamma}{a}.$$

An L-periodic solution r of (4.2) has a critical point x_0, i.e., $\partial_x r(x_0) = 0$. Since both $r(x + x_0)$ and $r(-x + x_0)$ satisfy (4.2) with coincidence of the Cauchy data at $x = 0$, by the uniqueness of solutions to the Cauchy problem for (4.2) we have $r(x + x_0) = r(-x + x_0)$, and therefore both are even functions. Thus, r is given by an appropriate shift of an even solution. In view of this fact, we seek even L-periodic solutions of (4.2).

To this end we make use of the relation between the period of a solution and the first integral. The first integral of (4.2), usually called the energy of the orbit, is given by

$$\mathcal{I} = \frac{1}{2}\partial_x r(x)^2 + \lambda F(r(x))$$

with

$$F(r) = \int_0^r f(s)ds = r - \frac{\gamma}{\gamma - 1}\left\{(1+r)^{1-1/\gamma} - 1\right\}, \quad r > -1.$$

F is monotone decreasing on $(-1, 0]$ and monotone increasing on $[0, \infty)$, having the limit at either end of the half line:

$$F(-1+0) = \frac{1}{\gamma - 1}, \qquad F(\infty) = \infty.$$

We can therefore find a unique closed orbit with energy $\mathcal{I}$ if and only if

$$0 < \mathcal{I} < \frac{\lambda}{\gamma - 1}.$$

The period l of the orbit with energy $\mathcal{I}$ is given by

$$l = 2\int_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{2(\mathcal{I} - \lambda F(r))}},$$

where $r_{\min} < 0$ and $r_{\max} > 0$ are the minimum and the maximum of the solution r, respectively. Notice that

$$F(r_{\min}) = F(r_{\max}) = \frac{\mathcal{I}}{\lambda}. \tag{4.3}$$

Dividing the integral into two parts, one over $(r_{\min}, 0)$ and the other over $(0, r_{\max})$, and changing the variables by $y = (\lambda/\mathcal{I})F(r)$, we get

$$l = \sqrt{2/\lambda}\, I_\gamma\left(\sqrt{\mathcal{I}/\lambda}\right), \tag{4.4}$$

where I_γ is a function on $(0, (\gamma - 1)^{-1/2})$ given by (2.1). The following lemma shows that the period of an orbit is a monotone increasing function of its energy provided $1 < \gamma < 2$.

Lemma 4 Assume $1 < \gamma < 2$. Then, $I'_\gamma(\theta) > 0$. Moreover we have

$$I_\gamma(+0) = \sqrt{2\gamma}\pi, \qquad I_\gamma((\gamma-1)^{-1/2}-0) < \infty.$$

Proof: Put

$$I_{\gamma,\pm}(\theta) = \int_0^1 \frac{1}{\sqrt{1-y}} \frac{\theta}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)} dy \tag{4.5}$$

and express $I_\gamma(\theta)$ as the sum of $I_{\gamma,+}(\theta)$ and $I_{\gamma,-}(\theta)$. Since

$$\frac{\partial}{\partial\theta}\left(F_\pm{}^{-1}(\theta^2 y)\right) = \frac{2\theta y}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)},$$

we have

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\frac{\theta}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)}\right) &= \frac{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^2 - 2\theta^2 y f'_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^3} \\ &= \frac{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^2 - 2F_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right) f'_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^3}.\end{aligned}$$

Noting that

$$\lim_{z\to+0} \frac{f_\pm(z)^2 - 2F_\pm(z)f'_\pm(z)}{f_\pm(z)^3} = -\frac{1}{3}\frac{f''_\pm(0)}{f'_\pm(0)^2},$$

we apply differentiation under the integral sign to (4.5) to obtain

$$I'_{\gamma,\pm}(\theta) = \int_0^1 \frac{1}{\sqrt{1-y}} \frac{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^2 - 2F_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right) f'_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^3} dy$$

and

$$\lim_{\theta\to+0} I'_{\gamma,\pm}(\theta) = -\frac{1}{3}\frac{f''_\pm(0)}{f'_\pm(0)^2}\int_0^1 \frac{dy}{\sqrt{1-y}} dy = -\frac{2}{3}\frac{f''_\pm(0)}{f'_\pm(0)^2}. \tag{4.6}$$

Similarly, we have

$$\begin{aligned}&\frac{\partial}{\partial\theta}\left(\frac{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^2 - 2F_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right) f'_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^3}\right) \\ &= \frac{2\theta y\, g\left(F_\pm{}^{-1}(\theta^2 y)\right)}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^5} \\ &= \frac{2\sqrt{y} F_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^{1/2} g\left(F_\pm{}^{-1}(\theta^2 y)\right)}{f_\pm\left(F_\pm{}^{-1}(\theta^2 y)\right)^5}\end{aligned}$$

with

$$g(z) = 2F_\pm(z)\left(3f'_\pm(z)^2 - f_\pm(z)f''_\pm(z)\right) - 3f_\pm(z)^2 f'_\pm(z).$$

Since

$$\lim_{z\to+0}\frac{F_\pm(z)^{1/2}g(z)}{f_\pm(z)^5} = \lim_{z\to+0}\left(\frac{F_\pm(z)}{f_\pm(z)^2}\right)^{1/2}\frac{2F_\pm(z)\left(3f'_\pm(z)^2 - f_\pm(z)f''_\pm(z)\right) - 3f_\pm(z)^2 f'_\pm(z)}{f_\pm(z)^4}$$

$$= \frac{5f''_{\pm}(0)^2 - 3f'_{\pm}(0)f'''_{\pm}(0)}{12\sqrt{2}f'_{\pm}(0)^{7/2}},$$

again by differentiation under the integral sign the second derivative of $I_{\gamma,\pm}$ is given by

$$I''_{\gamma,\pm}(\theta) = \int_0^1 \frac{1}{\sqrt{1-y}} \frac{2\theta y\, g\left({F_{\pm}}^{-1}(\theta^2 y)\right)}{f_{\pm}\left({F_{\pm}}^{-1}(\theta^2 y)\right)^5} dy, \tag{4.7}$$

a continuous function on $(0, (\gamma-1)^{-1/2})$. Now we put $\zeta = (1 \pm z)^{-1/\gamma}$ and express $g(z)$ as

$$g(z) = \frac{\zeta^{1+2\gamma}}{\gamma^2} h(\zeta)$$

with

$$h(\zeta) = \frac{\gamma(1+\gamma)}{1-\gamma}\zeta^{2-\gamma} + \frac{2(1+\gamma)(2-\gamma)}{1-\gamma}\zeta^{1-\gamma} + (2-\gamma)\zeta^{-\gamma} + \frac{2(\gamma-2)}{1-\gamma}\zeta - \frac{2(1+\gamma)}{1-\gamma}.$$

Since

$$h'(\zeta) = \frac{\gamma(1+\gamma)(2-\gamma)}{1-\gamma}\zeta^{1-\gamma} + 2(1+\gamma)(2-\gamma)\zeta^{-\gamma} - \gamma(2-\gamma)\zeta^{-1-\gamma} + \frac{2(\gamma-2)}{1-\gamma},$$

and since

$$\begin{aligned} h''(\zeta) &= \gamma(1+\gamma)(2-\gamma)\zeta^{-\gamma} - 2\gamma(1+\gamma)(2-\gamma)\zeta^{-1-\gamma} + \gamma(1+\gamma)(2-\gamma)\zeta^{-2-\gamma} \\ &= \gamma(1+\gamma)(2-\gamma)\zeta^{-2-\gamma}(\zeta-1)^2 \\ &\geq 0 \end{aligned}$$

for $1 < \gamma < 2$, we have $h(1) = h'(1) = 0$ and therefore $h(\zeta) > 0$, $\zeta \neq 1$. The integrand in (4.7) is positive, and so is $I''_{\gamma,\pm}(\theta)$ for $\theta \in (0, (\gamma-1)^{-1/2})$. This implies that $I'_\gamma = I'_{\gamma,+} + I'_{\gamma,-}$ as well as $I'_{\gamma,\pm}$ is monotone increasing on $(0, (\gamma-1)^{-1/2})$. Using $f'_{\pm}(0) = 1/\gamma$ and $f''_{\pm}(0) = \mp(1+1/\gamma)/\gamma$, from (4.6) we obtain $\lim_{\theta\to+0} I'_\gamma(\theta) = 0$, and hence I'_γ is positive on $(0, (\gamma-1)^{-1/2})$, as desired.

Since

$$\frac{\theta}{f_{\pm}\left({F_{\pm}}^{-1}(\theta^2 y)\right)} = \frac{1}{\sqrt{y}}\left(\frac{F_{\pm}\left({F_{\pm}}^{-1}(\theta^2 y)\right)}{f_{\pm}\left({F_{\pm}}^{-1}(\theta^2 y)\right)^2}\right)^{1/2},$$

and since the function $z \to F_{\pm}(z)/f_{\pm}(z)^2$ is bounded on $\left(0, {F_{\pm}}^{-1}\left((\gamma-1)^{-1} - 0\right)\right)$, we can take the limit of $I_{\gamma,\pm}(\theta)$ at either end of the interval $(0, (\gamma-1)^{-1/2})$ under the integral sign in (4.5). Thus we obtain

$$\lim_{\theta\to+0} I_{\gamma,\pm}(\theta) = \lim_{z\to+0}\left(\frac{F_{\pm}(z)}{f_{\pm}(z)^2}\right)^{1/2}\int_0^1 \frac{dy}{\sqrt{1-y}\sqrt{y}} = \frac{\pi}{(2f'_{\pm}(0))^{1/2}} = \left(\frac{\gamma}{2}\right)^{1/2}\pi$$

and

$$\lim_{\theta\to(\gamma-1)^{-1/2}-0} I_{\gamma,\pm}(\theta) = \int_0^1 \frac{1}{\sqrt{1-y}\sqrt{y}}\left(\frac{F_{\pm}\left({F_{\pm}}^{-1}(y/(\gamma-1))\right)}{f_{\pm}\left({F_{\pm}}^{-1}(y/(\gamma-1))\right)^2}\right)^{1/2} dy < \infty,$$

showing that $I_\gamma(+0) = \sqrt{2\gamma}\pi$ and $I_\gamma((\gamma-1)^{-1/2} - 0) < \infty$. $\square$

By **Lemma 4**, from the formula (4.4) we obtain a necessary and sufficient condition for the existence and uniqueness of l-periodic orbits for (4.2):

$$\sqrt{4\gamma/\lambda}\pi < l < \sqrt{2/\lambda} I_\gamma((\gamma-1)^{-1/2}-0). \tag{4.8}$$

Recalling $\lambda = 4\pi G V^\gamma / a$, we obtain the assertion of **Theorem 1** immediately.

We proceed to another task of finding out the stationary solutions with minimal value of the energy form. Assume that the stationary problem for (1.1)–(1.3) has a non-trivial solution on M_V. For $j = k_{\min}, \dots, k_{\max}$ choose a stationary solution $(\tilde{v}^{(j)}, 0) \in M_V$ with least period L/j as in **Theorem 1**, and put

$$S_V = \{(\tilde{v}^{(j)}, 0); j = k_{\min}, \dots, k_{\max}\} \cup \{(V, 0)\}. \tag{4.9}$$

We compare the values $\mathcal{E}(\tilde{v}^{(j)})$, $j = k_{\min}, \dots, k_{\max}$, and $\mathcal{E}(V) = 0$ with each other. To this end we introduce the following function with respect to the periods of stationary solutions:

$$\begin{aligned} \varepsilon(l) = &\int_0^l a\left(\frac{\tilde{v}^l(x) - V}{V^\gamma} - \frac{\tilde{v}^l(x)^{1-\gamma} - V^{1-\gamma}}{1-\gamma}\right) dx \\ &- \frac{2\pi G}{V}\int_0^l \int_0^l K_l(x,y)(\tilde{v}^l(x) - V)(\tilde{v}^l(y) - V) dx dy, \end{aligned}$$

where $(\tilde{v}^l, 0)$ is the non-trivial solution of the stationary problem (4.1) parametrized by $L = l$ with $\tilde{v}^l$ having the average V, the least period l, and the maximum at $x = 0$. In view of (4.8), $\tilde{v}^l$ as well as $\varepsilon(l)$ is well defined for l with

$$\left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2} < l < \frac{I_\gamma((\gamma-1)^{-1/2}-0)}{\sqrt{2\gamma}\pi}\left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}. \tag{4.10}$$

With this function the value $\mathcal{E}(\tilde{v}^{(j)})$ is expressed as follows.

Lemma 5 For $j = k_{\min}, \dots, k_{\max}$ we have

$$\mathcal{E}(\tilde{v}^{(j)}) = j\varepsilon(L/j). \tag{4.11}$$

Proof: Put $l_j = L/j$. Notice that $\mathcal{E}(\tilde{v}^{(j)}) = \mathcal{E}(\tilde{v}^{l_j})$. In the expression

$$\begin{aligned} \mathcal{E}(v^{l_j}) = &\int_0^L a\left(\frac{\tilde{v}^{l_j}(x) - V}{V^\gamma} - \frac{\tilde{v}^{l_j}(x)^{1-\gamma} - V^{1-\gamma}}{1-\gamma}\right) dx \\ &- \frac{2\pi G}{V}\int_0^L \int_0^L K_L(x,y)(\tilde{v}^{l_j}(x) - V)(\tilde{v}^{l_j}(y) - V) dx dy \end{aligned}$$

we divide every integral on the interval $[0, L]$ into the integrals on the subintervals $[ml_j, (m+1)l_j]$, $m = 0, \dots, j-1$, and rewrite every piece as an integral on $[0, l_j]$ by change of variables. By periodicity of $\tilde{v}^{l_j}$ we obtain

$$\begin{aligned} \mathcal{E}(\tilde{v}^{(j)}) = j &\int_0^{l_j} a\left(\frac{\tilde{v}^{l_j}(x) - V}{V^\gamma} - \frac{\tilde{v}^{l_j}(x)^{1-\gamma} - V^{1-\gamma}}{1-\gamma}\right) dx \\ &- \frac{2\pi G}{V}\int_0^{l_j} \int_0^{l_j} \sum_{m,n=0}^{j-1} K_L(x + ml_j, y + nl_j)(\tilde{v}^{l_j}(x) - V)(\tilde{v}^{l_j}(y) - V) dx dy. \end{aligned} \tag{4.12}$$

Noting that $0 \le x+ml_j, y+nl_j \le L$ for $0 \le x, y \le l_j$ and $m, n = 0, \dots, j-1$, we calculate the sum $\sum_{m,n=0}^{j-1} K_L(x+ml_j, y+nl_j)$ with the use of the expression (1.4) of K_L:

$$\begin{aligned}
&\sum_{m,n=0}^{j-1} K_L(x+ml_j, y+nl_j)\\
&= -\frac{1}{2}\sum_{m,n=0}^{j-1} |x-y+(m-n)l_j| + \frac{1}{2L}\sum_{m,n=0}^{j-1} \{x-y+(m-n)l_j\}^2 + \frac{L}{12}j^2\\
&= -\frac{1}{2}\left[\sum_{m=n}|x-y| + \sum_{m>n}\{x-y+(m-n)l_j\} - \sum_{m<n}\{x-y+(m-n)l_j\}\right]\\
&\quad + \frac{1}{2L}\sum_{m,n=0}^{j-1}\left\{(x-y)^2 + 2(m-n)(x-y)l_j + (m-n)^2 l_j^2\right\} + \frac{L}{12}j^2\\
&= -\frac{1}{2}\left(j|x-y| + 2\sum_{m>n}(m-n)l_j\right) + \frac{1}{2L}\left\{j^2(x-y)^2 + 2\sum_{m>n}(m-n)^2 l_j^2\right\} + \frac{L}{12}j^2\\
&= -j\frac{|x-y|}{2} + j\frac{(x-y)^2}{2l_j} + \sum_{m>n}\left\{\frac{(m-n)^2}{j} - (m-n)\right\}l_j + \frac{l_j}{12}j^3.
\end{aligned}$$

Here we have

$$\begin{aligned}
&\sum_{m>n}\left\{\frac{(m-n)^2}{j} - (m-n)\right\}\\
&= \sum_{m=1}^{j-1}\sum_{k=1}^{m}\left(\frac{k^2}{j} - k\right)\\
&= \sum_{m=1}^{j-1}\left(\frac{2m^3+3m^2+m}{6j} - \frac{m^2+m}{2}\right)\\
&= \frac{1}{12}\left[\left\{j(j-1)^2 + (j-1)(2j-1) + (j-1)\right\} - \{j(j-1)(2j-1) + 3j(j-1)\}\right]\\
&= \frac{1}{12}(-j^3 + j),
\end{aligned}$$

and hence,

$$\sum_{m,n=0}^{j-1} K_L(x+ml_j, y+nl_j) = j\left\{-\frac{|x-y|}{2} + \frac{(x-y)^2}{2l_j} + \frac{l_j}{12}\right\} = jK_{l_j}(x,y).$$

This together with (4.12) gives (4.11). □

Since

$$\mathcal{E}(\tilde{v}^{(j)}) = L\frac{\varepsilon(L/j)}{L/j}$$

by (4.11), the j-dependence of $\mathcal{E}(\tilde{v}^{(j)})$ would be known from the behavior of the function $l \mapsto \varepsilon(l)/l$ on the interval (4.10). We first show the differentiability of the function. Put $r^l(x) = (\tilde{v}^l(x)/V)^{-\gamma} - 1$. Notice that r^l is a solution of (4.2) having the least period l and the minimum at $x = 0$. Let us denote the minimum by $r^l_{\min}$, negative for l with (4.10). By (4.3) the energy of the orbit of r^l is $\lambda F(r^l_{\min})$. Therefore, from (4.4) we obtain

$$l = \sqrt{2/\lambda}\, I_\gamma \left(\sqrt{F(r^l_{\min})} \right).$$

By the monotonicity of I_γ due to **Lemma 4**,

$$F\left(r^l_{\min}\right) = \left({I_\gamma}^{-1} \left(l\sqrt{\lambda/2} \right) \right)^2 \tag{4.13}$$

holds. Since I_γ is continuously differentiable, so is the function $l \mapsto r^l_{\min}$ on (4.10). By continuous dependence on initial data in the Cauchy problem for (4.2) the correspondence $l \mapsto r^l$ defines a continuously differentiable function on (4.10) with values in the space of continuous functions on $\mathbf{R}$, and so does the correspondence $l \mapsto \tilde{v}^l$. From this together with the expression (1.4) of the kernel K_L with $L = l$ the differentiability of the function $l \mapsto \varepsilon(l)$ on (4.10) easily follows.

The following lemma shows that the function under consideration is monotonic and negative.

Lemma 6 We have $(\varepsilon(l)/l)' < 0$ and $\varepsilon(l) < 0$.

Proof: Put $\tilde{v}^l_0 = \tilde{v}^l(0)$. We take the derivative of $\varepsilon(l)$ and rewrite the result using $\tilde{v}^l(l) = \tilde{v}^l_0$, $\int_0^l \partial_l \tilde{v}^l(x)dx = V - \tilde{v}^l_0$ from $\int_0^l \tilde{v}^l(x)dx = Vl$, and the symmetry of the Green kernel $K_l(x, y)$. After rearrangement of terms we obtain

$$\begin{aligned}\varepsilon'(l) = &-a\frac{(\tilde{v}^l_0)^{1-\gamma} - V^{1-\gamma}}{1-\gamma} \\ &- \int_0^l a\tilde{v}^l(x)^{-\gamma}\partial_l\tilde{v}^l(x)dx - \frac{4\pi G}{V}\int_0^l\int_0^l K_l(x,y)(\tilde{v}^l(y) - V)dy\,\partial_l\tilde{v}^l(x)dx \\ &- \frac{4\pi G}{V}\int_0^l K_l(l,y)(\tilde{v}^l(y) - V)dy(\tilde{v}^l_0 - V) \\ &- \frac{2\pi G}{V}\int_0^l\int_0^l \partial_l K_l(x,y)(\tilde{v}^l(x) - V)(\tilde{v}^l(y) - V)dxdy.\end{aligned}$$

Here we notice that $\tilde{v}^l$ is subject to the following equation equivalent to the second one of (4.1) with $L = l$:

$$-a\tilde{v}^l(x)^{-\gamma} + \frac{1}{l}\int_0^l a\tilde{v}^l(x)^{-\gamma}dx - \frac{4\pi G}{V}\int_0^l K_l(x,y)(\tilde{v}^l(y) - V)dy = 0. \tag{4.14}$$

Then the sum of the second and the third terms on the right-hand side is

$$-\frac{1}{l}\int_0^l a\tilde{v}^l(x)^{-\gamma}dx \int_0^l \partial_l\tilde{v}^l(x)dx = \frac{1}{l}\int_0^l a\tilde{v}^l(x)^{-\gamma}dx\,(\tilde{v}^l_0 - V).$$

Adding the forth term to this expression and using (4.14) with $x = l$, we see that the sum of the above three terms turns out to be $a(\tilde{v}^l_0 - V)/(\tilde{v}^l_0)^\gamma$. Since $\tilde{v}^l$ is axially symmetric with respect to $x = l/2$, the last term on the right-hand side vanishes in view of

$$\partial_l K_l(x,y) = -\frac{(x-y)^2}{2l^2} + \frac{1}{12}$$

$$= -\frac{1}{2l^2}\left\{\left(x-\frac{l}{2}\right)^2+\left(y-\frac{l}{2}\right)^2-2\left(x-\frac{l}{2}\right)\left(y-\frac{l}{2}\right)\right\}+\frac{1}{12}, \quad 0\le x,y\le l.$$

Summing up, we obtain

$$\varepsilon'(l) = a\left\{-\frac{(\tilde v_0^l)^{1-\gamma}-V^{1-\gamma}}{1-\gamma}+\frac{\tilde v_0^l - V}{(\tilde v_0^l)^\gamma}\right\}.$$

From this expression the function $l\mapsto \varepsilon(l)$ is twice continuously differentiable and

$$\varepsilon''(l) = -a\gamma\frac{\partial_l \tilde v_0^l\,(\tilde v_0^l - V)}{(\tilde v_0^l)^{\gamma+1}}.$$

Since $\tilde v^l$ attains its maximum at $x = 0$, we have $\tilde v_0^l - V > 0$. Moreover, from $\tilde v_0^l = V\left(1+r_{\min}^l\right)^{-1/\gamma}$ with $r_{\min}^l$ as above, we obtain

$$\partial_l \tilde v_0^l = -\frac{V}{\gamma}(1+r_{\min}^l)^{-1/\gamma-1}\partial_l r_{\min}^l.$$

Thus, the sign of $\varepsilon''(l)$ coincides with that of $\partial_l r_{\min}^l$. Taking the derivatives of the both sides of (4.13), we obtain

$$f(r_{\min}^l)\partial_l r_{\min}^l = \sqrt{2\lambda}{I_\gamma}^{-1}\left(l\sqrt{\lambda/2}\right)({I_\gamma}^{-1})'\left(l\sqrt{\lambda/2}\right),$$

positive in view of **Lemma 4**. Since $r_{\min}^l$ is negative, so are $f\left(r_{\min}^l\right)$ and $\partial_l r_{\min}^l$. We thus conclude that $\varepsilon''(l) < 0$.

We next take the limit as $l \to \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}+0$ in (4.13). By **Lemma 4** we have $F(r_{\min}^l) \to ({I_\gamma}^{-1}(\sqrt{2\gamma}\pi+0))^2 = 0$, and hence $r_{\min}^l \to 0$. By continuous dependence on initial data in the Cauchy problem for (4.2) we obtain the uniform convergence of both r^l and $\tilde v^l$ as $l \to \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}+0$, showing $r^l(x)\to 0$ and $\tilde v^l(x)\to V$ on $\mathbf{R}$. Thus, $\varepsilon(l)$ as well as $\varepsilon'(l)$ tends to 0 as $l \to \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}+0$. From these in combination with $(\varepsilon(l)/l)' = (l\varepsilon'(l)-\varepsilon(l))/l^2$ and $(l\varepsilon'(l)-\varepsilon(l))' = l\varepsilon''(l) < 0$, we conclude that $(\varepsilon(l)/l)' < 0$ and $\varepsilon(l) < 0$. □

As a consequence of **Lemma 6** we obtain

Proposition 1 For $j_1, j_2 = k_{\min},\ldots,k_{\max}$ with $j_1 < j_2$, we have

$$\mathcal{E}(\tilde v^{(j_1)}) < \mathcal{E}(\tilde v^{(j_2)}) < \mathcal{E}(V) = 0.$$

In particular, $\mathcal{E}(\tilde v^{(k_{\min})})$ is minimal amongst the values of the energy form on S_V.

5 Initial condition for unbounded solutions **Proposition 1** claims that the subset A_V of $H^1\times H^1$ given by (2.3) consists of the states on M_V at which the energy form takes values smaller than any values of the energy form evaluated at the stationary solutions on M_V. As proved earlier, the orbit of a solution to (1.1)–(1.3) passing through A_V is necessarily unbounded, i.e., $\sup_{t,x} v(t,x) = \infty$. In this way we obtain an initial condition for unbounded solutions as presented by **Theorem 2**. The problem to be settled is to find a condition that ensures the non-emptiness of A_V. The final section is devoted to a partial answer to the problem, proving **Theorem 2**.

Our strategy is to find an element of A_V in a small neighborhood of a stationary solution giving the minimal value of the energy form on S_V given by (4.9). In order to introduce the idea we begin by examining the behavior of the energy form near an arbitrary stationary solution. Let $(\tilde{v},0)\in M_V$ be a stationary solution of (1.1)–(1.3), and $(v,u)\in M_V$ a state in a neighborhood of the stationary solution. We introduce the displacement from the stationary solution as

$$\phi(x)=v(x)-\tilde{v}(x),\quad \psi(x)=u(x).$$

Suppose the displacement is small enough in amplitude. Since

$$\frac{(\tilde{v}(x)+\phi(x))^{1-\gamma}-\tilde{v}(x)^{1-\gamma}}{1-\gamma}=\tilde{v}(x)^{-\gamma}\phi(x)-\frac{1}{2}\gamma\tilde{v}(x)^{-\gamma-1}\phi(x)^2+\mathcal{O}(|\phi(x)|^3),$$

evaluating the form (1.6) at $v=\tilde{v}+\phi$, we obtain

$$\begin{aligned}&\mathcal{E}(\tilde{v}+\phi)\\&=\mathcal{E}(\tilde{v})+\int_0^L\left(-a\tilde{v}(x)^{-\gamma}-\frac{4\pi G}{V}\int_0^L K_L(x,y)(\tilde{v}(y)-V)dy\right)\phi(x)dx\\&\quad+\frac{1}{2}\int_0^L a\frac{\gamma\phi(x)^2}{\tilde{v}(x)^{\gamma+1}}dx-\frac{2\pi G}{V}\int_0^L\int_0^L K_L(x,y)\phi(x)\phi(y)dxdy+\mathcal{O}(\|\phi\|_{L^\infty})\|\phi\|_{L^2}^2\end{aligned}$$

with $\|\phi\|_{L^\infty}$ the supremum norm of ϕ. As in (4.14), $\tilde{v}$ satisfies the equation

$$-a\tilde{v}(x)^{-\gamma}+\frac{1}{L}\int_0^L a\tilde{v}(x)^{-\gamma}dx-\frac{4\pi G}{V}\int_0^L K_L(x,y)(\tilde{v}(y)-V)dy=0. \tag{5.1}$$

Since the average of ϕ vanishes, this implies that

$$\mathcal{E}(\tilde{v}+\phi)=\mathcal{E}(\tilde{v})+\frac{1}{2}Q[\phi]+\mathcal{O}(\|\phi\|_{L^\infty})\|\phi\|_{L^2}^2, \tag{5.2}$$

where Q is the quadratic form on the Hilbert space $\mathcal{H}=\{\varphi\in L^2;\bar{\varphi}=0\}$ defined by

$$Q[\varphi]=\int_0^L a\frac{\gamma\varphi(x)^2}{\tilde{v}(x)^{\gamma+1}}dx-\frac{4\pi G}{V}\int_0^L\int_0^L K_L(x,y)\varphi(x)\varphi(y)dxdy.$$

Now suppose the quadratic form Q admits a negative value, i.e., $Q[\varphi_0]<0$ for some $\varphi_0\in\mathcal{H}$. By approximation of functions we may assume that φ_0 is smooth. Evaluating the energy form (1.5) with (1.6) at $(v,u)=(\tilde{v}+\varepsilon\varphi_0,0)$ for small $|\varepsilon|$, from (5.2) we obtain

$$\mathcal{E}(\tilde{v}+\varepsilon\varphi_0,0)=\mathcal{E}(\tilde{v})+\frac{1}{2}\varepsilon^2Q[\varphi_0]+\mathcal{O}(|\varepsilon|^3).$$

This shows that the energy form takes a value smaller than its value at the stationary solution in any small neighborhood of that stationary solution.

In order to examine the sign of Q we make use of the expression

$$Q[\varphi]=\frac{a}{V^{\gamma+1}}(T\varphi,\varphi)_{L^2},$$

where T is the self-adjoint operator on $\mathcal{H}$ given by

$$(T\varphi)(x)=\frac{\gamma\varphi(x)}{(1+\tilde{w}(x))^{\gamma+1}}-\frac{1}{L}\int_0^L\frac{\gamma\varphi(x)}{(1+\tilde{w}(x))^{\gamma+1}}dx-\lambda\int_0^L K_L(x,y)\varphi(y)dy \tag{5.3}$$

with $\tilde{w}=\tilde{v}/V-1$ and $\lambda=4\pi GV^\gamma/a$. We are concerned with the spectrum $\sigma(T)$ of T since the lower bound of $\sigma(T)$ gives $\inf_{\|\varphi\|_{L^2}=1}(T\varphi,\varphi)_{L^2}$.

In case $\tilde{v} = V$, that is, $\tilde{w} = 0$ the spectrum of T is easily obtained from that of the Green operator of $-d^2/dx^2$ on $\mathcal{H}$. The spectrum consists of double eigenvalues $\gamma - (\lambda L^2)/(4\pi^2 j^2)$ with two independent eigenvectors $\cos(2\pi j/L)x$ and $\sin(2\pi j/L)x$, $j = 1, 2, \dots,$ and the accumulation point γ of them. Thus, we obtain

$$\inf \sigma(T) = \gamma - \frac{\lambda L^2}{4\pi^2} \tag{5.4}$$

immediately.

In considering the spectrum of T corresponding to a non-trivial stationary solution $(\tilde{v}, 0)$ some preliminary observations are in order. Since $(T\varphi, \varphi)_{L^2} \le \int_0^L \frac{\gamma\varphi(x)^2}{(1+\tilde{w}(x))^{\gamma+1}}dx$ for $\varphi \in \mathcal{H}$, we have $\inf \sigma(T) \le \gamma(1 + \max \tilde{w})^{-\gamma-1}$. In the region below $\gamma(1 + \max \tilde{w})^{-\gamma-1}$ the spectrum in fact consists of eigenvalues of T. This follows from rewriting an equation $T\varphi - \Lambda\varphi = \psi$ in $\mathcal{H}$ with parameter $\Lambda < \gamma(1 + \max \tilde{w})^{-\gamma-1}$ as $P_\Lambda\varphi - \lambda K_L\varphi = \psi$ with

$$\begin{aligned}
(P_\Lambda\varphi)(x) &= \left\{ \frac{\gamma}{(1 + \tilde{w}(x))^{\gamma+1}} - \Lambda \right\} \varphi(x) - \frac{1}{L}\int_0^L \frac{\gamma\varphi(x)}{(1 + \tilde{w}(x))^{\gamma+1}}dx, \\
(K_L\varphi)(x) &= \int_0^L K_L(x, y)\varphi(y)dy,
\end{aligned}$$

noting the positivity of P_Λ and the compactness of K_L, and applying the the Riesz-Schauder theory to the compact operator ${P_\Lambda}^{-1}K_L$ on $\mathcal{H}$. We next remark that $(\tilde{v}(\,\cdot\, - \alpha), 0)$ is also a stationary solution of (1.1)–(1.3) for any $\alpha \in \mathbf{R}$, and hence

$$-\frac{1}{(1 + \tilde{w}(x - \alpha))^\gamma} + \frac{1}{L}\int_0^L \frac{dx}{(1 + \tilde{w}(x - \alpha))^\gamma} - \lambda\int_0^L K_L(x, y)\tilde{w}(y - \alpha)dy$$

holds by (5.1). Differentiating this relation with respect to α and evaluating the result at $\alpha = 0$, we obtain

$$\frac{\gamma\tilde{w}'(x)}{(1 + \tilde{w}(x))^{\gamma+1}} - \frac{1}{L}\int_0^L \frac{\gamma\tilde{w}'(x)}{(1 + \tilde{w}(x))^{\gamma+1}}dx - \lambda\int_0^L K_L(x, y)\tilde{w}'(y)dy = 0,$$

that is, $T\tilde{w}' = 0$. This shows that T has a non-trivial null space with an eigenvector $\tilde{w}' \neq 0$. Let us define a self-adjoint operator on $\mathcal{H}$ corresponding to the stationary solution $(\tilde{v}(\,\cdot\, - \alpha), 0)$ by (5.3):

$$(T^\alpha\varphi)(x) = \frac{\gamma\varphi(x)}{(1 + \tilde{w}(x - \alpha))^{\gamma+1}} - \frac{1}{L}\int_0^L \frac{\gamma\varphi(x)}{(1 + \tilde{w}(x - \alpha))^{\gamma+1}}dx - \lambda\int_0^L K_L(x, y)\varphi(y)dy.$$

Our last remark here is that the point spectrum of T^α coincides with that of T for any $\alpha \in \mathbf{R}$ with the correspondence of associating eigenspaces given by the shift of functions $\varphi \mapsto \varphi(\,\cdot\, - \alpha)$, for from the equation $T\varphi = \Lambda\varphi$ we have

$$\begin{aligned}
&\frac{\gamma\varphi(x - \alpha)}{(1 + \tilde{w}(x - \alpha))^{\gamma+1}} - \frac{1}{L}\int_\alpha^{L+\alpha} \frac{\gamma\varphi(x - \alpha)}{(1 + \tilde{w}(x - \alpha))^{\gamma+1}}dx \\
&\quad - \lambda\int_\alpha^{L+\alpha} K_L(x - \alpha, y - \alpha)\varphi(y - \alpha)dy = \Lambda\varphi(x - \alpha),
\end{aligned}$$

and hence $T^\alpha\varphi(\,\cdot\, - \alpha) = \Lambda\varphi(\,\cdot\, - \alpha)$ in view of $K_L(x - \alpha, y - \alpha) = K_L(x, y)$ and the L-periodicity of $\tilde{w}$, φ and $K_L(x, \cdot\,)$. To sum up, we are allowed to study the lower bound of T focusing on the nonpositive eigenvalues of T after a favorable shift of $\tilde{v}$.

With the above considerations in mind we prove the following.

Lemma 7 Let k be an integer satisfying (2.2), and $\tilde{v}^{(k)}$ as in **Theorem 1**. Let T be the self-adjoint operator on $\mathcal{H}$ that corresponds to the stationary solution $(\tilde{v}^{(k)}, 0)$ by (5.3). If $k \geq 2$, then the lower bound of T is a negative eigenvalue.

Proof: As shown just before the statement of the lemma, we may assume that $\tilde{v}^{(k)}$ is even and attains its maximum at $x = 0$. Such a stationary solution with least period L/k is unique. Rewriting (2.2) with a parameter $\lambda = 4\pi G V^\gamma / a$, we consider the stationary solution as parametrized over the interval

$$\gamma \left(\frac{2\pi k}{L}\right)^2 < \lambda < 2\left(\frac{I_\gamma\left((\gamma-1)^{-1/2} - 0\right)k}{L}\right)^2, \tag{5.5}$$

and denote $\tilde{v}^{(k)}/V - 1$ by $\tilde{w}_\lambda$. We first notice that $\lambda \mapsto \tilde{w}_\lambda$ is a continuous function with values in the space of continuously differentiable L-periodic functions on (5.5) with uniform limit

$$\lim_{\lambda \to \gamma(2\pi k/L)^2 + 0} \tilde{w}_\lambda(x) = 0.$$

To show this put $r_\lambda = (1 + \tilde{w}_\lambda)^{-\gamma} - 1$ and notice that r_λ is a solution of (4.2) attaining its minimum $r_{\lambda,\min}$, which is negative, at $x = 0$. Since the energy of the orbit of r_λ is given by $\lambda\left({I_\gamma}^{-1}\left((L/k)\sqrt{\lambda/2}\right)\right)^2$, we have $F(r_{\lambda,\min}) = \left({I_\gamma}^{-1}\left((L/k)\sqrt{\lambda/2}\right)\right)^2$. See (4.3) and (4.4). This together with

$$\lim_{\lambda \to \gamma(2\pi k/L)^2 + 0} {I_\gamma}^{-1}\left((L/k)\sqrt{\lambda/2}\right) = {I_\gamma}^{-1}\left(\sqrt{2\gamma}\pi + 0\right) = 0,$$

coming from **Lemma 4**, implies that $r_{\lambda,\min}$ depends continuously on λ with $r_{\lambda,\min} \to 0$ as $\lambda \to \gamma\,(2\pi k/L)^2 + 0$. The continuity of r_λ with respect to λ as well as the uniform convergence $r_\lambda(x) \to 0$ as $\lambda \to \gamma\,(2\pi k/L)^2 + 0$ follows from continuous dependence on initial data in the Cauchy problem for (4.2). Thus, the map $\lambda \mapsto \tilde{w}_\lambda$ enjoys the continuity as desired. Now put

$$(T_\lambda \varphi)(x) = \frac{\gamma\varphi(x)}{(1+\tilde{w}_\lambda(x))^{\gamma+1}} - \frac{1}{L}\int_0^L \frac{\gamma\varphi(x)}{(1+\tilde{w}_\lambda(x))^{\gamma+1}}dx - \lambda\int_0^L K_L(x,y)\varphi(y)dy.$$

In view of $K_L(-x, y) = K_L(x, -y)$ and the L-periodicity of $\tilde{w}_\lambda$ and $K_L(x, \cdot)$, T_λ maps an odd function into an odd one. The restriction of T_λ onto the subspace $\mathcal{H}^{(o)} = \{\varphi \in \mathcal{H}; \varphi(-x) = -\varphi(x)\}$ of $\mathcal{H}$ is denoted by $T^{(o)}_\lambda$. As shown above, $T^{(o)}_\lambda$ as well as T_λ has the eigenvalue 0 with eigenvector $\tilde{w}'_\lambda \in \mathcal{H}^{(o)}$. Moreover, the eigenvalue 0 is simple. This is because the equation $T^{(o)}_\lambda \varphi = 0$ is equivalent to the second order linear differential equation $\partial_x^2\left\{\gamma(1+\tilde{w}_\lambda(x))^{-\gamma-1}\varphi(x)\right\} + \lambda\varphi(x) = 0$ and any odd solution of the differential equation must be proportional to the solution $\tilde{w}'_\lambda$ by the uniqueness of solutions to the Cauchy problem. Noting that $\inf\sigma(T_\lambda) \leq \inf\sigma(T^{(o)}_\lambda)$, we show that the lower bound of $T^{(o)}_\lambda$ is negative.

From the continuous dependence of $\tilde{w}_\lambda$ on λ we see that the correspondence $\lambda \mapsto T^{(o)}_\lambda$ is continuous on the interval (5.5) up to the left end $\gamma\,(2\pi k/L)^2$ with respect to the operator norm. The limit $T^{(o)}_{\gamma(2\pi k/L)^2+0}$ is the restriction onto $\mathcal{H}^{(o)}$ of the operator (5.3) with $\lambda = \gamma\,(2\pi k/L)^2$ and $\tilde{w} = 0$, and its lower bound is the eigenvalue $\gamma(1-k^2)$ with eigenvector

$\sin(2\pi/L)x$, as shown by (5.4). Thus, the correspondence $\lambda \mapsto \inf \sigma(T_\lambda^{(\mathrm{o})})$ gives a continuous function on (5.5) with

$$\lim_{\lambda \to \gamma(2\pi k/L)^2+0} \inf \sigma(T_\lambda^{(\mathrm{o})}) = \inf \sigma(T_{\gamma(2\pi k/L)^2+0}^{(\mathrm{o})}) = \gamma(1-k^2),$$

which is negative by the assumption $k \geq 2$. Put $c(\lambda) = \inf \sigma(T_\lambda^{(\mathrm{o})})$ and suppose the function $\lambda \mapsto c(\lambda)$ admits a nonnegative value on (5.5). In view of the continuity of the function and $c(\gamma\,(2\pi k/L)^2 + 0) < 0$ as proved above, there does exist a zero of the function. The smallest zero is denoted by λ_*. For $\gamma\,(2\pi k/L)^2 < \lambda < \lambda_*$, since $c(\lambda) < 0$, $c(\lambda)$ is proved to be an eigenvalue as the lower bound of T_λ is. Let φ_λ be an eigenvector associated with $c(\lambda)$ satisfying $\|\varphi_\lambda\|_{L^2} = 1$. By the boundedness of $\{\varphi_\lambda; \gamma\,(2\pi k/L)^2 < \lambda < \lambda_*\}$ in $\mathcal{H}^{(\mathrm{o})}$ we can choose a sequence $\{\lambda_n; n = 1, 2, \dots\}$ and an element φ_{λ_*} of $\mathcal{H}^{(\mathrm{o})}$ so that $\gamma\,(2\pi k/L)^2 < \lambda_n < \lambda_*$, $\lambda_n \to \lambda_*$ as $n \to \infty$, and the sequence $\{\varphi_{\lambda_n}; n = 1, 2, \dots\}$ converges to φ_{λ_*} weakly in $\mathcal{H}^{(\mathrm{o})}$ as $n \to \infty$. Noting that $\varphi_\lambda(1 + \tilde{w}_\lambda)^{-\gamma-1}$ is an odd function, we rewrite $T_\lambda^{(\mathrm{o})}\varphi_\lambda = c(\lambda)\varphi_\lambda$ as

$$\varphi_\lambda(x) = \frac{(1 + \tilde{w}_\lambda(x))^{\gamma+1}}{\gamma} \left(\lambda \int_0^L K_L(x,y)\varphi_\lambda(y)dy + c(\lambda)\varphi_\lambda(x) \right)$$

and then take the limit along the sequence. Since, as $\lambda \to \lambda_*$, $\tilde{w}_\lambda$ converges uniformly to $\tilde{w}_{\lambda_*}$ and $c(\lambda) \to c(\lambda_*) = 0$, and since the integral operator with kernel K_L is compact on $\mathcal{H}^{(\mathrm{o})}$, the sequence $\{\varphi_{\lambda_n}; n = 1, 2, \dots,\}$ converges strongly in L^2 and also in $\mathcal{H}^{(\mathrm{o})}$. Therefore, $\|\varphi_{\lambda_*}\|_{L^2} = 1$ and $T_{\lambda_*}^{(\mathrm{o})}\varphi_{\lambda_*} = 0$ hold. This shows that φ_{λ_*} is an eigenvector of $T_{\lambda_*}^{(\mathrm{o})}$ associated with the eigenvalue 0. Since $\tilde{w}'_\lambda$ and φ_λ are orthogonal to each other for $\gamma\,(2\pi k/L)^2 < \lambda < \lambda_*$, so are $\tilde{w}'_{\lambda_*}$ and φ_{λ_*} by passage to the limit along the sequence and the continuity of $\tilde{w}'_\lambda$ with respect to λ. In particular, $\tilde{w}'_{\lambda_*}$ and φ_{λ_*} are independent, however, this contradicts the simplicity of the eigenvalue 0.

Thus, the lower bound of $T_\lambda^{(\mathrm{o})}$ must be negative over the interval (5.5), as desired. □

We are now in position to present a condition for A_V to be nonempty. Given **Proposition 1** and **Lemma 7**, we think it reasonable to pick up the cases in which the minimal value of the energy form on S_V as in (4.9) is attained either at the stationary solution $(\tilde{v}^{(k_{\min})}, 0)$ with $k_{\min} \geq 2$ or at the trivial solution $(V, 0)$ with $\inf \sigma(T) < 0$. In view of **Remark 1** and (5.4), the condition that we propose turns out to be

$$V \geq \left(\frac{aI_\gamma((\gamma-1)^{-1/2} - 0)^2}{2\pi G L^2} \right)^{1/\gamma},$$

the assumption of **Theorem 2**. Now the proof of the theorem is completed.

Remark 3 In the proof of **Lemma 7** we rely on the fact that the lower bound of the operator $T_\lambda^{(\mathrm{o})}$ is somewhere negative on the interval (5.5). Here we essentially make use of the assumption $k \geq 2$. In case $k = 1$, however, the situation is subtle, and in fact the lower bound of T corresponding to the stationary solution $(\tilde{v}^{(1)}, 0)$ proves the eigenvalue 0, which is isolated and simple. An outline of the reasoning is given by [6], where the spectrum of the restriction of T_λ onto the space of even functions are considered with the use of **Lemma 4** and the result of Crandall and Rabinowitz [2] on the perturbation of simple eigenvalues along bifurcation curves of stationary solutions. The result shows, in some sense, the stability of the set of stationary solutions having a profile in common, and in order to find an element of A_V for $\left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma} < V < \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$ we are forced to study the behavior of the energy form beyond a small neighborhood of the set of stationary solutions, which is a global and therefore difficult problem. The situation is quite similar in case $V \leq \left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma}$ since the trivial solution $(V, 0)$ is the unique stationary solution on M_V with stability in some sense.

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Communicated by *Atsushi Yagi*

Masahiro Sawada
Graduate School of Information Science and Technology
Osaka University
Suita, 565-0871 Osaka, Japan
E-mail address: masahiro.sawada@ist.osaka-u.ac.jp

Yoshitaka Yamamoto
Graduate School of Information Science and Technology
Osaka University
Suita, 565-0871 Osaka, Japan
E-mail address: yamamoto@ist.osaka-u.ac.jp