SOME INTEGRALS BETWEEN THE LEBESGUE INTEGRAL AND THE DENJOY INTEGRAL

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Abstract. B. Bongiorno, Di Piazza and Preiss gave a minimal constructive integration process of Riemann type, called the C-integral, which contains the Lebesgue integral and the Newton integral. D. Bongiorno gave a minimal constructive integration process of Riemann type, called the $\tilde{C}$-integral, which contains the Lebesgue integral and the improper Newton integral. On the other hand, Nakanishi gave criteria for the restricted Denjoy integrability. Motivated by the results of Nakanishi, Kawasaki and Suzuki gave criteria for the C-integrability, and Kawasaki gave criteria for the $\tilde{C}$-integrability. In this paper, motivated by the results above, we give new integrals between the Lebesgue integral and the restricted Denjoy integral. Moreover we give criteria for the integrability of one of them in the style of Nakanishi.

1 Introduction Throughout this paper we denote by $\mathbf{(L)}(S)$, $\mathbf{(L^*)}(S)$ and $\mathbf{(D^*)}(S)$ the class of all Lebesgue integrable functions, the class of all improper Lebesgue integrable functions and the class of all restricted Denjoy integrable functions from a measurable set $S \subset \mathbb{R}$ into $\mathbb{R}$, respectively, and we denote by $|A|$ the measure of a measurable set A. We recall that a gauge δ is a function from an interval $[a,b]$ into $(0,\infty)$ and a δ-fine McShane partition of an interval $[a,b] \subset \mathbb{R}$ is a collection $\{(I_k, x_k) \mid k = 1,\dots,k_0\}$ of non-overlapping intervals $I_k \subset [a,b]$ and $x_k \in [a,b]$ satisfying $I_k \subset (x_k - \delta(x_k), x_k + \delta(x_k))$ and $\sum_{k=1}^{k_0} |I_k| = b-a$. If $\sum_{k=1}^{k_0} |I_k| \leq b-a$, then we say that the collection is a δ-fine partial McShane partition. Moreover, if $x_k \in I_k$ for any $k = 1,\dots,k_0$, then a δ-fine McShane partition and a δ-fine partial McShane partition are called a δ-fine Perron partition and a δ-fine partial Perron partition, respectively. We say that a function f from an interval $[a,b]$ into $\mathbb{R}$ is Newton integrable if there exists a differentiable function F from $[a,b]$ into $\mathbb{R}$ such that $F' = f$ on $[a,b]$. We denote by $\mathbf{(N)}([a,b])$ the class of all Newton integrable functions from $[a,b]$ into $\mathbb{R}$. In [3] B. Bongiorno, Di Piazza and Preiss gave a minimal constructive integration process of Riemann type, called the C-integral, which contains the Lebesgue integral and the Newton integral. Furthermore in [1–3] B. Bongiorno et al. gave some criteria for the C-integrability. We denote by $\mathbf{(C)}([a,b])$ the class of all C-integrable functions from $[a,b]$ into $\mathbb{R}$. We say that a function f from an interval $[a,b]$ into $\mathbb{R}$ is improper Newton integrable if there exist a countable subset $N \subset [a,b]$ and a function F from $[a,b]$ into $\mathbb{R}$ such that $F' = f$ on $[a,b] \setminus N$. We denote by $\mathbf{(N^*)}([a,b])$ the class of all improper Newton integrable functions from $[a,b]$ into $\mathbb{R}$. In [4] D. Bongiorno gave a minimal constructive integration process of Riemann type, called the $\tilde{C}$-integral, which contains the Lebesgue integral and the improper Newton integral. Furthermore in [4] D. Bongiorno gave some criteria for the $\tilde{C}$-integrability. We denote by $(\tilde{C})([a,b])$ the class of all $\tilde{C}$-integrable functions from $[a,b]$ into $\mathbb{R}$. The improper Lebesgue integral, the C-integral and the $\tilde{C}$-integral are between the Lebesgue integral and the restricted Denjoy integral.

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On the other hand, in [11, 14] Nakanishi gave criteria for the restricted Denjoy integrability. Motivated by the results of Nakanishi, in [10] Kawasaki and Suzuki gave criteria for the C-integrability, and in [9] Kawasaki gave criteria for the $\tilde{C}$-integrability.

In this paper, motivated by the results above, we give new integrals between the Lebesgue integral and the restricted Denjoy integral. Moreover we give criteria for the integrability of one of them in the style of Nakanishi.

2 Preliminaries We know that the Lebesgue integral and the restricted Denjoy integral are equivalent to the McShane integral and the Henstock-Kurzweil integral, respectively. The McShane integral and the Henstock-Kurzweil integral are Riemann type integrals and these definitions are as follows.

Definition 2.1. A function f from an interval $[a, b]$ into $\mathbb{R}$ is McShane integrable if there exists a constant A such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$. The constant A is the value of the McShane integral of f and we denote by

$$A = (MS)\int_{[a,b]} f(x)dx = (L)\int_{[a,b]} f(x)dx.$$

We denote by **(MS)**$([a, b])$ the class of all McShane integrable functions from $[a, b]$ into $\mathbb{R}$.

Definition 2.2. A function f from an interval $[a, b]$ into $\mathbb{R}$ is Henstock-Kurzweil integrable if there exists a constant A such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ with $x_k \in I_k$, that is, δ-fine Perron partition. The constant A is the value of the Henstock-Kurzweil integral of f and we denote by

$$A = (HK)\int_{[a,b]} f(x)dx = (D^*)\int_{[a,b]} f(x)dx.$$

We denote by **(HK)**$([a, b])$ the class of all Henstock-Kurzweil integrable functions from $[a, b]$ into $\mathbb{R}$.

In [5] D. Bongiorno showed a criterion for the improper Lebesgue integral as follows.

Theorem 2.1. *A function f from an interval $[a, b]$ into $\mathbb{R}$ is improper Lebesgue integrable if and only if there exist a constant A and a finite subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that*

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$. Moreover

$$A = (L^*) \int_{[a,b]} f(x)dx.$$

The theorem above gives a Riemann type definition for the improper Lebesgue integral. In [1], see also [2,3], B. Bongiorno gave the C-integral, which is also a Riemann type integral, as follows.

Definition 2.3. A function f from an interval $[a, b]$ into $\mathbb{R}$ is C-integrable if there exists a constant A such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ satisfying $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$, where $d(I, x) = \inf_{y \in I} |y - x|$. The constant A is the value of the C-integral of f and we denote by

$$A = (C) \int_{[a,b]} f(x)dx.$$

We denote by $\mathbf{(C)}([a, b])$ the class of all C-integrable functions from $[a, b]$ into $\mathbb{R}$.

In [4] D. Bongiorno gave the $\tilde{C}$-integral, which is also a Riemann type integral, as follows.

Definition 2.4. A function f from an interval $[a, b]$ into $\mathbb{R}$ is $\tilde{C}$-integrable if there exist a constant A and a countable subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ whenever $x_k \in N$.

The constant A is the value of the $\tilde{C}$-integral of f and we denote by

$$A = (\tilde{C}) \int_{[a,b]} f(x)dx.$$

We denote by $(\tilde{C})([a, b])$ the class of all $\tilde{C}$-integrable functions from $[a, b]$ into $\mathbb{R}$.

Throughout this paper, we say that a function defined on the class of all intervals in $[a, b]$ is an interval function on $[a, b]$. If an interval function F on $[a, b]$ satisfies $F(I_1 \cup I_2) = F(I_1) + F(I_2)$ for any intervals $I_1, I_2 \subset [a, b]$ with ${I_1}^i \cap {I_2}^i = \emptyset$, where I^i is the interior of I, then it is said to be additive. In [11, 14] Nakanishi gave the following criteria for the restricted Denjoy integrability. Firstly Nakanishi considered the following four criteria for the pair of a function f from $[a, b]$ into $\mathbb{R}$ and an additive interval function F on $[a, b]$.

(A) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a, b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) $\left|\sum_{k=1}^{k_0} \left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1, \dots, k_0\}$ of non-overlapping intervals in $[a, b]$ with $I_k \cap F_n \neq \emptyset$.

(B) For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a, b]$;

(2) $F_n \subset M_n$ for any n and $|[a, b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) $\left|\sum_{k=1}^{k_0} \left(F(I_k) - (L)\int_{I_k \cap F_n} f(x)dx\right)\right| < \varepsilon_n$ for any n and for any finite family $\{I_k \mid k = 1, \dots, k_0\}$ of non-overlapping intervals in $[a, b]$ with $I_k \cap M_n \neq \emptyset$.

(C) There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a, b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) for any n and for any positive number ε there exists a positive number η such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1, \dots, k_0\}$ of non-overlapping intervals in $[a, b]$ satisfying

(3.1) $I_k \cap F_n \neq \emptyset$ for any k;

(3.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a, b]$

$$F(I) = (L)\int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p \in \mathbb{N}\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

(D) There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a, b]$;

(2) $F_n \subset M_n$ for any n and $|[a, b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) for any n and for any positive number ε there exists a positive number η such that

$$\left|\sum_{k=1}^{k_0} F(I_k)\right| < \varepsilon$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a, b]$ satisfying

(4.1) $I_k \cap M_n \neq \emptyset$ for any k;

(4.2) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a, b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p \in \mathbb{N}\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

Next Nakanishi gave the following theorem for the restricted Denjoy integrability.

Theorem 2.2. *A function f from an interval $[a, b]$ into $\mathbb{R}$ is restricted Denjoy integrable if and only if there exists an additive interval function F on $[a, b]$ such that the pair of f and F satisfies one of* (A), (B), (C) *and* (D). *Moreover, if the pair of f and F satisfies one of* (A), (B), (C) *and* (D), *then*

$$F(I) = (D^*)\int_I f(x)dx$$

holds for any interval $I \subset [a, b]$.

Motivated by the results of Nakanishi, in [10] Kawasaki and Suzuki gave similar criteria and theorems for the C-integrability, and in [9] Kawasaki give similar criteria and theorems for the $\tilde{C}$-integrability.

3 Definitions of new integrals In this section firstly we define new integrals. By observing the definitions of the McShane, the improper Lebesgue in the sense of Theorem 2.1, the Henstock-Kurzweil integrals, C-integral and $\tilde{C}$-integral, we become aware of the following two integrals.

Definition 3.1. A function f from an interval $[a, b]$ into $\mathbb{R}$ is C*-integrable if there exist a constant A and a finite subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ whenever $x_k \in N$.

The constant A is the value of the C*-integral of f and we denote by

$$A = (C^*) \int_{[a,b]} f(x)dx.$$

We denote by $(\mathbf{C}^*)([a,b])$ the class of all C*-integrable functions from $[a,b]$ into $\mathbb{R}$.

Definition 3.2. A function f from an interval $[a,b]$ into $\mathbb{R}$ is $\tilde{L}$-integrable if there exist a constant A and a countable subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$. The constant A is the value of the $\tilde{L}$-integral of f and we denote by

$$A = (\tilde{L}) \int_{[a,b]} f(x)dx.$$

We denote by $(\tilde{L})([a,b])$ the class of all $\tilde{L}$-integrable functions from $[a,b]$ into $\mathbb{R}$.

By the definitions of these integrals we obtain the following relations.

$$\begin{array}{ccccccc}
(\mathbf{N}) & & \subset & & (\mathbf{N}^*) & & (\mathbf{D}^*) \\
\cap & & & & \cap & & \\
(\mathbf{C}) & \subset & (\mathbf{C}^*) & \subset & (\tilde{C}) & & \| \\
\cup & & & & & \subset & \\
(\mathbf{MS}) & & \cup & & \cup & & (\mathbf{HK}) \\
\| & & & & & & \\
(\mathbf{L}) & \subset & (\mathbf{L}^*) & \subset & (\tilde{L}) & &
\end{array}$$

The above relations of inclusion are proper. We give some examples to check these. To show these, we provide the Saks-Henstock type lemmas. The following is the Saks-Henstock type lemma for the C*-integral.

Theorem 3.1. *If $f \in (\mathbf{C}^*)([a,b])$, then there exists a finite subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (C^*) \int_{I_k} f(x)dx \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ *whenever* $x_k \in N$.

Proof. Since $f \in (\mathbf{C}^*)([a,b])$, there exists a finite subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} f(x_k)|I_k| - (C^*) \int_{[a,b]} f(x)dx \right| < \frac{\varepsilon}{4}$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_1\}$ satisfying

$$\sum_{k=1}^{k_1} d(I_k, x_k) < \frac{2}{\varepsilon}$$

and $x_k \in I_k$ whenever $x_k \in N$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a δ-fine partial McShane partition satisfying

$$\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$$

and $x_k \in I_k$ whenever $x_k \in N$, and let $\{I_k \mid k = k_0 + 1, \ldots, k_1\}$ be the sequence of intervals satisfying

$$\bigcup_{k=1}^{k_1} I_k = [a, b]$$

and $I^i_{k_2} \cap I^i_{k_3} = \emptyset$ if $k_2 \neq k_3$. Since f is C*-integrable on each I_k $(k = k_0 + 1, \ldots, k_1)$, there exists a gauge δ_k such that

$$\left| \sum_{\ell=1}^{\ell(k)} \left(f(x_{k,\ell})|I_{k,\ell}| - (C^*) \int_{I_{k,\ell}} f(x)dx \right) \right| < \frac{\varepsilon}{4(k_1 - k_0)}$$

for any δ_k-fine McShane partition $\{(I_{k,\ell}, x_{k,\ell}) \mid \ell = 1, \ldots, \ell(k)\}$ satisfying

$$\sum_{\ell=1}^{\ell(k)} d(I_{k,\ell}, x_{k,\ell}) < \frac{1}{\varepsilon(k_1 - k_0)}$$

and $x_{k,\ell} \in I_{k,\ell}$ whenever $x_{k,\ell} \in N$. Without loss of generality, it may be assumed that $\delta_k \leq \delta$ for any $k = k_0 + 1, \ldots, k_1$. Note that

$$\sum_{k=1}^{k_0} d(I_k, x_k) + \sum_{k=k_0+1}^{k_1} \sum_{\ell=1}^{\ell(k)} d(I_{k,\ell}, x_{k,\ell}) < \frac{1}{\varepsilon} + \sum_{k=k_0+1}^{k_1} \frac{1}{\varepsilon(k_1 - k_0)} = \frac{2}{\varepsilon}.$$

Therefore we obtain

$$\begin{aligned}
&\left| \sum_{k=1}^{k_0} \left(f(x_k)|I_k| - (C^*) \int_{I_k} f(x)dx \right) \right| \\
&\quad \leq \left| \sum_{k=1}^{k_1} f(x_k)|I_k| - (C^*) \int_{[a,b]} f(x)dx \right| \\
&\qquad + \sum_{k=k_0+1}^{k_1} \left| \sum_{\ell=1}^{\ell(k)} \left(f(x_{k,\ell})|I_{k,\ell}| - (C^*) \int_{I_{k,\ell}} f(x)dx \right) \right| \\
&\quad < \frac{\varepsilon}{4} + \sum_{k=k_0+1}^{k_1} \frac{\varepsilon}{4(k_1 - k_0)} = \frac{\varepsilon}{2}.
\end{aligned}$$

Moreover we obtain

$$
\begin{aligned}
&\sum_{k=1}^{k_0}\left|f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx\right| \\
&= \left|\sum_{f(x_k)|I_k|-(C^*)\int_{I_k} f(x)dx>0}\left(f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx\right)\right| \\
&\quad + \left|\sum_{f(x_k)|I_k|-(C^*)\int_{I_k} f(x)dx<0}\left(f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx\right)\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

□

The following is the Saks-Henstock type lemma for the $\tilde{L}$-integral. The proof is similar to Theorem 3.1.

Theorem 3.2. *If $f \in (\tilde{L})([a,b])$, then there exists a countable subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0}\left|f(x_k)|I_k| - (\tilde{L})\int_{I_k} f(x)dx\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$.

The Saks-Henstock type lemma for the improper Lebesgue integral also holds, see [5].

Theorem 3.3. *If $f \in (\mathbf{L}^*)([a,b])$, then there exists a finite subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0}\left|f(x_k)|I_k| - (L^*)\int_{I_k} f(x)dx\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$.

We show that the above relations of inclusion are proper.

Theorem 3.4. *There exists a function f such that $f \in (\mathbf{C}^*)([0,1])$ but $f \notin (\mathbf{C})([0,1])$.*

Proof. Let f_1 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_1(x) = \begin{cases} (1-2x)\left(\sin\frac{1}{x(1-x)} - \frac{1}{x(1-x)}\cos\frac{1}{x(1-x)}\right), & \text{if } x \in (0,1), \\ 0, & \text{if } x \in \{0,1\}, \end{cases}$$

and let F_1 be a function defined by

$$F_1(x) = \begin{cases} x(1-x)\sin\frac{1}{x(1-x)}, & \text{if } x \in (0,1), \\ 0, & \text{if } x \in \{0,1\}. \end{cases}$$

Since f_1 is continuous on $(0,1)$ and

$$\lim_{\alpha\downarrow 0,\beta\uparrow 1}(L)\int_{[\alpha,\beta]} f_1(x)dx = \lim_{\alpha\downarrow 0,\beta\uparrow 1}(F_1(\beta)-F_1(\alpha)) = 0,$$

we obtain $f_1 \in \mathbf{(L^*)}([0,1])$ and hence $f_1 \in \mathbf{(C^*)}([0,1])$. However $f_1 \notin \mathbf{(C)}([0,1])$. Indeed, assume that $f_1 \in \mathbf{(C)}([0,1])$. Then by [2, Lemma 6] for any positive number ε with $\varepsilon < 1$ there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f_1(x_k)(b_k-a_k) - (F_1(b_k)-F_1(a_k))| < \varepsilon$$

for any δ-fine partial McShane partition $\{([a_k,b_k],x_k) \mid k=1,\ldots,k_0\}$ satisfying

$$\sum_{k=1}^{k_0} d([a_k,b_k],x_k) < \frac{1}{\varepsilon}.$$

For any natural number n let

$$\begin{aligned} a_n &= \frac{1-\sqrt{1-\frac{4}{\frac{3}{2}\pi+2n\pi}}}{2}, \\ b_n &= \frac{1-\sqrt{1-\frac{4}{\frac{\pi}{2}+2n\pi}}}{2}. \end{aligned}$$

Note that $\{[a_n,b_n]\}$ is mutually disjoint and

$$\begin{aligned} F_1(a_n) &= -a_n(1-a_n) = -\frac{1}{\frac{3}{2}\pi+2n\pi}, \\ F_1(b_n) &= b_n(1-b_n) = \frac{1}{\frac{\pi}{2}+2n\pi}. \end{aligned}$$

Since the sequence $\{b_n(1-b_n)+a_n(1-a_n) \mid n \in \mathbb{N}\}$ is a strictly decreasing sequence tending to 0 and

$$\begin{aligned} &0 < b_n(1-b_n)+a_n(1-a_n), \\ &\sum_{n=1}^{\infty}(b_n(1-b_n)+a_n(1-a_n)) = \infty, \end{aligned}$$

we can take a strictly increasing finite sequence $\{n(k) \mid k=1,\ldots,k_0\}$ satisfying $b_{n(1)} < \delta(0)$ and

$$\varepsilon < \sum_{k=1}^{k_0}(b_{n(k)}(1-b_{n(k)})+a_{n(k)}(1-a_{n(k)})) < \frac{1}{\varepsilon}.$$

Then $\{([a_{n(k)},b_{n(k)}],0) \mid k=1,\ldots,k_0\}$ is a δ-fine partial McShane partition and satisfies

$$\sum_{k=1}^{k_0} d([a_{n(k)},b_{n(k)}],0) = \sum_{k=1}^{k_0} a_{n(k)} < \sum_{k=1}^{k_0}(b_{n(k)}(1-b_{n(k)})+a_{n(k)}(1-a_{n(k)})) < \frac{1}{\varepsilon}.$$

However

$$\begin{aligned}
&\sum_{k=1}^{k_0} |f_1(0)(b_{n(k)} - a_{n(k)}) - (F_1(b_{n(k)}) - F_1(a_{n(k)}))| \\
&\quad = \sum_{k=1}^{k_0} |F_1(b_{n(k)}) - F_1(a_{n(k)})| \\
&\quad = \sum_{k=1}^{k_0} (b_{n(k)}(1 - b_{n(k)}) + a_{n(k)}(1 - a_{n(k)})) \\
&\quad > \varepsilon
\end{aligned}$$

and hence it is a contradiction. □

Theorem 3.5. *There exists a function f such that $f \in (\tilde{C})([0,1])$ but $f \notin (\mathbf{C}^*)([0,1])$.*

Proof. Let f_2 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_2(x) = \begin{cases} n(n+1)f_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \;\middle|\; n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

and let F_2 be a function defined by

$$F_2(x) = \begin{cases} F_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \;\middle|\; n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

where f_1 and F_1 are the functions in Theorem 3.4. Since $F_2'(x) = f_2(x)$ for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $n \in \mathbb{N}$, we obtain $f_2 \in (\mathbf{N}^*)([0,1])$ and hence $f_2 \in (\tilde{C})([0,1])$. However $f_2 \notin (\mathbf{C}^*)([0,1])$. Indeed, assume that $f_2 \in (\mathbf{C}^*)([0,1])$. Then by Theorem 3.1 there exists a finite subset $N \subset [0,1]$ such that for any positive number ε with $\varepsilon < 1$ there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f_2(x_k)(b_k - a_k) - (F_2(b_k) - F_2(a_k))| < \varepsilon$$

for any δ-fine partial McShane partition $\{([a_k, b_k], x_k) \mid k = 1, \dots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d([a_k, b_k], x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in [a_k, b_k]$ whenever $x_k \in N$.

Since N is finite, there exists a natural number p such that $\left[\frac{1}{p+1}, \frac{1}{p}\right] \cap N = \emptyset$. For any natural number n let

$$\begin{aligned}
a_n &= \frac{1}{p+1} + \frac{1 - \sqrt{1 - \frac{4}{\frac{3}{2}\pi + 2n\pi}}}{2p(p+1)}, \\
b_n &= \frac{1}{p+1} + \frac{1 - \sqrt{1 - \frac{4}{\frac{\pi}{2} + 2n\pi}}}{2p(p+1)}.
\end{aligned}$$

Note that $\{[a_n, b_n]\}$ is mutually disjoint and

$$\begin{aligned}
F_2(a_n) &= -(p(p+1)a_n - p)(p+1-p(p+1)a_n) \\
&= -p(p+1)((p+1)a_n - 1)(1 - pa_n) \\
&= -\frac{1}{\frac{3}{2}\pi + 2n\pi}, \\
F_2(b_n) &= (p(p+1)b_n - p)(p+1-p(p+1)b_n) \\
&= p(p+1)((p+1)b_n - 1)(1 - pb_n) \\
&= \frac{1}{\frac{\pi}{2} + 2n\pi}.
\end{aligned}$$

Since the sequence $\{p(p+1)(((p+1)b_n - 1)(1-pb_n) + ((p+1)a_n - 1)(1-pa_n)) \mid n \in \mathbb{N}\}$ is a strictly decreasing sequence tending to 0 and

$$\begin{aligned}
&0 < p(p+1)(((p+1)b_n - 1)(1-pb_n) + ((p+1)a_n - 1)(1-pa_n)), \\
&\sum_{n=1}^{\infty} p(p+1)(((p+1)b_n - 1)(1-pb_n) + ((p+1)a_n - 1)(1-pa_n)) = \infty,
\end{aligned}$$

we can take a strictly increasing finite sequence $\{n(k) \mid k = 1, \ldots, k_0\}$ satisfying $b_{n(1)} < \frac{1}{p+1} + \delta\left(\frac{1}{p+1}\right)$ and

$$\varepsilon < \sum_{k=1}^{k_0} p(p+1)(((p+1)b_{n(k)} - 1)(1-pb_{n(k)}) + ((p+1)a_{n(k)} - 1)(1-pa_{n(k)})) < \frac{1}{\varepsilon}.$$

Then $\left\{ \left([a_{n(k)}, b_{n(k)}], \frac{1}{p+1}\right) \,\middle|\, k = 1, \ldots, k_0 \right\}$ is a δ-fine partial McShane partition and

$$\begin{aligned}
&\sum_{k=1}^{k_0} d\left([a_{n(k)}, b_{n(k)}], \frac{1}{p+1}\right) \\
&= \sum_{k=1}^{k_0} \left(a_{n(k)} - \frac{1}{p+1}\right) \\
&< \sum_{k=1}^{k_0} p(p+1)(((p+1)b_{n(k)} - 1)(1-pb_{n(k)}) + ((p+1)a_{n(k)} - 1)(1-pa_{n(k)})) \\
&< \frac{1}{\varepsilon}.
\end{aligned}$$

However

$$\begin{aligned}
&\sum_{k=1}^{k_0} \left| f_2\left(\frac{1}{p+1}\right)(b_{n(k)} - a_{n(k)}) - (F_2(b_{n(k)}) - F_2(a_{n(k)})) \right| \\
&= \sum_{k=1}^{k_0} |F_2(b_{n(k)}) - F_2(a_{n(k)})| \\
&= \sum_{k=1}^{k_0} p(p+1)(((p+1)b_{n(k)} - 1)(1-pb_{n(k)}) + ((p+1)a_{n(k)} - 1)(1-pa_{n(k)})) \\
&> \varepsilon
\end{aligned}$$

and hence it is a contradiction. □

Theorem 3.6. *There exists a function f such that $f \in (\tilde{L})([0,1])$ but $f \notin (\mathbf{L}^*)([0,1])$.*

Proof. Let f_3 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_3(x) = \begin{cases} f_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

and let F_3 be a function defined by

$$F_3(x) = \begin{cases} \frac{1}{n(n+1)} F_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

where f_1 and F_1 are the functions in Theorem 3.4. Then $f_3 \in (\tilde{L})([0,1])$ but $f_3 \notin (\mathbf{L}^*)([0,1])$. Indeed, since f_3 is improper Lebesgue integrable on each $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ and

$$(L^*) \int_{\left[\frac{1}{n+1}, \frac{1}{n}\right]} f_3(x) dx = 0,$$

by Theorem 2.1 there exists a finite subset $N_n \subset \left[\frac{1}{n+1}, \frac{1}{n}\right]$ such that for any positive number ε there exists a gauge δ_n such that

$$\left| \sum_{k=1}^{k_n} f_3(x_{n,k}) |I_{n,k}| \right| < \frac{\varepsilon}{2^{n+1}}$$

for any δ_n-fine McShane partition $\{(I_{n,k}, x_{n,k}) \mid k = 1, \ldots, k_n\}$ of $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ satisfying $x_{n,k} \in I_{n,k}$ whenever $x_{n,k} \in N_n$. It is obvious that $N_n = \left\{\frac{1}{n+1}, \frac{1}{n}\right\}$. Let

$$M_n = \max\left\{ |F_3(x)| \,\middle|\, x \in \left[\frac{1}{n+1}, \frac{1}{n}\right] \right\}.$$

It holds that $M_n = \frac{2}{n(n+1)} M_1$. Without loss of generality, it may be assumed that $(x - \delta_n(x), x + \delta_n(x)) \subset \left(\frac{1}{n+1}, \frac{1}{n}\right)$ for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$. Let $N = \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\} \cup \{0\}$, $\delta(x) = \delta_n(x)$ for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $\delta\left(\frac{1}{n}\right) = \min\left\{\delta_n\left(\frac{1}{n}\right), \delta_{n-1}\left(\frac{1}{n}\right)\right\}$ for any $n \in \mathbb{N}$ with $n \geq 2$ and $\delta(0) < \frac{1}{p}$ with $M_p < \frac{\varepsilon}{2}$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$. Let

$q=\min\left\{n \;\middle|\; I_1\cap\left[\frac{1}{n+1},\frac{1}{n}\right)\neq\emptyset\right\}$. Then

$$\begin{aligned}
&\left|\sum_{k=1}^{k_0} f_3(x_k)|I_k|\right| \\
&\quad=\left|f_3(0)|I_1|+\sum_{n=1}^{q}\sum_{I_k\subset\left[\frac{1}{n+1},\frac{1}{n}\right]} f_3(x_k)|I_k|+\sum_{n=2}^{q}\sum_{\frac{1}{n}\in I_k} f_3\left(\frac{1}{n}\right)|I_k|\right| \\
&\quad\le |f_3(0)|I_1|| \\
&\qquad+\left|\sum_{I_k\subset\left[\frac{1}{q+1},\frac{1}{q}\right]} f_3(x_k)|I_k|+\sum_{\frac{1}{q}\in I_k} f_3\left(\frac{1}{q}\right)\left|I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]\right|\right| \\
&\qquad+\sum_{n=2}^{q-1}\left|\sum_{\frac{1}{n+1}\in I_k} f_3\left(\frac{1}{n+1}\right)\left|I_k\cap\left[\frac{1}{n+1},\frac{1}{n}\right]\right|\right. \\
&\qquad\quad+\sum_{I_k\subset\left[\frac{1}{n+1},\frac{1}{n}\right]} f_3(x_k)|I_k| \\
&\qquad\quad\left.+\sum_{\frac{1}{n}\in I_k} f_3\left(\frac{1}{n}\right)\left|I_k\cap\left[\frac{1}{n+1},\frac{1}{n}\right]\right|\right| \\
&\qquad+\left|\sum_{\frac{1}{2}\in I_k} f_3\left(\frac{1}{2}\right)\left|I_k\cap\left[\frac{1}{2},1\right]\right|+\sum_{I_k\subset\left[\frac{1}{2},1\right]} f_3(x_k)|I_k|\right| \\
&\quad\le 0+\left|\sum_{I_k\subset\left[\frac{1}{q+1},\frac{1}{q}\right]} f_3(x_k)|I_k|+\sum_{\frac{1}{q}\in I_k} f_3\left(\frac{1}{q}\right)\left|I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]\right|\right|+\sum_{n=2}^{q-1}\frac{\varepsilon}{2^{n+1}}+\frac{\varepsilon}{2^2}.
\end{aligned}$$

By Theorem 3.3 we obtain

$$\begin{aligned}
&\left|\sum_{I_k\subset\left[\frac{1}{q+1},\frac{1}{q}\right]} f_3(x_k)|I_k|+\sum_{\frac{1}{q}\in I_k} f_3\left(\frac{1}{q}\right)\left|I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]\right|\right| \\
&\quad\le\sum_{I_k\subset\left[\frac{1}{q+1},\frac{1}{q}\right]}\left|f_3(x_k)|I_k|-(L^*)\int_{I_k} f_3(x)dx\right| \\
&\qquad+\sum_{\frac{1}{q}\in I_k}\left|f_3(x_k)\left|I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]\right|-(L^*)\int_{I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]} f_3(x)dx\right| \\
&\qquad+\left|(L^*)\int_{\left(\bigcup_{I_k\subset\left[\frac{1}{q+1},\frac{1}{q}\right]} I_k\right)\cup\left(\bigcup_{\frac{1}{q}\in I_k} I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]\right)} f_3(x)dx\right| \\
&\quad<\frac{\varepsilon}{2^{q+1}}+M_q.
\end{aligned}$$

Therefore

$$\begin{aligned}\left|\sum_{k=1}^{k_0} f_3(x_k)|I_k|\right| &< \frac{\varepsilon}{2^{q+1}} + M_q + \sum_{n=2}^{q-1} \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2^2} \\ &< M_p + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} \\ &< \varepsilon\end{aligned}$$

and hence $f_3 \in (\tilde{L})([0,1])$. However, since it can be shown similarly to Theorem 3.5 that $f_3 \notin (\mathbf{C}^*)([0,1])$, we obtain $f_3 \notin (\mathbf{L}^*)([0,1])$. □

Theorem 3.7. *There exists a function f such that $f \in (\mathbf{C}^*)([0,1])$ but $f \notin (\mathbf{L}^*)([0,1])$.*

Proof. Let C be the Cantor set in $[0,1]$, let $\{(\alpha_p, \beta_p) \mid p \in \mathbb{N}\}$ be the sequence of all connected components of $[0,1] \setminus C$, let f_4 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_4(x) = \begin{cases} \frac{2(\alpha_p+\beta_p-2x)}{(\beta_p-\alpha_p)^2}\left(\frac{(x-\alpha_p)(\beta_p-x)}{(\beta_p-\alpha_p)^2}\sin\frac{(\beta_p-\alpha_p)^2}{(x-\alpha_p)(\beta_p-x)} - \cos\frac{(\beta_p-\alpha_p)^2}{(x-\alpha_p)(\beta_p-x)}\right), \\ \qquad\qquad \text{if } x \in (\alpha_p, \beta_p), p \in \mathbb{N}, \\ 0, \\ \qquad\qquad \text{if } x \in C, \end{cases}$$

and let F_4 be a function defined by

$$F_4(x) = \begin{cases} \frac{(x-\alpha_p)^2(\beta_p-x)^2}{(\beta_p-\alpha_p)^4}\sin\frac{(\beta_p-\alpha_p)^2}{(x-\alpha_p)(\beta_p-x)}, & \text{if } x \in (\alpha_p, \beta_p), p \in \mathbb{N}, \\ 0, & \text{if } x \in C. \end{cases}$$

Since $F_4'(x) = f_4(x)$ for any $x \in [0,1]$, we obtain $f_4 \in (\mathbf{N})([0,1])$ and hence $f_4 \in (\mathbf{C}^*)([0,1])$. However $f_4 \notin (\tilde{L})([0,1])$ and hence $f_4 \notin (\mathbf{L}^*)([0,1])$. We show $f_4 \notin (\tilde{L})([0,1])$. Assume that $f_4 \in (\tilde{L})([0,1])$. Then by Theorem 3.2 there exists a countable subset $N \subset [0,1]$ such that for any positive number ε there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f_4(x_k)(b_k - a_k) - (F_4(b_k) - F_4(a_k))| < \varepsilon$$

for any δ-fine partial McShane partition $\{([a_k, b_k], x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in [a_k, b_k]$ whenever $x_k \in N$. Since N is countable and C is perfect, there exist $z \in C$ and $\{(\alpha_{p(q)}, \beta_{p(q)}) \mid q \in \mathbb{N}\} \subset \{(\alpha_p, \beta_p) \mid p \in \mathbb{N}\}$ such that $z \notin N$ and $\left(\alpha_{p(q)}, \frac{\alpha_{p(q)}+\beta_{p(q)}}{2}\right) \subset [z, z+\delta(z))$ for any q. For any natural numbers q and n let

$$\begin{aligned} a_{q,n} &= \alpha_{p(q)} + \frac{(\beta_{p(q)} - \alpha_{p(q)})\left(1 - \sqrt{1 - \frac{4}{\frac{3}{2}\pi + 2n\pi}}\right)}{2}, \\ b_{q,n} &= \alpha_{p(q)} + \frac{(\beta_{p(q)} - \alpha_{p(q)})\left(1 - \sqrt{1 - \frac{4}{\frac{\pi}{2} + 2n\pi}}\right)}{2}. \end{aligned}$$

Note that $\{[a_{q,n}, b_{q,n}]\}$ is mutually disjoint and

$$\begin{aligned} F_4(a_{q,n}) &= -\frac{(a_{q,n}-\alpha_{p(q)})^2(\beta_{p(q)}-a_{q,n})^2}{(\beta_{p(q)}-\alpha_{p(q)})^4} \\ &= -\frac{1}{\left(\frac{3}{2}\pi+2n\pi\right)^2}, \\ F_4(b_{q,n}) &= \frac{(b_{q,n}-\alpha_{p(q)})^2(\beta_{p(q)}-b_{q,n})^2}{(\beta_{p(q)}-\alpha_{p(q)})^4} \\ &= \frac{1}{\left(\frac{\pi}{2}+2n\pi\right)^2}. \end{aligned}$$

Since $\{([a_{q,n}, b_{q,n}], z) \mid q, n \in \mathbb{N}\}$ is a δ-fine partial McShane partition and

$$\sum_{q=1}^{\infty}\sum_{n=1}^{\infty} |f_4(z)(b_{q,n}-a_{q,n}) - (F_4(b_{q,n}) - F_4(a_{q,n}))| = \sum_{q=1}^{\infty}\sum_{n=1}^{\infty} |F_4(b_{q,n}) - F_4(a_{q,n})| = \infty,$$

there exists $\{([a_k, b_k], z) \mid k = 1, \dots, k_0\} \subset \{([a_{q,n}, b_{q,n}], z) \mid q, n \in \mathbb{N}\}$ such that

$$\sum_{k=1}^{k_0} |f_4(z)(b_k - a_k) - (F_4(b_k) - F_4(a_k))| > \varepsilon.$$

It is a contradiction. □

Theorem 3.8. *There exists a function f such that $f \in (\tilde{C})([0,1])$ but $f \notin (\tilde{L})([0,1])$.*

Proof. We show in the proof of Theorem 3.7 that $f_4 \in \mathbf{(N)}([0,1])$ and hence $f_4 \in (\tilde{C})([0,1])$ but $f_4 \notin (\tilde{L})([0,1])$. □

Theorem 3.9. *There exists a function f such that $f \in \mathbf{(C^*)}([0,1])$ but $f \notin (\tilde{L})([0,1])$.*

Proof. We show in the proof of Theorem 3.7 that $f_4 \in \mathbf{(N)}([0,1])$ and hence $f_4 \in \mathbf{(C^*)}([0,1])$ but $f_4 \notin (\tilde{L})([0,1])$. □

Theorem 3.10. *There exists a function f such that $f \in (\tilde{L})([0,1])$ but $f \notin \mathbf{(C^*)}([0,1])$.*

Proof. We show in the proof of Theorem 3.6 that $f_3 \in (\tilde{L})([0,1])$ but $f_3 \notin \mathbf{(C^*)}([0,1])$. □

4 Properties of the C*-integral In this section we give a criterion for the C*-integrability.

Definition 4.1. Let F be an interval function on $[a,b]$ and let N be a finite subset of $[a,b]$. Then F is said to be C*-absolutely continuous on $E \subset [a,b]$ with respect to N if for any positive number ε there exist a gauge δ and a positive number η such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in I_k$ whenever $x_k \in N$;

(4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

We denote by $\mathbf{AC}_{\mathrm{C}^*}(E, N)$ the class of all C*-absolutely continuous interval functions on E with respect to N. Moreover F is said to be C*-generalized absolutely continuous on $[a, b]$ if there exist a finite subset N and a sequence $\{E_m\}$ of measurable sets such that $\bigcup_{m=1}^{\infty} E_m = [a, b]$ and $F \in \mathbf{AC}_{\mathrm{C}^*}(E_m, N)$ for any m. We denote by $\mathbf{ACG}_{\mathrm{C}^*}([a, b])$ the class of all C*-generalized absolutely continuous interval functions on $[a, b]$.

Lemma 4.1. *If $F \in \mathbf{ACG}_{C^*}([a, b])$ and $E \subset [a, b]$ with $|E| = 0$, then there exists a finite subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} |F(I_k)| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ *for any* k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in I_k$ *whenever* $x_k \in N$.

Proof. Since $F \in \mathbf{ACG}_{\mathrm{C}^*}([a, b])$, there exist a finite subset $N \subset [a, b]$ and a sequence $\{E_m\}$ of measurable sets such that $\bigcup_{m=1}^{\infty} E_m = [a, b]$ and $F \in \mathbf{AC}_{\mathrm{C}^*}(E_m, N)$ for any m. Therefore for any positive number ε and for any natural number m there exist a gauge δ_m and a positive number η_m such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \frac{\varepsilon}{2^{m+1}}$$

for any δ_m-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E_m$ for any k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in I_k$ whenever $x_k \in N$;

(4) $\sum_{k=1}^{k_0} |I_k| < \eta_m$.

Since $|E \cap E_m| = 0$, there exists an open set $O_m \supset E \cap E_m$ such that $|O_m| < \eta_m$. Define $\delta_m^*(x) = \min\{\delta_m(x), d(O_m^c, x)\}$, where O_m^c is the complement of O_m. Then we obtain

$$\sum_{k=1}^{k_0} |F(I_k)| < \frac{\varepsilon}{2^{m+1}}$$

for any δ_m^*-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying (1), (2), (3) and (4). Define $\delta(x) = \delta_m^*(x)$ for any $x \in E \cap E_m$ $(m \in \mathbb{N})$. Then we obtain

$$\sum_{k=1}^{k_0} |F(I_k)| = \sum_{n=1}^{\infty} \sum_{x_k \in E_m} |F(I_k)| \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^{m+1}} = \frac{\varepsilon}{2} < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in I_k$ whenever $x_k \in N$.

□

Lemma 4.2. *If F is differentiable at $x \in [a,b]$, then for any positive number ε there exists a positive number δ such that*

$$|F(t) - F(s) - F'(x)(t-s)| < \varepsilon(2d([s,t],x) + t - s)$$

for any interval $[s,t] \subset (x-\delta, x+\delta) \cap [a,b]$.

Proof. Since F is differentiable at $x \in [a,b]$, there exists a positive number δ such that

$$|F(\xi) - F(x) - F'(x)(\xi - x)|| < \varepsilon|\xi - x|$$

for any $\xi \in (x-\delta, x+\delta) \cap [a,b]$. Therefore for any interval $[s,t] \subset (x-\delta, x+\delta) \cap [a,b]$ we obtain

$$\begin{aligned}
&|F(t) - F(s) - F'(x)(t-s)| \\
&\quad \leq |F(t) - F(x) - F'(x)(t-x)| + |F(x) - F(s) - F'(x)(x-s)| \\
&\quad < \varepsilon|t-x| + \varepsilon|s-x| \\
&\quad = \varepsilon(2d([s,t],x) + t - s).
\end{aligned}$$

□

Theorem 4.1. *For any $F \in \mathbf{ACG}_{C^*}([a,b])$ there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$, and there exists $f \in \mathbf{(C^*)}([a,b])$ such that $f(x) = \frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$ and*

$$F(I) = (C^*)\int_I f(x)dx$$

for any interval $I \subset [a,b]$.

Conversely the interval function F defined above for any $f \in \mathbf{(C^*)}([a,b])$ satisfies $F \in \mathbf{ACG}_{C^*}([a,b])$.

Proof. Note that, if $F \in \mathbf{ACG}_{\mathrm{C}^*}([a,b])$, then $F \in \mathbf{ACG}_\delta([a,b])$, see [7, Definition 9.14]. By [7, Theorem 9.17] there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$. Let

$$E = \left\{ x \;\middle|\; \frac{d}{dx}F([a,x]) \text{ does not exist at } x \in [a,b] \right\}.$$

Then $|E| = 0$, and by Lemma 4.1 there exists a finite subset $N \subset [a,b]$ such that for any positive number ε with $\varepsilon < \frac{4}{b-a}$ there exists a gauge δ_1 such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \frac{\varepsilon}{4}$$

for any δ_1-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;

(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(3) $x_k \in I_k$ whenever $x_k \in N$.

If $x \notin E$, then by Lemma 4.2 there exists a positive number $\delta_2(x)$ such that

$$\left|F(t) - F(s) - \frac{d}{dx}F([a,x])(t-s)\right| < \frac{\varepsilon^2}{8}(2d([s,t],x) + t - s)$$

for any interval $[s,t] \subset (x - \delta_2(x), x + \delta_2(x)) \cap [a,b]$. Let

$$\delta(x) = \begin{cases} \delta_1(x), & \text{if } x \in E, \\ \delta_2(x), & \text{if } x \notin E, \end{cases}$$

and let

$$f(x) = \begin{cases} 0, & \text{if } x \in E, \\ \frac{d}{dx}F([a,x]), & \text{if } x \notin E. \end{cases}$$

Then we obtain

$$\begin{aligned} \left|\sum_{k=1}^{k_0} f(x_k)|I_k| - F(I)\right| &\leq \left|\sum_{x_k \in E} F(I_k)\right| + \left|\sum_{x_k \notin E} f(x_k)|I_k| - F(I_k)\right| \\ &\leq \sum_{x_k \in E} |F(I_k)| + \sum_{x_k \notin E} |f(x_k)|I_k| - F(I_k)| \\ &< \frac{\varepsilon}{4} + \sum_{x_k \notin E} \frac{\varepsilon^2}{8}(2d(I_k, x_k) + |I_k|) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon^2}{8} \cdot 2 \cdot \frac{1}{\varepsilon} + \frac{\varepsilon^2}{8}(b-a) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

for any interval $I \subset [a,b]$ and for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ of I satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ whenever $x_k \in N$.

Conversely let $f \in (\mathbf{C}^*)([a,b])$ and let

$$F(I) = (C^*) \int_I f(x) dx$$

for any interval $I \subset [a,b]$. For any natural number m let $E_m = \{x \mid x \in [a,b], |f(x)| \leq m\}$. Then $\bigcup_{m=1}^{\infty} E_m = [a,b]$. We show that $F \in \mathbf{AC}_{\mathrm{C}^*}(E_m, N)$, where N is an excepting finite subset of $[a,b]$ in the definition of the C*-integral of f. Let ε be a positive number. By Theorem 3.1 there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f(x_k)|I_k| - F(I_k)| < \frac{\varepsilon}{2}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ whenever $x_k \in N$.

Let $\eta = \frac{\varepsilon}{2m}$. If $x_k \in E_m$ for any k and $\sum_{k=1}^{k_0} |I_k| < \eta$, then we obtain

$$\begin{aligned} \sum_{k=1}^{k_0} |F(I_k)| &\leq \sum_{k=1}^{k_0} |f(x_k)||I_k| + \sum_{k=1}^{k_0} |f(x_k)|I_k| - F(I_k)| \\ &< m \sum_{k=1}^{k_0} |I_k| + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

□

5 Criteria for the C*-integrability We consider the following four criteria for the pair of a function f from $[a, b]$ into $\mathbb{R}$ and an additive interval function F on $[a, b]$.

$(A)_{C^*}$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a, b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) there exists a finite subset $N \subset [a, b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} \left(F(I_k) - (L) \int_{I_k \cap F_n} f(x)dx \right) \right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0 + 1, \ldots, k_1,\ 0 \leq k_0 \leq k_1\}$ of non-overlapping intervals in $[a, b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap F_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0 + 1, \ldots, k_1\}$ satisfying

(3.1) $x_k \in F_n$ for any $k = k_0 + 1, \ldots, k_1$;

(3.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$;

(3.3) $x_k \in I_k$ whenever $x_k \in N$.

$(B)_{C^*}$ For any decreasing sequence $\{\varepsilon_n\}$ tending to 0 there exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a, b]$;

(2) $F_n \subset M_n$ for any n and $|[a, b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) there exists a finite subset $N \subset [a, b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} \left(F(I_k) - (L) \int_{I_k \cap F_n} f(x)dx \right) \right| < \varepsilon_n$$

for any finite family $\{I_k \mid k = 1, \dots, k_0, k_0+1, \dots, k_1,\ 0 \leq k_0 \leq k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1, \dots, k_0\}$ with $I_k \cap M_n \neq \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \dots, k_1\}$ satisfying

(4.1) $x_k \in M_n$ for any $k = k_0+1, \dots, k_1$;

(4.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$;

(4.3) $x_k \in I_k$ whenever $x_k \in N$.

$(\mathrm{C})_{\mathrm{C}^*}$ There exists an increasing sequence $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} F_n = [a,b]$;

(2) $f \in \mathbf{(L)}(F_n)$ for any n;

(3) there exists a finite subset $N \subset [a,b]$ such that for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left| \sum_{k=1}^{k_0} F(I_k) \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ in $[a,b]$ satisfying

(3.1) $x_k \in F_n$ for any k;

(3.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$;

(3.3) $x_k \in I_k$ whenever $x_k \in N$;

(3.4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(4) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p \in \mathbb{N}\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

$(\mathrm{D})_{\mathrm{C}^*}$ There exist increasing sequences $\{M_n\}$ of non-empty measurable sets and $\{F_n\}$ of closed sets such that

(1) $\bigcup_{n=1}^{\infty} M_n = [a,b]$;

(2) $F_n \subset M_n$ for any n and $|[a,b] \setminus \bigcup_{n=1}^{\infty} F_n| = 0$;

(3) $f \in \mathbf{(L)}(F_n)$ for any n;

(4) there exists a finite subset $N \subset [a,b]$ such that for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left| \sum_{k=1}^{k_0} F(I_k) \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \dots, k_0\}$ in $[a,b]$ satisfying

(4.1) $x_k \in M_n$ for any k;

(4.2) $\sum_{k=k_0+1}^{k_1} d(I_k, x_k) < \frac{1}{\varepsilon_n}$;
(4.3) $x_k \in I_k$ whenever $x_k \in N$;
(4.4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

(5) for any n and for any interval $I \subset [a,b]$

$$F(I) = (L)\int_{I\cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),$$

where I^i is the interior of I, $\{J_p \mid p \in \mathbb{N}\}$ is the sequence of all connected components of $I^i \setminus F_n$ and $\overline{J_p}$ is the closure of J_p.

It is clear that $(\mathrm{A})_{\mathrm{C}^*}$ implies $(\mathrm{B})_{\mathrm{C}^*}$ and $(\mathrm{C})_{\mathrm{C}^*}$ implies $(\mathrm{D})_{\mathrm{C}^*}$. Now we give the following theorems for the C*-integral.

Theorem 5.1. *Let $f \in (\mathbf{C}^*)([a,b])$ and let F be an additive interval function on $[a,b]$ defined by*

$$F(I) = (C^*)\int_I f(x)dx$$

for any interval $I \subset [a,b]$. Then the pair of f and F satisfies $(\mathrm{A})_{\mathrm{C}^*}$.

Proof. Since $f \in (\mathbf{C}^*)([a,b])$, we obtain $f \in (\mathbf{D}^*)([a,b])$. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. Since by Theorem 2.2 the pair of f and F satisfies (A), for $\left\{\frac{\varepsilon_n}{2}\right\}$ there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) hold. Moreover

$$\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \frac{\varepsilon_n}{2}$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals in $[a,b]$ with $I_k \cap F_n \neq \emptyset$. By Theorem 3.1 there exists a finite subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for any n there exists a gauge δ such that

$$\left|\sum_{k=k_0+1}^{k_1}(f(x_k)|I_k| - F(I_k))\right| < \frac{\varepsilon_n}{4}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ in $[a,b]$ satisfying (3.2) and (3.3). Since $f\chi_{F_n} \in (\mathbf{L})([a,b])$, where χ_{F_n} means the characteristic function of F_n, by the Saks-Henstock lemma for the McShane integral, for instance see [7, Lemma 10.6], for any n there exists a gauge δ such that

$$\left|\sum_{k=k_0+1}^{k_1}\left(f(x_k)\chi_{F_n}(x_k)|I_k| - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| < \frac{\varepsilon_n}{4}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ in $[a,b]$. Since $f = f\chi_{F_n}$ on F_n, for any n there exists a gauge δ such that

$$\begin{aligned}
&\left|\sum_{k=k_0+1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad = \left|\sum_{k=k_0+1}^{k_1}(F(I_k) - f(x_k)|I_k|)\right| + \left|\sum_{k=k_0+1}^{k_1}\left(f(x_k)\chi_{F_n}(x_k)|I_k| - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad < \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} = \frac{\varepsilon_n}{2}
\end{aligned}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ in $[a,b]$ satisfying (3.1), (3.2) and (3.3). Therefore

$$\begin{aligned}
&\left|\sum_{k=1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad \le \left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| + \left|\sum_{k=k_0+1}^{k_1}\left(F(I_k) - (L)\int_{I_k\cap F_n} f(x)dx\right)\right| \\
&\quad < \frac{\varepsilon_n}{2} + \frac{\varepsilon_n}{2} = \varepsilon_n
\end{aligned}$$

for any finite family $\{I_k \mid k = 1, \ldots, k_0, k_0+1, \ldots, k_1,\ 0 \le k_0 \le k_1\}$ of non-overlapping intervals in $[a,b]$ which consists of a finite family $\{I_k \mid k = 1, \ldots, k_0\}$ with $I_k \cap F_n \ne \emptyset$ and a δ-fine partial McShane partition $\{(I_k, x_k) \mid k = k_0+1, \ldots, k_1\}$ satisfying (3.1), (3.2) and (3.3), that is, (3) holds. □

Theorem 5.2. *If the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* $(\mathrm{A})_{\mathrm{C}^*}$*, then the pair of f and F satisfies* $(\mathrm{C})_{\mathrm{C}^*}$*. Similarly, if the pair of a function f from an inteval $[a,b]$ into $\mathbb{R}$ and an additive interval function F on $[a,b]$ satisfies* $(\mathrm{B})_{\mathrm{C}^*}$*, then the pair of f and F satisfies* $(\mathrm{D})_{\mathrm{C}^*}$*.*

Proof. Let $\{\varepsilon_n\}$ be a decreasing sequence tending to 0. Then there exists an increasing sequence $\{F_n\}$ of closed sets such that (1) and (2) of $(\mathrm{C})_{\mathrm{C}^*}$ hold. We show (3) of $(\mathrm{C})_{\mathrm{C}^*}$. Let n be a natural number and let ε be a positive number. Since $f \in \mathbf{(L)}(F_n)$, there exists a positive number $\rho(n,\varepsilon)$ such that, if $|E| < \rho(n,\varepsilon)$, then

$$\left|(L)\int_{E\cap F_n} f(x)dx\right| < \frac{\varepsilon}{2}.$$

Take a natural number $m(n,\varepsilon)$ such that $\varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}$ and $m(n,\varepsilon) \ge n$, and put $\eta = \rho(m(n,\varepsilon),\varepsilon)$. By (3) of $(\mathrm{A})_{\mathrm{C}^*}$ there exists a subset $N \subset [a,b]$ independent of $\{\varepsilon_n\}$ such that for $m(n,\varepsilon)$ there exists a gauge $\delta_{m(n,\varepsilon)}$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a $\delta_{m(n,\varepsilon)}$-fine partial McShane partition in $[a,b]$ satisfying (3.1), (3.2), (3.3) and (3.4) of $(\mathrm{C})_{\mathrm{C}^*}$. Then we obtain

$$\left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right)\right| < \varepsilon_{m(n,\varepsilon)} < \frac{\varepsilon}{2}.$$

Moreover, since $\sum_{k=1}^{k_0}|I_k| < \eta = \rho(m(n,\varepsilon),\varepsilon)$, we obtain

$$\left|\sum_{k=1}^{k_0}(L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right| < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned}
\left|\sum_{k=1}^{k_0} F(I_k)\right| &\le \left|\sum_{k=1}^{k_0}\left(F(I_k) - (L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right)\right| + \left|\sum_{k=1}^{k_0}(L)\int_{I_k\cap F_{m(n,\varepsilon)}} f(x)dx\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Next we show (4) of $(\mathrm{C})_{\mathrm{C}^*}$. Let I be a subinterval of $[a,b]$. In the case of $I \cap F_n = \emptyset$ (4) of $(\mathrm{C})_{\tilde{C}}$ is clear. Consider the case of $I \cap F_n \ne \emptyset$. Let $\{J_p \mid p = 1, 2, \ldots\}$ be the sequence of all connected components of $I^i \setminus F_n$. Since $I \cap F_m \neq \emptyset$ holds for any $m \geq n$, by (3) of $(\mathrm{A})_{\mathrm{C}^*}$ we obtain

$$\left| F(I) - (L) \int_{I \cap F_m} f(x)dx \right| < \varepsilon_m.$$

Since $\overline{J_p} \cap F_m \neq \emptyset$ holds for any p, by (3) of $(\mathrm{A})_{\tilde{C}}$ we obtain

$$\left| \sum_{p=1}^{\infty} \left(F(\overline{J_p}) - (L) \int_{\overline{J_p} \cap F_m} f(x)dx \right) \right| \leq \varepsilon_m$$

for any $m \geq n$. On the other hand, we obtain

$$(L) \int_{I \cap F_m} f(x)dx = (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} (L) \int_{\overline{J_p} \cap F_m} f(x)dx$$

for any $m \geq n$. Therefore we obtain

$$\begin{aligned}
&\left| F(I) - \left((L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}) \right) \right| \\
&\quad \leq \left| F(I) - (L) \int_{I \cap F_m} f(x)dx \right| \\
&\qquad + \left| (L) \int_{I \cap F_m} f(x)dx - \left((L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} (L) \int_{\overline{J_p} \cap F_m} f(x)dx \right) \right| \\
&\qquad + \left| - \sum_{p=1}^{\infty} F(\overline{J_p}) + \sum_{p=1}^{\infty} (L) \int_{\overline{J_p} \cap F_m} f(x)dx \right| \\
&\quad < \varepsilon_m + 0 + \varepsilon_m = 2\varepsilon_m
\end{aligned}$$

for any $m \geq n$ and hence

$$F(I) = (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}).$$

Similarly, we can prove that, if the pair of f and F satisfies $(\mathrm{B})_{\mathrm{C}^*}$, then the pair of f and F satisfies $(\mathrm{D})_{\mathrm{C}^*}$. $\square$

Theorem 5.3. *If the pair of a function f from an inteval $[a, b]$ into $\mathbb{R}$ and an additive interval function F on $[a, b]$ satisfies $(\mathrm{D})_{\mathrm{C}^*}$, then $f \in (\mathbf{C}^*)([a, b])$ and*

$$F(I) = (C^*) \int_I f(x)dx$$

holds for any interval $I \subset [a, b]$.

Proof. By (1) and (4) there exist a finite subset $N \subset [a, b]$ and a increasing sequence $\{M_n\}$ of non-empty measurable sets such that $\bigcup_{n=1}^{\infty} M_n = [a, b]$ and for any n and for any positive number ε there exist a positive number η and a gauge δ such that

$$\left| \sum_{k=1}^{k_0} F(I_k) \right| < \frac{\varepsilon}{2}$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ in $[a, b]$ satisfying (4.1), (4.2), (4.3) and (4.4). Therefore we obtain

$$\begin{aligned}\sum_{k=1}^{k_0} |F(I_k)| &= \left|\sum_{F(x_k)>0} F(I_k)\right| + \left|\sum_{F(x_k)<0} F(I_k)\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and hence $F \in \mathbf{ACG}_{\mathrm{C}^*}([a, b])$. By Theorem 4.1 there exists $\frac{d}{dx}F([a, x])$ for almost every $x \in [a, b]$, and there exists $g \in \mathbf{(C^*)}([a, b])$ such that

$$F(I) = (C^*) \int_I g(x)dx$$

for any interval $I \subset [a, b]$. We show that $g = f$ almost everywhere. To show this, we consider a function

$$g_n(x) = \begin{cases} f(x), & \text{if } x \in F_n, \\ g(x), & \text{if } x \notin F_n. \end{cases}$$

By [16, Theorem (5.1)] $g_n \in \mathbf{(D^*)}(I)$ for any interval $I \subset [a, b]$ and by (3)

$$\begin{aligned}(D^*) \int_I g_n(x)dx &= (D^*) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} (D^*) \int_{\overline{J_p}} g(x)dx \\ &= (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} (C^*) \int_{\overline{J_p}} g(x)dx \\ &= (L) \int_{I \cap F_n} f(x)dx + \sum_{p=1}^{\infty} F(\overline{J_p}),\end{aligned}$$

where $\{J_p \mid p = 1, 2, \ldots\}$ is the sequence of all connected components of $I^i \setminus F_n$. By comparing the equation above with (5), we obtain

$$F(I) = (D^*) \int_I g_n(x)dx.$$

Therefore we obtain $\frac{d}{dx}F([a, x]) = g_n(x) = f(x)$ for almost every $x \in F_n$. By (2) we obtain $g(x) = \frac{d}{dx}F([a, x]) = f(x)$ for almost every $x \in [a, b]$. $\square$

By Theorems 5.1, 5.2 and 5.3 we obtain the following criteria for the C*-integrability.

Theorem 5.4. *A function f from an interval $[a, b]$ into $\mathbb{R}$ is C*-integrable if and only if there exists an additive interval function F on $[a, b]$ such that the pair of f and F satisfies one of* $(\mathrm{A})_{\mathrm{C}^*}$, $(\mathrm{B})_{\mathrm{C}^*}$, $(\mathrm{C})_{\mathrm{C}^*}$ *and* $(\mathrm{D})_{\mathrm{C}^*}$. *Moreover, if the pair of f and F satisfies one of* $(\mathrm{A})_{\mathrm{C}^*}$, $(\mathrm{B})_{\mathrm{C}^*}$, $(\mathrm{C})_{\mathrm{C}^*}$ *and* $(\mathrm{D})_{\mathrm{C}^*}$, *then*

$$F(I) = (C^*) \int_I f(x)dx$$

holds for any interval $I \subset [a, b]$.

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