# OPTIMAL GROUPING OF STUDENTS IN COALITIONAL GAMES WITH THE SHAPLEY VALUE

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#### Abstract

In this paper, the classroom consists of three different kinds of students, and we discuss the problem how to divide these students into three person groups. The benefit of one group is the sum of three students' benefit by cooperation game. The benefit of each person is given by the Shapley value from the characteristic function we defined. Our goal is how to divide 18 students into subgroups with three persons to make the total benefit of the classroom maximal.

It is impossible to get the maximal score by using different 18 students having different potentials and six different coefficients for the combinations of three different levels of potentials. Especially, the number of combinations for dividing 18 students by 3 persons evenly is tremendous. Therefore, we can investigate some numerical examples under some limited conditions. Finally, we can obtain the theorem to make the total benefit of the classroom maximal under the limited condition.

The authors believe that this research can apply to group learning and the field of Education in the real life.

#### 1. Introduction

There is a proverb, "Two heads are better than one". In school life, groups form spontaneously, and usually smart people study with other smart people. People who cannot be in that smart team gather and construct other groups. Seen from a big picture, in Japan, every university, high school, and even private junior high school is ranked. Each student is sent to a specific school based on their score on a paper examination which is given by each school.

I am not sure if it is good to divide students ordered by smartness for the classroom and for society, or not. The way to divide proper groups is affected by what is considered as priority. If your purpose is to make the smartest student smarter, the way we are adopting the structure of the deviation value now is obviously correct. However, to make the benefit of the entire classroom or entire society biggest, we are not sure if it is correct that the deviation value structure we have now in Japan is the best way. Therefore, we are going to talk about the structure, which makes the benefit of the entire group the biggest.

In this paper, the classroom has three different kinds of students, and we divide these students into three person groups. We assume that they cooperate and study together in groups. We anticipate that three smart students compete or work together with each other and their score should go up. Conversely, we assume, if three students who don't like to study gather, they will not gain anything. On this paper, we set one classroom with 6 smart students, 6 neutral students, and 6 not good students. We divide them into 3 persons groups, so there are 6 groups in the classroom. The benefit

Mathematics Subject classification 2010. 91A12 Key words and phrases. coalitional game, the Shapley value, characteristic function of one group is the sum of three students' benefit by cooperation game. The benefit of each person is given by the Shapley value from the characteristic function we defined. Also, in this model, we think and simulate how and where to put these groups in the classroom. If a group talks to another group, possibly they will gain something by conveying and receiving information.

#### 2.1 Model of the classroom with 18 students

We figured out some dispositions from this problem when we construct the problem as a general form. After giving the concrete numbers to the functions and others, we find the proper structure of the classroom after finding proper groups by computational simulations. Let  $X = \{ X_1, X_2, X_3, X_4, X_5, X_6 \}$  be the set of 6 smart students. Let  $Y = \{ Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 \}$  be the set of 6 neutral students. Let  $Z=\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$  be the set of 6 not good students. Let  $S = {s_1, s_2, s_3, s_4, s_5, s_6}$  be the set of relationship between two students. It is assumed that  $s_i \gg s_j$  for any i < j, so  $s_1 \gg s_2 \gg s_3 \gg s_4 \gg s_5 \gg s_6$ .  $\gg$  means that the relationship of  $s_i$  is better than that of  $s_j$ .  $W_i \in W = \{X, Y, Z\} \ni (j=1,2,\dots,18)$  W is the set of all students.  $W_i$ ,  $X_i$ ,  $Y_i$ , and  $Z_i$  represent people.  $w_i$ ,  $x_i$ ,  $y_j$ , and  $z_i$  represent values  $v(W_i)$ ,  $v(X_i)$ ,  $v(Y_i)$ , and  $v(Z_i)$  respectively. Let  $s_1$  be the state of the relationships between  $X_i$  and  $X_i$ . Let  $s_2$  be the state of the relationships between  $X_i$  and  $Y_j$ . Let  $s_3$  be the state of the relationships between  $Y_i$  and  $Y_i$ . Let  $s_4$  be the state of the relationships between  $X_i$  and  $Z_i$ .

Let  $s_5$  be the state of the relationships between  $Y_i$  and  $Z_j$ . Let  $s_6$  be the state of the relationships between  $Z_i$  and  $Z_j$ .

So we have assumed the quality of the relationship is highest for good students with good students and lowest for poor students with poor students. This is probably the strongest assumption in the paper and only reasonable in some situations. In some situations, it is possible that two average (neutral) students will be able to combine and really both grow, but good students will not have much room for growth. In other situations (modeled here), two average students gain but less than two good students. So, we have the following value ordering. Another part of this assumption is that two average or neutral students gain more than a good student combined with a poor student. This would not always be true.

[Definition I]

We define the characteristic function of the reward that both  $W_i$  and  $W_j$  corporate together.  $v(W_i \cup W_j, s) = s(w_i + w_j)$ , where  $W_i$  represents an arbitrary person with value  $w_i = v(W_i)$ . s is an arbitrary element of  $S = \{s_1 \dots s_6\}$ . s represents the state of the relationship between two persons and is real value.

Let  $G_1 = (W_1, W_2, W_3)$ ,  $G_2 = (W_4, W_5, W_6)$ ,...,  $G_6 = (W_{16}, W_{17}, W_{18})$ , where  $W_i \in W = \{X, Y, Z\}$   $(j=1,2, \dots, 18)$ .

In this coalitional game of three players, the Shapley value of player  $W_i$  in  $G_i = (W_i, W_{i+1,i}, W_{i+2})$  is

$$f(W_{i}) = \frac{2!}{3!} \{ v(W_{i}) - v(\Phi) \} + \frac{1}{3!} \{ v(W_{i} \cup W_{i+1}, s') - v(W_{i+1}) \}$$
  
+  $\frac{1}{3!} \{ v(W_{i} \cup W_{i+2}, s'') - v(W_{i+2}) \}$   
+  $\frac{2!}{3!} \{ v(W_{i} \cup W_{i+1} \cup W_{i+2}) - v(W_{i+1} \cup W_{i+2}, s''') \} \cdots \cdots \cdots \cdots (2-1-1)$   
s', s'', s'''  $\in S$  ( $\Phi$  is an empty set.)

s' depends on combination of W<sub>i</sub> and W<sub>i</sub>, so there are 6 possible values.

 $v(W_i \cup W_{i+1} \cup W_{i+2})$  is defined as

 $\frac{1}{2} \{ v(W_i \cup W_{i+1}, s') + v(W_i \cup W_{i+2}, s'') + v(W_{i+1} \cup W_{i+2}, s''') \}.$ 

From (2-1-1),

$$f(W_{i}) = \frac{1}{6} \{ 2W_{i} - (W_{i+1} + W_{i+2}) \} + \frac{1}{6} \{ 2\{ v(W_{i} \cup W_{i+1}, s') + v(W_{i} \cup W_{i+2}, s'') \} - v(W_{i+1} \cup W_{i+2}, s''') \}$$
(2-1-2)

G<sub>i</sub>'s group value is defined as  $F(G_i) = f(W_i) + f(W_{i+1}) + f(W_{i+2})$ The Sum of Group Values:  $SGV = \sum_{i=1}^{6} F(G_i)$  .....(2-1-3) SGV represents the total score of whole classroom. Our objective is to find the grouping the make the SGV maximal.

#### 2.2 The comparison of two kinds of the classroom

There are so many ways to make six groups with three people each having different values. We will observe the total value of classroom with two examples. We let  $\sum_{i=1}^{6} x_i > \sum_{i=1}^{6} y_i > \sum_{i=1}^{6} z_i$  and  $s_1 \ge s_2 \ge s_3 \ge s_4 \ge s_5 \ge s_6$ .

[Example I]

Let us consider the situation where the groups are divided by matching abilities.

X<sub>i</sub> and X<sub>j</sub> have different numbers.

We let 
$$G_1 = (X_1, X_2, X_3)$$
,  $G_2 = (X_4, X_5, X_6)$ ,  $G_3 = (Y_1, Y_2, Y_3)$ ,  $G_4 = (Y_4, Y_5, Y_6)$ ,  
 $G_5 = (Z_1, Z_2, Z_3)$ , and  $G_6 = (Z_4, Z_5, Z_6)$ .  
 $f(X_1) = \frac{1}{2} \{ 2x_1 - (x_2 + x_3) \} + \frac{1}{6} \{ 2\{ (x_1 + x_2)s_1 + (x_1 + x_3)s_1 \} - (x_2 + x_3)s_1 \}$   
 $f(X_2) = \frac{1}{2} \{ 2x_2 - (x_1 + x_3) \} + \frac{1}{6} \{ 2\{ (x_2 + x_1)s_1 + (x_2 + x_3)s_1 \} - (x_1 + x_3)s_1 \}$   
 $f(X_3) = \frac{1}{2} \{ 2x_3 - (x_1 + x_2) \} + \frac{1}{6} \{ 2\{ (x_3 + x_1)s_1 + (x_3 + x_2)s_1 \} - (x_1 + x_2)s_1 \}$   
 $F(G_1) = f(X_1) + f(X_2) + f(X_3) = \frac{1}{2} \{ (x_1 + x_2)s_1 + (x_1 + x_3)s_1 + (x_2 + x_3)s_1 \} = (x_1 + x_2 + x_3)s_1$   
 $F(G_2) = (x_4 + x_5 + x_6)s_1$ ,  $F(G_3) = (y_1 + y_2 + y_3)s_3$ ,  $F(G_4) = (y_4 + y_5 + y_6)s_3$   
 $F(G_5) = (z_1 + z_2 + z_3)s_6$ ,  $F(G_6) = (z_4 + z_5 + z_6)s_6$ .

Call the total value SGV<sub>1</sub> =  $s_1 \sum_{i=1}^{6} x_i + s_3 \sum_{i=1}^{6} y_i + s_6 \sum_{i=1}^{6} z_i \cdots (2-2-1)$ 

[Example II]

Next, let 
$$G_1 = (X_1, Y_1, Z_1)$$
,  $G_2 = (X_2, Y_2, Z_2)$ ,..., and  $G_6 = (X_6, Y_6, Z_6)$ .  

$$f(X_1) = \frac{1}{2} \{ 2x_1 - (y_1 + z_1) \} + \frac{1}{6} \{ 2\{ (x_1 + y_1)s_2 + (x_1 + z_1)s_4 \} - (y_1 + z_1)s_5 \}$$

$$f(Y_1) = \frac{1}{2} \{ 2y_1 - (x_1 + z_1) \} + \frac{1}{6} \{ 2\{ (x_1 + y_1)s_2 + (y_1 + z_1)s_5 \} - (x_1 + z_1)s_4 \}$$

$$f(Z_1) = \frac{1}{2} \{ 2z_1 - (x_1 + y_1) \} + \frac{1}{6} \{ 2\{ (y_1 + z_1)s_5 + (x_1 + z_1)s_4 \} - (x_1 + y_1)s_2 \}$$

$$F(G_1) = \frac{1}{2} \{ (x_1 + y_1)s_2 + (x_1 + z_1)s_4 + (y_1 + z_1)s_5 \}$$

Call the total value for these groups

$$\begin{split} & SGV_2 = \frac{s2+s4}{2} \sum_{i=1}^6 x_i + \frac{s2+s5}{2} \sum_{i=1}^6 y_i + \frac{s4+s5}{2} \sum_{i=1}^6 z_i \quad \cdots \cdots (2-2-2) \\ & \text{We show a situation where } SGV_1 > SGV_2 \text{ and one where } SGV_1 < SGV_2. \\ & SGV_1 - SGV_2 = (s_1 - \frac{s2+s4}{2}) \sum_{i=1}^6 x_i + (s_3 - \frac{s2+s5}{2}) \sum_{i=1}^6 y_i + (s_6 - \frac{s4+s5}{2}) \sum_{i=1}^6 z_i \\ & \text{The first coefficient, } s_1 - \frac{s2+s4}{2}, \text{ is bigger than 0. The third coefficient, } s_6 - \frac{s4+s5}{2}, \text{ is smaller than 0.} \\ & \text{But the second coefficient, } s_3 - \frac{s2+s5}{2}, \text{ is the thing we cannot know in this setting.} \end{split}$$

1) If we let the  $s_i$ -values differ by a constant increment,

then  $s_6 < s_5 = s_6 + \Delta < s_4 = s_6 + 2\Delta < s_3 = s_6 + 3\Delta < s_2 = s_6 + 4\Delta < s_1 = s_6 + 5\Delta$ . And we get,

$$SGV_{1} - SGV_{2} = (5\Delta - \frac{4\Delta + 2\Delta}{2})\sum_{i=1}^{6} x_{i} + (3\Delta - \frac{4\Delta + \Delta}{2})\sum_{i=1}^{6} y_{i} + (0 - \frac{2\Delta + \Delta}{2})\sum_{i=1}^{6} z_{i}$$
$$= 2\Delta \sum_{i=1}^{6} x_{i} + \frac{\Delta}{2} \sum_{i=1}^{6} y_{i} - \frac{3\Delta}{2} \sum_{i=1}^{6} z_{i} > 0$$

Therefore  $SGV_1 > SGV_2$ .

2) However, if we let  $s_1 = s_2 = s_3 = s_4 = s_5 > s_6$ ,

 $SGV_1 - SGV_2 = (s_6 - s_5) \sum_{i=1}^6 z_i \ <\!\! 0.$ 

 $Therefore \quad SGV_1 < SGV_2.$ 

The two examples above show that it is hard to find the maximal SGV. That is because the total value of the classroom changes with s-values.

### 2.3 Finding all possible groupings

We want to make these models simple. So, now we examine all possible groupings where all the  $X_i$ 's have the same value. The  $Y_i$ 's and  $Z_i$ 's have a single y-value and z-value respectively. X = (X, X, X, X, X, X), Y = (Y, Y, Y, Y, Y, Y), Z = (Z, Z, Z, Z, Z, Z)

To represent a group's makeup, we use the following notation:

(number of x members in the group, number of y members, number of z members).

Group  $H_1$ : (X,X,X)=(3,0,0), Group  $H_2$ : (X,X,Y)=(2,1,0),

Group  $H_3$ : (X,X,Z)=(2,0,1), Group  $H_4$ : (X,Y,Y)=(1,2,0),

$$\begin{split} & \text{Group } H_5 : (Y,Y,Y) {=} (0,3,0), \quad \text{Group } H_6 : (Y,Y,Z) {=} (0,2,1), \\ & \text{Group } H_7 : (Y,Z,Z) {=} (0,1,2), \quad \text{Group } H_8 : (X,Y,Z) {=} (1,1,1), \\ & \text{Group } H_9 : (X,Z,Z) {=} (1,0,2), \quad \text{Group } H_{10} : (Z,Z,Z) {=} (0,0,3). \end{split}$$

The alphas in the following equation represent the number of groups of each makeup.

Equation (2-3-2) represents that the sum of alphas needs to be 6 because we have six groups. Equations (2-3-3), (2-3-4), and (2-3-5) come from the equation (2-3-1).

We will solve these with matrices.

$$\alpha_{1} + \alpha_{2} + \alpha_{3} + \alpha_{4} + \alpha_{5} + \alpha_{6} + \alpha_{7} + \alpha_{8} + \alpha_{9} + \alpha_{10} = 6 \qquad (2-3-2)$$

$$0 \leq \alpha_{i} \leq 6, \ \alpha_{i} \in \mathbb{N}$$

$$3\alpha_{1} + 2\alpha_{2} + 2\alpha_{3} + \alpha_{4} \qquad + \alpha_{8} + \alpha_{9} \qquad = 6 \qquad (2-3-3)$$

$$\alpha_{2} \qquad + 2\alpha_{4} + 3\alpha_{5} + 2\alpha_{6} + \alpha_{7} + \alpha_{8} \qquad = 6 \qquad (2-3-4)$$

$$\alpha_{3} \qquad + \alpha_{6} + 2\alpha_{7} + \alpha_{8} + 2\alpha_{9} + 3\alpha_{10} = 6 \qquad (2-3-5)$$

By using a computer programming language VBA, we found there are 103 solutions meeting (2-3-2) through (2-3-5). (Appendix) These solutions correspond to possible groupings. As we did before, we find the group values for different group makeups.

$$\begin{pmatrix} 3\\0\\0 \end{pmatrix} \dots F(H_1) = 3xs_1 \qquad \begin{pmatrix} 2\\1\\0 \end{pmatrix} \dots F(H_2) = xs_1 + xs_2 + ys_2 \begin{pmatrix} 2\\0\\1 \end{pmatrix} \dots F(H_3) = xs_1 + xs_4 + zs_4 \qquad \begin{pmatrix} 1\\2\\0 \end{pmatrix} \dots F(H_4) = xs_2 + ys_2 + ys_3 \begin{pmatrix} 0\\3\\0 \end{pmatrix} \dots F(H_5) = 3ys_3 \qquad \begin{pmatrix} 0\\2\\1 \end{pmatrix} \dots F(H_6) = ys_3 + ys_5 + zs_5 \begin{pmatrix} 0\\1\\2 \end{pmatrix} \dots F(H_7) = ys_5 + zs_5 + zs_6 \qquad \begin{pmatrix} 1\\1\\1 \end{pmatrix} \dots F(H_8) = 1/2 \{x(s_2 + s_4) + y(s_2 + s_5) + z(s_4 + s_5) \} \begin{pmatrix} 1\\0\\2 \end{pmatrix} \dots F(H_9) = xs_4 + zs_4 + zs_6 \qquad \begin{pmatrix} 0\\0\\3 \end{pmatrix} \dots F(H_{10}) = 3zs_6$$

The possible groupings given by the alpha values are in the following tables. They were found by Program 3 which is given an appendix.

Also, we have the chart of all possible groupings given by alphas at an appendix. We sort the numbers of groupings by ascending order.

This chart is all Possible Groupings Given by Alphas. We set through  $s_1$  to  $s_6$  the characteristic numbers because this case, we assume that good student and good student can help each other the most. In other words, we assume that poor student and poor student don't cooperate each other much because they don't know the material they need to do. Finally, we sort this by highest score to lowest score.

	<b>S</b> 1	<b>S</b> 2	<b>S</b> 3	<b>S</b> 4	<b>S</b> 5	<b>S</b> 6		Х	у	Z	
	1.25	1.2	1.15	1.1	1.05	1		80	60	40	
NO											CON
NO	$\alpha_1$	$\alpha_2$	$\alpha_3$	α <sub>4</sub>	$\alpha_{5}$	$\alpha_{6}$	$\alpha_7$	α <sub>8</sub>	$\alpha_9$	α <sub>10</sub>	SGV
99	2	0	0	0	2	0	0	0	0	2	1254
94	1	1	0	1	1	0	0	0	0	2	1252
60	0	3	0	0	1	0	0	0	0	2	1251
76	1	0	0	3	0	0	0	0	0	2	1251
54	0	2	0	2	0	0	0	0	0	2	1250
98	2	0	0	0	1	1	1	0	0	1	1246
93	1	1	0	1	0	1	1	0	0	1	1244
59	0	3	0	0	0	1	1	0	0	1	1243
89	1	1	0	0	1	0	1	1	0	1	1242.5
96	2	0	0	0	0	3	0	0	0	1	1242
97	2	0	0	0	1	0	3	0	0	0	1242
73	1	0	0	2	0	0	1	1	0	1	1241.5
85	1	0	1	1	1	0	1	0	0	1	1241
91	1	1	0	0	1	1	0	0	1	1	1241
51	0	2	0	1	0	0	1	1	0	1	1240.5
57	0	2	1	0	1	0	1	0	0	1	1240
75	1	0	0	2	0	1	0	0	1	1	1240
92	1	1	0	1	0	0	3	0	0	0	1240
41	0	1	1	2	0	0	1	0	0	1	1239
53	0	2	0	1	0	1	0	0	1	1	1239
58	0	3	0	0	0	0	3	0	0	0	1239
71	1	0	0	1	1	0	0	1	1	1	1238.5
87	1	1	0	0	0	2	0	1	0	1	1238.5
82	1	0	1	0	2	0	0	0	1	1	1238
95	2	0	0	0	0	2	2	0	0	0	1238

		-									
49	0	2	0	0	1	0	0	1	1	1	1237.5
84	1	0	1	1	0	2	0	0	0	1	1237
90	1	1	0	0	1	0	2	0	1	0	1237
30	0	1	0	2	0	0	0	1	1	1	1236.5
40	0	1	1	1	1	0	0	0	1	1	1236
56	0	2	1	0	0	2	0	0	0	1	1236
68	1	0	0	1	0	1	0	2	0	1	1236
74	1	0	0	2	0	0	2	0	1	0	1236
80	1	0	1	0	1	1	0	1	0	1	1235.5
10	0	0	1	3	0	0	0	0	1	1	1235
46	0	2	0	0	0	1	0	2	0	1	1235
52	0	2	0	1	0	0	2	0	1	0	1235
63	1	0	0	0	1	0	0	3	0	1	1234.5
86	1	1	0	0	0	1	2	1	0	0	1234.5
38	0	1	1	1	0	1	0	1	0	1	1233.5
44	0	1	2	0	1	1	0	0	0	1	1233
72	1	0	0	1	1	0	1	0	2	0	1233
83	1	0	1	1	0	1	2	0	0	0	1233
88	1	1	0	0	0	2	1	0	1	0	1233
26	0	1	0	1	0	0	0	3	0	1	1232.5
20	0	0	2	2	0	1	0	0	0	1	1232
34	0	1	1	0	1	0	0	2	0	1	1232
50	0	2	0	0	1	0	1	0	2	0	1232
55	0	2	1	0	0	1	2	0	0	0	1232
67	1	0	0	1	0	0	2	2	0	0	1232
79	1	0	1	0	1	0	2	1	0	0	1231.5
7	0	0	1	2	0	0	0	2	0	1	1231
31	0	1	0	2	0	0	1	0	2	0	1231
45	0	2	0	0	0	0	2	2	0	0	1231
17	0	0	2	1	1	0	0	1	0	1	1230.5
69	1	0	0	1	0	1	1	1	1	0	1230.5
23	0	0	3	0	2	0	0	0	0	1	1230
66	1	0	0	0	2	0	0	0	3	0	1230
81	1	0	1	0	1	1	1	0	1	0	1230
37	0	1	1	1	0	0	2	1	0	0	1229.5

47	0	2	0	0	0	1	1	1	1	0	1229.5
43	0	1	2	0	1	0	2	0	0	0	1229
64	1	0	0	0	1	0	1	2	1	0	1229
70	1	0	0	1	0	2	0	0	2	0	1229
19	0	0	2	2	0	0	2	0	0	0	1228
29	0	1	0	1	1	0	0	0	3	0	1228
39	0	1	1	1	0	1	1	0	1	0	1228
48	0	2	0	0	0	2	0	0	2	0	1228
65	1	0	0	0	1	1	0	1	2	0	1227.5
77	1	0	1	0	0	2	1	1	0	0	1227.5
2	0	0	0	3	0	0	0	0	3	0	1227
27	0	1	0	1	0	0	1	2	1	0	1227
35	0	1	1	0	1	0	1	1	1	0	1226.5
61	1	0	0	0	0	1	1	3	0	0	1226.5
78	1	0	1	0	0	3	0	0	1	0	1226
8	0	0	1	2	0	0	1	1	1	0	1225.5
28	0	1	0	1	0	1	0	1	2	0	1225.5
18	0	0	2	1	1	0	1	0	1	0	1225
36	0	1	1	0	1	1	0	0	2	0	1225
42	0	1	2	0	0	2	1	0	0	0	1225
62	1	0	0	0	0	2	0	2	1	0	1225
9	0	0	1	2	0	1	0	0	2	0	1224
25	0	1	0	0	1	0	0	2	2	0	1224
32	0	1	1	0	0	1	1	2	0	0	1224
1	0	0	0	2	0	0	0	2	2	0	1223
102	0	1	0	0	0	0	1	4	0	0	1223
6	0	0	1	1	1	0	0	1	2	0	1222.5
15	0	0	2	1	0	1	1	1	0	0	1222.5
33	0	1	1	0	0	2	0	1	1	0	1222.5
14	0	0	2	0	2	0	0	0	2	0	1222
22	0	0	3	0	1	1	1	0	0	0	1222
4	0	0	1	1	0	0	1	3	0	0	1221.5
24	0	1	0	0	0	1	0	3	1	0	1221.5
12	0	0	2	0	1	0	1	2	0	0	1221
16	0	0	2	1	0	2	0	0	1	0	1221

5	0	0	1	1	0	1	0	2	1	0	1220
13	0	0	2	0	1	1	0	1	1	0	1219.5
101	0	0	0	1	0	0	0	4	1	0	1219
3	0	0	1	0	1	0	0	3	1	0	1218.5
21	0	0	3	0	0	3	0	0	0	0	1218
11	0	0	2	0	0	2	0	2	0	0	1217
100	0	0	1	0	0	1	0	4	0	0	1216
103	0	0	0	0	0	0	0	6	0	0	1215

When you see the difference between top 5 groups,

Highest	No.99	(X,X,X),(X,X,X),(Y,Y,Y),(Y,Y,Y),(Z,Z,Z),(Z,Z,Z)
Second highest	No.94	(X,X,X),(X,X,Y),(X,Y,Y),(Y,Y,Y),(Z,Z,Z),(Z,Z,Z)
Third highest	No.60	(X,X,Y),(X,X,Y),(X,X,Y),(Y,Y,Y),(Z,Z,Z),(Z,Z,Z)
Fourth highest	No.76	(X,X,X),(X,Y,Y),(X,Y,Y),(X,Y,Y),(Z,Z,Z),(Z,Z,Z)
Fifth highest	No.54	(X,X,Y),(X,X,Y),(X,Y,Y),(X,Y,Y),(Z,Z,Z),(Z,Z,Z)

As you can see, for creating the group with second highest score in this situation, you need to exchange one of X for one of Y on the highest grouping. And then, No.94 is created from No.99 with one exchange. Next, No.60 is created by No.94 with an exchange X for Y. By fifth highest group in this situation, the ranking changes only by exchanging X for Y.

From  $6^{th}$  highest to lower, the ranking changes by exchanging something for Z. The group with lowest score, 103th, is (X,Y,Z),(X,Y,Z), (X,Y,Z), (X,Y,Z), (X,Y,Z).

## 2-4 Some numerical calculations with different conditions

## [Numerical example I]

We selected a simple constant decrease in the s-values favoring the good students working together. We also selected values for the X and the Z. For getting the different result, we change the value of Y from 50 to 70 by 10.

s1		s2	s3	s4	s5	s6	х	у	z
	1.25	1.2	1.15	1.1	1.05	1	80	60	40

Ranking	y=50	y=60	y=70
1	99	99	99
2	94	94	94
3	60	60	60
4	76	76	76
5	54	54	54
6	98	98	98
7	93	93	93
8	96**	59*	59*
9	97	89	89
10	59*	96**	73
11	89	97	51
12	92	73	85
13	95	85	91
14	73	91	57
15	85	51	75
16	91	57	41
17	58	75	53
18	51	92	96**
19	87	41	97
20	57	53	71
21	75	58	49
22	41	71	82
23	53	87	92
24	84	82	30
25	90	95	58

This chart is describing the ranking by ys', respectively. For example, 99 in this chart means grouping No.99, which is (X,X,X),(X,X,X),(Y,Y,Y),(Y,Y,Y),(Z,Z,Z),(Z,Z,Z).

When we change ys' from 50 to 70 by 10, we describe the top 25 groupings by descending order. We cannot see the difference above  $7^{\text{th}}$  but we can see the difference under the  $8^{\text{th}}$ . The No.59 goes up from  $10^{\text{th}}$ ,  $8^{\text{th}}$  to  $8^{\text{th}}$ , respectively. On the other hand, No.96 goes down from  $8^{\text{th}}$ ,  $10^{\text{th}}$  to  $18^{\text{th}}$ , respectively.

No.59 is (X,X,Y),(X,X,Y),(X,X,Y),(Y,Y,Z),(Y,Z,Z),(Z,Z,Z). No.96 is (X,X,X),(X,X,X), (Y,Y,Z),(Y,Y,Z),(Y,Y,Z),(Z,Z,Z). By giving y the difference, y has more advantage when Y is with Xs. Therefore, the rankings change. We can see that the degree of tops and lows don't change at all. But we can also see that some groupings around middle of the rankings change much.

# [Numerical example II]

We didn't change anything but the value of  $s_3$ . Numerical example II changes the value of  $s_3$  from 1.12 to 1.18 by 0.03.

NO	$s_3=1.12$	$s_3=1.15$	$s_3 = 1.18$
1	54**	99*	99*
2	60	94	94
3	76	60	60
4	94	76	76
5	99*	54**	<b>54**</b>
6	59	98	98
7	93	93	82
8	58	59	91
9	98	89	85
10	51	96	89
11	92	97	93
12	73	73	96
13	89	85	97
14	96	91	71
15	97	51	57
16	41	57	75
17	53	75	73
18	87	92	59
19	57	41	40
20	75	53	49
21	95	58	80
22	85	71	41
23	91	87	53
24	46	82	84
25	52	95	90

Since  $s_3$  is the coefficient for the relationship between Y and Y, when the value of  $s_3$  decreases, the value of (Y,Y,Y) decreases as well. Therefore, the ranking of No.99 goes down. When  $s_3$  is 1.12, the

grouping having the highest score is No.54, which is (X,X,Y),(X,X,Y),(X,Y,Y),(X,Y,Y),(Z,Y,Y),(Z,Z,Z). The grouping of just Ys was disappeared, and groupings with X and Y have advantage more than Ys.

### [Numerical example III]

Numerical example III changes the values of coefficients s's. There are three ways to change coefficients, which are concave, linear, and convex. This chart below is how we set them.

State	concave	Linear	Convex
<b>S</b> 1	1.25	1.25	1.25
$\mathbf{S}_2$	1.15	1.2	1.24
<b>S</b> 3	1.07	1.15	1.22
$\mathbf{S}_4$	1.03	1.1	1.18
$\mathbf{S}_5$	1.01	1.05	1.1
<b>S</b> 6	1	1	1



This chart describes the ranking by ascending order from top to 25th.

No.99 is the top when we use concave and linear ways. No.14 is the top with convex way.

NO	concave	linear	convex
1	99	99	14
2	94	94	2
3	98	60	29
4	60	76	23
5	76	54	66
6	96	98	6
7	97	93	10
8	54	59	40
9	93	89	82
10	95	96	17
11	59	97	1
12	92	73	9
13	89	85	25
14	58	91	30
15	73	51	36

16	87	57	18
17	85	75	49
18	91	92	54
19	51	41	3
20	86	53	71
21	57	58	7
22	75	71	31
23	84	87	60
24	90	82	76
25	41	95	13

We cannot see the difference a lot between concave and linear. But when we apply the convex way, the rankings change a lot.

What we need to check out is No.14. No.14 is changed from 100<sup>th</sup>, 90<sup>th</sup>, to 1<sup>st</sup> by different settings, respectively.

No.14 is (X,X,Z), (X,X,Z), (Y,Y,Y), (Y,Y,Y), (X,Z,Z), (X,Z,Z).

 $s_5$  (relationship between Y and Z) and  $s_4$  (relationship between X and Z) causes this result because the values of  $s_4$  and  $s_5$  go up drastically. No.99 which is the top at other's setting becomes  $34^{th}$  with convex situation.

We have observed just groupings whose rankings are increasing or decreasing, but we found the groupings doing weird movement in rankings. For example, No.60 places 4<sup>th</sup>, 3<sup>rd</sup>, and 23th, respectively.

No.60 is (X,X,Y), (X,X,Y), (X,X,Y), (Y,Y,Y), (Z,Z,Z), (Z,Z,Z).

### 2-5 The Model with limited sequence $\{s_i\}$

In this part, X<sub>i</sub> and X<sub>j</sub> have different numbers which is x<sub>i</sub> and x<sub>j</sub> respectively.

Let  $\{s_i\}$  be sequence of numbers with common difference d (constant),

 $s_i = s + (6 - i)d$ , and  $x_1 \ge x_2 \ge \cdots \ge x_6 > y_1 \ge y_2 \ge \cdots \ge y_6 > z_1 \ge z_2 \ge \cdots \ge z_6$ .

We let  $G_1 = (X_1, X_2, X_3)$ ,  $G_2 = (X_4, X_5, X_6)$ ,  $G_3 = (Y_1, Y_2, Y_3)$ ,  $G_4 = (Y_4, Y_5, Y_6)$ ,

 $G_5=(Z_1, Z_2, Z_3)$ , and  $G_6=(Z_4, Z_5, Z_6)$  and call this grouping "the group of likes".

The SGV<sub>max</sub> denotes the sum of group values of "the group of likes".

[Theorem]

The SGV<sub>max</sub> is the maximum in the all groupings.  $\blacksquare$ 

Proof:

No matter how you exchange arbitrary  $X_i$  and  $X_j$  between the two X-only groups  $\{X_1, X_2, X_3\}$  and  $\{X_4, X_5, X_6\}$ , the value of SGV<sub>max</sub> doesn't change. It is the same for  $\{Y_1, Y_2, \dots, Y_6\}$  and  $\{Z_1, Z_2, \dots, Z_6\}$ . When you exchange an arbitrary  $X_i$  in  $G_1$  or  $G_2$  for an arbitrary  $Y_j$  in  $G_3$  or  $G_4$ , we let

SGV'. We have already obtained SGV<sub>max</sub> from Example I.

We let  $\alpha = \sum_{i=1}^{6} x_i$ ,  $\beta = \sum_{i=1}^{6} y_i$ , and  $\gamma = \sum_{i=1}^{6} z_i$ . SGV<sub>max</sub>=  $s_1 \sum_{i=1}^{6} x_i + s_3 \sum_{i=1}^{6} y_i + s_6 \sum_{i=1}^{6} z_i$ 

$$= s_1\alpha + s_3\beta + s_6\gamma = s(\alpha + \beta + \gamma) + 5d\alpha + 3d\beta \quad (by \quad s_i = s + (6 - i)d)$$

If  $X_1$  and  $Y_m$  belong to the same group with  $X_i$  and  $Y_p$  and  $Y_q$  belong to the same group with  $Y_j$ , we exchange  $X_i$  for  $Y_j$  to be able to get SGV' easily, where l, m, p, and q  $\epsilon$  {1, 2, 3, 4, 5, 6}.

$$\begin{split} SGV' &= s(\alpha + \beta + \gamma) + 5d\alpha - (d/2)(x_1 + x_m) - dx_i + 3d\beta + dy_j + (d/2)(y_p + y_q) \\ SGV_{max} - SGV' &= dx_i + (d/2)(x_1 + x_m) - dy_j - (d/2)(y_p + y_q) \\ &= d(x_i - y_j) + (d/2)\{(x_1 + x_m) - (y_p + y_q)\} > 0 \quad (since x_i > y_j) \end{split}$$

In "the group of likes", exchanging one of X for one of Y makes the value of  $SGV_{max}$  small. From the same process, we can tell easily that exchanging one of X for one of Z makes the value of  $SGV_{max}$  small. Therefore, we can show that any exchange to "the group of likes" reduce  $SGV_{max}$ .

From numerical example I, II, and III, we have predicted that the sum of group values of "likes grouping" became the maximum. But under  $\sum_{i=1}^{6} x_i > \sum_{i=1}^{6} y_i > \sum_{i=1}^{6} z_i$  and  $s_1 \ge s_2 \ge s_3 \ge s_4 \ge s_5 \ge s_6$ , we can't prove the theorem. This proof is done with giving  $\{s_i\}$  the condition of sequence of numbers with common difference.

#### 3. Conclusion

We divide 18 students into six groups. We let each group do a coalitional game. Our purpose is that we find the benefit of whole classroom maximal. As we saw the results of numerical example I and Theorem of chapter 2-5, No.99 makes the highest benefit. That means we make groups from better students in order. However, we noticed that we have the different proper groupings from numerical example II and III with the different coefficient  $s_i$ . The rank of No.99 has chances to become not the highest under the condition,  $s_1 > s_2 > ... > s_6$ . We want to focus on the result with the convex way. No.14 became the highest with the convex way. Under the condition that the only relationship between Z and Z creates less benefit than others, to group Z and X creates the whole benefit bigger.

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### Appendix

Sub MIPS() k = -1 For a = 0 To 6 For b = 0 To 6 For c = 0 To 6 For d = 0 To 6 For e = 0 To 6 For f = 0 To 6 For g = 0 To 6 For h = 0 To 6 For i = 0 To 6 For j = 0 To 6 If WorksheetFunction.And(3 \* a + 2 \* b + 2 \* c + d + h + i = 6, b + 2 \* d + 3 \* e + 2 \* f + g + h = 6, b + 2 \* d + 3 \* e + 2 \* f + g + 2 \* f + 2 \* fc + f + 2 \* g + h + 2 \* i + 3 \* j = 6, a + b + c + d + e + f + g + h + i + j = 6) = True Then Cells(1, 1). Activate k = k + 1With Application.WorksheetFunction ActiveCell.Offset(0, k) = aActiveCell.Offset(1, k) = b

ActiveCell.Offset(2, k) = cActiveCell.Offset(3, k) = dActiveCell.Offset(4, k) = eActiveCell.Offset(5, k) = fActiveCell.Offset(6, k) = g ActiveCell.Offset(7, k) = hActiveCell.Offset(8, k) = iActiveCell.Offset(9, k) = jEnd With End If Next j Next i Next h Next g Next f Next e Next d Next c Next b Next a End Sub