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COLORINGS FOR SET-VALUED MAPS

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Abstract. We know many results about colorability for single-valued maps. But we know a few results about colorability for set-valued maps. In this paper we generalize some results on colorability for single-valued maps to those for set-valued maps. Especially, our main result is a generalization of E. K. van Douwen's result, which insists that every fixed-point free continuous closed map $f : X \to X$ with $\sup\{|f^{-1}(x)| : x \in X\} < \infty$ on a finite-dimensional paracompact space X is colorable. In fact, we prove the following: Let X be a finite-dimensional paracompact space and $f : X \to \mathcal{F}_k(X)$ a fixed-point free upper semi-continuous map, where $\mathcal{F}_k(X)$ is the family of non-empty subsets of X with at most k elements. Suppose that $\sup\{|f^{-1}(x)| : x \in X\} < \infty$ and $\bigcup\{f(x) : x \in F\}$ is closed in X for any closed subset F of X. Then f is colorable.

1 Introduction

All spaces under discussion are regular. We will discuss some set-valued versions of results about colorability for single-valued maps.

We define some notions about colorability of single-valued maps as follows: Let X be a subset of a space Y and $f : X \to Y$ a single-valued map. For a subset A of X, A is called a *color* of f if $A \cap f(A) = \emptyset$ and a *bright color* of f if $\overline{A}^Y \cap \overline{f(A)}^Y = \emptyset$, where $\overline{A}^Y$ denotes the closure of A in Y. Also we call a finite closed cover of X consisting of colors of f a *coloring* of f and we say that f is *colorable* if there is a coloring of f. Similarly, we define a *bright coloring* of f and say that f is *brightly colorable* if there is a bright coloring of f.

The following shows the essential meaning of colorability for single-valued maps:

Proposition 1.1. *Let X be a closed subspace of a normal space Y and let $f : X \to Y$ be a fixed-point free continuous map. Then, the following are equivalent:*

(1) *f is brightly colorable.*

(2) *The Stone-Čech extension $\beta f : \beta X \to \beta Y$ of f is fixed-point free.*

Also the following results for single-valued maps are known:

Proposition 1.2. *Let X be a compact subspace of a space Y and let $f : X \to Y$ be a fixed-point free continuous map. Then f is colorable.*

Theorem 1.3. *([5]) Let X be a closed subspace of a locally compact separable metrizable space Y with $\dim Y \leq n$ and let $f : X \to Y$ be a fixed-point free continuous map. Then, f is brightly colorable.*

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Theorem 1.4. *([6]) Let X be a paracompact space with $\dim X \leq n$ and let $f : X \to X$ be a fixed-point free continuous closed map such that $l = \sup\{|f^{-1}(x)| : x \in X\} < \infty$. Then, f is colorable with at most $(l+1)(n+1)+1$ colors.*

Theorem 1.5. *([2]) Let X be a separable metrizable space with $\dim X \leq n$ and let $f : X \to X$ be a fixed-point free homeomorphism. Then, f is colorable with at most $n+3$ colors.*

In this paper we generalize these results for single-valued maps to some results for set-valued maps. To start our discussion we give a topology of the space consisting of closed subsets (see [9] in detail).

For a space X we define the hyperspace 2^X of X as the family of all non-empty closed subsets of X and endow 2^X with the Vietoris topology, which has

$$\langle \mathcal{U} \rangle = \Big\{ A \in 2^X : A \subset \bigcup \mathcal{U} \text{ and } A \cap U \neq \emptyset \text{ for any } U \in \mathcal{U} \Big\},$$

where $\mathcal{U}$ is a finite family of open subsets of X, as the basic open subsets of 2^X. Also let $\mathcal{K}(X)$ and $\mathcal{F}_k(X)$ for $k \in \mathbb{N}$ denote the family of non-empty compact subsets of X and the family of non-empty finite subsets of X with at most k elements, respectively.

Let X and Y be spaces and $f : X \to 2^Y$ a set-valued map. For $A \subset X$ we write $f(A) = \bigcup\{f(x) : x \in A\}$. Also for $y \in Y$, $B \subset Y$ and $\mathcal{B} \subset 2^Y$ we write $f^{-1}(y) = \{x \in X : y \in f(x)\}$, $f^{-1}(B) = \{x \in X : f(x) \cap B \neq \emptyset\}$ and $f^{-1}[\mathcal{B}] = \{x \in X : f(x) \in \mathcal{B}\}$. Also $f : X \to 2^Y$ is upper semi-continuous if for $x \in X$ and an open set V of Y with $f(x) \subset V$, $f^{-1}[\langle\{V\}\rangle](= \{x' \in X : f(x') \subset V\})$ is open in X.

When $X \subset Y$ we define some notions about colorability of set-valued maps as follows: A map $f : X \to 2^Y$ is called a *fixed-point free* map if $x \notin f(x)$ for any $x \in X$. For a subset A of X, A is called a *color* of f if $A \cap f(A) = \emptyset$ and called a *bright color* of f if $\overline{A}^Y \cap \overline{f(A)}^Y = \emptyset$. Also we call a finite closed cover of X consisting of colors of f a *coloring* of f and we say that f is *colorable* if there is a coloring of f. Similarly, we define a *bright coloring* of f and say that f is *brightly colorable* if there is a bright coloring of f.

Any space X can be embedded to 2^X by the inclusion $\iota : X \to 2^X$ defined by $x \mapsto \{x\}$. Hence all results for set-valued maps are also true for single-valued maps. The proofs are modifications of proofs for single-valued versions in [5], [6] and [2].

Also let (A, B) be a pair of disjoint closed subsets of a space X. A subset S of X is called a partition between A and B if there is a pair (U, V) of disjoint open subsets of X such that $A \subset U$, $B \subset V$ and $X \backslash S = U \cup V$.

2 Results

First, we present a generalization of Proposition 1.2.

Proposition 2.1. *Let X be a compact subspace of a space Y and let $f : X \to 2^Y$ be a fixed-point free and upper semi-continuous map. Then, f is colorable.*

Proof. By compactness of X it is sufficient to show that for each $x \in X$ there is an open neighborhood of x in X such that its closure is a color of f. Take $x \in X$. Then $x \notin f(x)$ since f is fixed-point free. By regularity of Y there are two open neighborhoods U and V of x and $f(x)$ in Y, respectively, such that $U \cap V = \emptyset$. Since f is upper semi-continuous, $f^{-1}[\langle\{V\}\rangle]$ is open in X. By regularity of X there is an open neighborhood W of x in X such that $\overline{W} \subset U \cap f^{-1}[\langle\{V\}\rangle]$. This is as required. □

Next, we consider a generalization of Theorem 1.3.

Theorem 2.2. *([4]) Let X be a closed subspace of $\mathbb{R}^n$ and let $f : X \to \mathcal{F}_k(\mathbb{R}^n)$ be a fixed-point free continuous map. Then, f is brightly colorable.*

Applying Theorem 2.2, we obtain a generalization of Theorem 1.3 as follows.

Theorem 2.3. *Let X be a closed subset of a locally compact separable metrizable space Y with $\dim Y \leq n$ and let $f : X \to \mathcal{F}_k(Y)$ be a fixed-point free continuous map. Then, f is brightly colorable.*

Proof. We may assume that Y is closed in $\mathbb{R}^{2n+1}$ since any n-dimensional locally compact separable metrizable space can be embedded in $\mathbb{R}^{2n+1}$ as a closed subset. Therefore, this proof is completed by Theorem 2.2. □

Remark. For Theorem 2.2 we know that for $n, k \in \mathbb{N}$ there is a minimal integer $K(n, k)$ such that every fixed point free continuous map $f : X \to \mathcal{F}_k(\mathbb{R}^n)$ is colorable with at most $K(n, k)$ colors (see [4]). So we can see that $K(2n + 1, k)$ plays the same part for Theorem 2.3. But it is not clear about the exact values.

To show our main result we define the *order* and give a lemma.

Let X be a space and $\mathcal{U}$ a family of subsets of X and $n \in \{0, 1, 2, ...\}$. We define the order of $\mathcal{U}$, which is denoted by ord $\mathcal{U}$, as follows:

$$\mathrm{ord}\,\mathcal{U} \leq n \text{ if } \sup\Big\{\big|\{U \in \mathcal{U} : x \in U\}\big| : x \in X\Big\} \leq n.$$

Remark. In many books ord $\mathcal{U} \leq n$ is defined by $\big|\{U \in \mathcal{U} : x \in U\}\big| \leq n+1$ for any $x \in X$. But in this paper we use the above definition to see inequalities about the order easily.

Lemma 2.4. *([6]) Let X be a normal space. Let $\{G_i : i = 1, ..., k\}$ be a family of closed subsets of X with $\mathrm{ord}\{G_i : i = 1, ..., k\} \leq \dim X + 1$ and $\{W_i : i = 1, ..., k\}$ an open cover of X such that $G_i \subset W_i$ for $i = 1, ..., k$. Then, there is an open cover $\{V_i : i = 1, ..., k\}$ of X such that $\mathrm{ord}\{\overline{V_i} : i = 1, ..., k\} \leq \dim X + 1$ and $G_i \subset V_i$ and $\overline{V_i} \subset W_i$ for $i = 1, ..., k$.*

The following theorem is a generalization of Theorem 1.4.

Theorem 2.5. *Let X be a paracompact space with $\dim X \leq n$ and let $f : X \to \mathcal{F}_k(X)$ be a fixed-point free upper semi-continuous map. Suppose that $l = \sup\big\{|f^{-1}(x)| : x \in X\big\} < \infty$ and $f(F)$ is closed in X for any closed subset F of X. Then, f is colorable with at most $(k + l)(n + 1) + 1$ colors.*

Proof. First, fix $x \in X$. Since f is fixed-point free, there are two open neighborhoods U_x and V_x of x and $f(x)$ in X, respectively, such that $U_x \cap V_x = \emptyset$. $f^{-1}[\langle\{V_x\}\rangle]$ is an open neighborhood of x in X since f is upper semi-continuous. Put $W_x = U_x \cap f^{-1}[\langle\{V_x\}\rangle]$. Then $W_x \cap f(W_x) = \emptyset$.

Put $\mathcal{W} = \{W_x : x \in X\}$. Then $\mathcal{W}$ covers X. So by paracompactness of X there is a locally finite closed refinement $\mathcal{A}$ of $\mathcal{W}$. List $\mathcal{A}$ as $\{A_\xi : \xi < \kappa\}$ for some ordinal number κ. Observe that $A_\xi \cup f(A_\xi) = \emptyset$ for each $\xi < \kappa$.

Next, put $p = (k+l)(n+1)+1$ and for each $\xi < \kappa$ we will construct inductively a closed cover $\{B_{\xi,i} : i = 1, ..., p\}$ of A_ξ in a way such that if

$$C_{\eta,i} = \bigcup_{\xi<\eta} B_{\xi,i} \quad \text{for } i = 1, ..., p$$

then for all $\eta < \kappa$ we have

$$(1_\eta) \qquad C_{\eta,i} \cap f(C_{\eta,i}) = \emptyset \quad \text{for } i = 1, ..., p,$$

$$(2_\eta) \qquad \operatorname{ord}\big\{C_{\eta,i} : i = 1, ..., p\big\} \le n+1.$$

We note that $C_{\eta,i}$ is closed in X for each $\eta < \kappa$ and $i = 1, ..., p$ since $\mathcal{A}$ is locally finite.

The construction: For $\eta = 0$ (1_0) and (2_0) hold since $C_{0,i} = \emptyset$ for $i = 1, ..., p$.

When constructing $\{B_{\xi,i} : i = 1, ..., p\}$ for an $\eta < \kappa$ and each $\xi < \eta$, we may assume (1_η) and (2_η) to hold. Now we will construct $\{B_{\eta,i} : i = 1, ..., p\}$. For $i = 1, ..., p$ define

$$D_i = f^{-1}(C_{\eta,i}) \cup f(C_{\eta,i}).$$

Then D_i is closed in X since f is upper semi-continuous. To see that

(a) $\{A_\eta \backslash D_i : i = 1, ..., p\}$ covers A_η

we claim that

$$\bigcap_{i=1}^{p} D_i = \emptyset.$$

By (2_η) and $|f(x)| \le k$, $|f^{-1}(x)| \le l$ for all $x \in X$ we have

$$\operatorname{ord}\{f^{-1}(C_{\eta,i}) : i = 1, ..., p\} \le k(n+1),$$
$$\operatorname{ord}\{f(C_{\eta,i}) : i = 1, ..., p\} \le l(n+1).$$

Indeed, for the first when we put $f(x) = \{x_1, ..., x_k\}$ for each $x \in X$, $\big|\{i : x_j \in C_{\eta,i}\}\big| \le n+1$ for $j = 1, ..., k$ by (2_η). Hence

$$\begin{aligned}\big|\{i : x \in f^{-1}(C_{\eta,i})\}\big| &= \Big|\bigcup_{j=1}^{k}\{i : x_j \in C_{\eta,i}\}\Big| \\ &\le \sum_{j=1}^{k}\big|\{i : x_j \in C_{\eta,i}\}\big| \\ &\le k(n+1).\end{aligned}$$

Similarly, we can verify the second.

Thus, from the definition of D_i

$$\begin{aligned}\operatorname{ord}\{D_i : i = 1, ..., p\} &\le \operatorname{ord}\{f^{-1}(C_{\eta,i}) \cup f(C_{\eta,i}) : i = 1, ..., p\} \\ &\le k(n+1) + l(n+1) \\ &= (k+l)(n+1).\end{aligned}$$

So $\bigcap_{i=1}^{p} D_i = \emptyset$ and (a) holds.

By (1_η)

(b) $C_{\eta,i} \cap D_i = \emptyset$ for $i = 1, ..., p$.

So because $\dim A_\eta \leq n$, $\mathrm{ord}\{A_\eta \cap C_{\eta,i} : i = 1, ..., p\} \leq n + 1$ and $A_\eta \cap C_{\eta,i} \subset A_\eta \backslash D_i$ for $i = 1, ..., p$, by Lemma 2.4 there is a relatively open cover $\{O_i : i = 1, ..., p\}$ of A_η such that

(c) $A_\eta \cap C_{\eta,i} \subset O_i$, $\overline{O_i} \subset A_\eta \backslash D_i$ for $i = 1, ..., p$,

(d) $\mathrm{ord}\big\{\overline{O_i} : i = 1, ..., p\big\} \leq n + 1$.

Define $B_{\eta,i} = \overline{O_i}$ for $i = 1, ..., p$. Then, $C_{\eta+1,i} = C_{\eta,i} \cup B_{\eta,i} = C_{\eta,i} \cup \overline{O_i}$ for $i = 1, ..., p$. We check $(1_{\eta+1})$ and $(2_{\eta+1})$. For $(2_{\eta+1})$ we obtain

$$\begin{aligned}\mathrm{ord}\{C_{\eta+1,i} : i = 1, ..., p\} &= \mathrm{ord}\{C_{\eta,i} \cup \overline{O_i} : i = 1, ..., p\} \\ &= \mathrm{ord}\{(C_{\eta,i} \backslash A_\eta) \cup \overline{O_i} : i = 1, ..., p\} \\ &\leq n + 1\end{aligned}$$

by (2_η), (d) and the first part of (c). For $(1_{\eta+1})$ it is sufficient to prove that

$$\begin{aligned}&C_{\eta,i} \cap f(C_{\eta,i}) = \emptyset, \\ &C_{\eta,i} \cap f(B_{\eta,i}) = \emptyset, \\ &B_{\eta,i} \cap f(C_{\eta,i}) = \emptyset, \\ &B_{\eta,i} \cap f(B_{\eta,i}) = \emptyset\end{aligned}$$

for $i = 1, ..., p$. The first and fourth are trivial from (1_η) and the property of A_η. Also $B_{\eta,i} \cap f^{-1}(C_{\eta,i}) = \emptyset$ if and only if $C_{\eta,i} \cap f(B_{\eta,i}) = \emptyset$. Thus, the second and third hold by the second part of (c). This completes the construction of $B_{\xi,i}$.

Finally, define

$$C_i = \bigcup_{\eta < \kappa} C_{\eta,i} \quad \text{for } i = 1, ..., p.$$

It is easy to see that $\mathcal{C} = \{C_i : i = 1, ..., p\}$ is a closed cover of X consisting of colors of f. Consequently, $\mathcal{C}$ is as required. □

When X is compact, Theorem 2.5 implies the following corollary.

Corollary 2.6. *Let X be a compact space with $\dim X \leq n$ and let $f : X \to \mathcal{F}_k(X)$ be a fixed-point free and upper semi-continuous map. Suppose that $l = \sup\big\{|f^{-1}(x)| : x \in X\big\} < \infty$. Then f is colorable with at most $(k + l)(n + 1) + 1$ colors.*

Proof. By compactness of X, $f(F)$ is closed in X for any closed subset F of X. So this is shown from Theorem 2.5. □

The numbers of colors in the above results are not sharp. Here we consider reducing the numbers of colors.

Lemma 2.7. *Let X be a separable metrizable space with $\dim X \leq n$ and let $f : X \to 2^X$ be an upper semi-continuous map such that $f(F)$ is closed in X and $\dim f^{-1}(F) = \dim F$ for any closed subset F of X. Let φ and $\varphi_i (i = 1, 2, ...)$ denote one of the map f and the inclusion ι. Assume that $\mathcal{S} = \{S_i : i \in \mathbb{N}\}$ is a family of closed subsets of X such that*

$$\dim(\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_k}(S_{i_k})) \leq n - k$$

whenever $i_1 < \cdots < i_k$ *and* $k = 1, ..., n+1$. *Then for every pair* (G, H) *of disjoint closed subsets of* X *there is a partition* S *between* G *and* H *in* X *such that*

$$\dim(\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_{k-1}}(S_{i_{k-1}}) \cap \varphi(S)) \leq n - k$$

whenever $i_1 < \cdots < i_{k-1}$ *and* $k = 1, ..., n+1$.

Proof. Put $X_k = \bigcup\{\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_k}(S_{i_k}) : i_1 < \cdots < i_k\}$ for $k = 1, ..., n$. Write $X_0 = X$. Then X_k is an F_σ-subset of X. By assumptions of $\mathcal{S}$ we have $\dim X_k \leq n - k$. So there is an F_σ-subset Z of X with $\dim Z = 0$ and $\dim(X_k \backslash Z) \leq n - k - 1$ for $k = 1, ..., n$. Since f is upper semi-continuous, $f^{-1}(Z)$ is an F_σ-subset of X. By assumption of f we have $\dim(Z \cup f^{-1}(Z)) = 0$. Hence there is a partition S between G and H in X such that $S \cap (Z \cup f^{-1}(Z)) = \emptyset$.

Then

$$\begin{gathered}\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_{k-1}}(S_{i_{k-1}}) \cap S \subset X_{k-1} \backslash Z, \\ \varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_{k-1}}(S_{i_{k-1}}) \cap f(S) \subset X_{k-1} \backslash Z,\end{gathered}$$

whenever $i_1 < \cdots < i_{k-1}$ and $k = 1, ..., n+1$. Therefore,

$$\begin{aligned}\dim((\varphi_{i_1}(S_{i_1}) \cap \cdots \cap \varphi_{i_{k-1}}(S_{i_{k-1}}) \cap \varphi(S)) &\leq n - (k-1) - 1 \\ &= n - k.\end{aligned}$$

So S is as required. □

Lemma 2.8. *Let* X *be a separable metrizable space with* $\dim X \leq n$ *and let* $f : X \to 2^X$ *be an upper semi-continuous map such that* $f(F)$ *is closed in* X *and* $\dim f^{-1}(F) = \dim F = \dim f(F)$ *for any closed subset* F *of* X. *Let* φ *and* $\varphi_i (i = 1, 2, ...)$ *denote one of the map* f *and the inclusion* ι. *Let* $\mathcal{U} = \{U_i : i = 1, ..., m\}$ *be an open cover of* X *and* $\mathcal{K} = \{K_i : i = 1, ..., m\}$ *be an closed shrinking of* $\mathcal{U}$. *Then there is a closed cover* $\mathcal{L} = \{L_i : i = 1, ..., m\}$ *of* X *such that* $K_i \subset L_i \subset U_i$ *for* $i = 1, ..., m$ *and*

$$\varphi_{i_1}(\partial L_{i_1}) \cap \cdots \cap \varphi_{i_{n+1}}(\partial L_{i_{n+1}}) = \emptyset$$

whenever $1 \leq i_1 < \cdots < i_{n+1} \leq m$.

Proof. The proof will be done by induction.

First, we define L_1. Since $\dim X \leq n$, there is a partition S_1 between K_1 and $X \backslash U_1$ in X such that $\dim S_1 \leq n - 1$. By assumption of f we have $\dim f(S_1) \leq n - 1$. Now $X \backslash S_1$ is the disjoint union of two open subsets V_1 and W_1 in X such that $K_1 \subset V_1$ and $X \backslash U_1 \subset W_1$. Define $L_1 = \overline{V_1}$. Then $\partial L_1 \subset S_1$ and so $\dim \varphi(\partial L_1) \leq n - 1$.

Next, assume that for some $r \in \{1, ..., m\}$ L_i is defined for $i = 1, ..., r-1$ such that the family $\{\partial L_i : i = 1, ..., r-1\}$ has the property

$$(*) \qquad \dim(\varphi_{i_1}(\partial L_{i_1}) \cap \cdots \cap \varphi_{i_k}(\partial L_{i_k})) \leq n - k,$$

whenever $1 \leq i_1 < \cdots < i_k \leq r - 1$ and $k = 1, ..., n+1$. From Lemma 2.7 there is a partition S_r between K_r and $X \backslash U_r$ in X such that the property $(*)$ holds for the family $\{\partial L_i : i = 1, ..., r-1\} \cup \{S_r\}$. Now $X \backslash S_r$ is the disjoint union of two open subsets V_r and W_r in X such that $K_r \subset V_r$ and $X \backslash U_r \subset W_r$. Define $L_r = \overline{V_r}$. Then $\partial L_r \subset S_r$ and so the property $(*)$ holds for $\{\partial L_i : i = 1, ..., r\}$. This completes the construction of L_i.

Take $1 \le i_1 < \cdots < i_{n+1} \le m$. Then

$$\dim(\varphi_{i_1}(\partial L_{i_1}) \cap \cdots \cap \varphi_{i_{n+1}}(\partial L_{i_{n+1}})) \le n - (n+1) = -1$$

and so

$$\varphi_{i_1}(\partial L_{i_1}) \cap \cdots \cap \varphi_{i_{n+1}}(\partial L_{i_{n+1}}) = \emptyset.$$

Consequently, $\mathcal{L} = \{L_i : i = 1, ..., m\}$ is as required. $\square$

Lemma 2.9. *([2]) Let $\mathcal{K} = \{K_i : i = 1, ..., k\}$ be a finite closed cover of a space X. Define $L_i = \overline{K_i \backslash (\bigcup_{j=1}^{i-1} K_j)}$ for $i = 1, ..., m$. Then $\mathcal{L} = \{L_i : i = 1, ..., k\}$ has the following properties:*

(1) *$L_s \cap L_t = \partial L_s \cap \partial L_t$ for $s \neq t$.*

(2) *If $\partial L_{i_1} \cap \cdots \cap \partial L_{i_m} \neq \emptyset$, then $\partial K_{i_1} \cap \cdots \cap \partial K_{i_{m-1}} \neq \emptyset$ whenever $1 \le i_1 < \cdots < i_m \le k$.*

The following theorem is a generalization of Theorem 1.5.

Theorem 2.10. *Let X be a separable metrizable space with $\dim X \le n$ and let $f : X \to \mathcal{F}_k(X)$ be a fixed-point free upper semi-continuous map such that $l = \sup\{|f^{-1}(x)| : x \in X\} < \infty$. Suppose that $f(F)$ is closed in X and $\dim f^{-1}(F) = \dim F = \dim f(F)$ for any closed subset F of X. Then f is colorable with at most $kn + k + l + 1$ colors.*

Proof. f is colorable by Theorem 2.5. So there is a coloring $\mathcal{A} = \{A_i : i = 1, ..., r\}$ of f for some $r \in \mathbb{N}$. Assume that $r > kn + k + l + 1$. Because A_i and $f(A_i)$ are disjoint closed subsets of X for each $i = 1, ..., r$ and X is normal, there are two open neighborhoods U_i and V_i of A_i and $f(A)$ in X, respectively, such that $U_i \cap V_i = \emptyset$ for each $i = 1, ..., r$. Since f is upper semi-continuous, $f^{-1}[\langle\{V_i\}\rangle]$ is an open neighborhood of A_i in X for each $i = 1, ..., r$. Put $B_i = U_i \cap f^{-1}[\langle\{V_i\}\rangle]$ for each $i = 1, ..., r$. Then $\mathcal{B} = \{B_i | i = 1, ..., r\}$ is an open cover of X such that $A_i \subset B_i$ and $B_i \cap f(B_i) = \emptyset$ for $i = 1, ..., r$.

Define $g : X \to 2^X$ by $g(x) = f(f(x))$ for $x \in X$. Since f is upper semi-continuous, g is upper semi-continuous. For any closed subset F of X, $g(F) = f(f(F))$ and $g^{-1}(F) = f^{-1}(f^{-1}(F))$. Hence $g(F)$ is closed in X and $\dim g^{-1}(F) = \dim F = \dim g(F)$ by assumptions of f. These enable us to apply Lemma 2.8 as φ and $\varphi_i (i = 1, 2, ...)$ denote one of the map g and the inclusion ι. So there is a closed cover $\mathcal{C} = \{C_i : i = 1, ..., r\}$ of X such that $A_i \subset C_i \subset B_i$ for $i = 1, ..., r$ and

$$(\sharp) \qquad \varphi_{i_1}(\partial C_{i_1}) \cap \cdots \cap \varphi_{i_{n+1}}(\partial C_{i_{n+1}}) = \emptyset,$$

whenever $1 \le i_1 < \cdots < i_{n+1} \le r$. Define $D_i = \overline{C_i \backslash (\bigcup_{j=1}^{i-1} C_j)}$ and let $\mathcal{D} = \{D_i : i = 1, ..., r\}$. Observe that $\mathcal{D}$ is a coloring of f.

Take $x \in D_r$ and put $f^{-1}(x) = \{y_1, ..., y_l\}$ and $f(x) = \{z_1, ..., z_k\}$. Define m, p_a and q_b for $a = 1, ..., l$, $b = 1, ..., k$ as follows:

$$\begin{aligned}
&m = |\{i : (f^{-1}(x) \cup f(x)) \cap D_i \neq \emptyset\}|,\\
&p_1 = |\{i : y_1 \in D_i\}|,\\
&p_a = |\{i : \{y_1, ..., y_{a-1}\} \cap D_i = \emptyset \text{ and } y_a \in D_i\}| \ (a \ge 2),\\
&q_1 = |\{i : f^{-1}(x) \cap D_i = \emptyset \text{ and } z_1 \in D_i\}|,\\
&q_b = |\{i : (f^{-1}(x) \cup \{z_1, ..., z_{b-1}\}) \cap D_i = \emptyset \text{ and } z_b \in D_i\}| \ (b \ge 2).
\end{aligned}$$

Without lost of generality we may assume that $p_a \geq 1$, $q_b \geq 1$ for $a = 1, ..., l$, $b = 1, ..., k$. Note that no indices i overlap in the definition of p_a and q_b for $a = 1, ..., l$, $b = 1, ..., k$ i.e.

$$m = \sum_{a=1}^{l} p_a + \sum_{b=1}^{k} q_b$$

By Lemma 2.9

$$\begin{aligned} y_a \in \partial C_i &\quad \text{for at least } p_a - 1 \text{ indices } i, \\ z_b \in \partial C_i &\quad \text{for at least } q_b - 1 \text{ indices } i, \end{aligned}$$

for $a = 1, ..., l$, $b = 1, ..., k$. So

$$f(x) \subset g(\partial C_i) \quad \text{for at least } \sum_{a=1}^{l} (p_a - 1) \text{ indices } i.$$

Hence for $b = 1, ..., k$

$$z_b \in \varphi(\partial C_i) \quad \text{for at least } \sum_{a=1}^{l} (p_a - 1) + (q_b - 1) \text{ indices } i.$$

By the property ($\sharp$) for $b = 1, ..., k$

$$\sum_{a=1}^{l} (p_a - 1) + (q_b - 1) \leq n.$$

Since $p_a - 1 \geq 0$ for $a = 1, ..., l$,

$$\begin{aligned} m &= \sum_{a=1}^{l} p_a + \sum_{b=1}^{k} q_b \\ &= \sum_{a=1}^{l} (p_a - 1) + l + \sum_{b=1}^{k} (q_b - 1) + k \\ &\leq \sum_{b=1}^{k} \Big(\sum_{a=1}^{l} (p_a - 1) + (q_b - 1) \Big) + k + l \\ &\leq \sum_{b=1}^{k} n + k + l \\ &= kn + k + l. \end{aligned}$$

Now since $r > kn + k + l + 1$, there is a $j(x) \in \{1, ..., r-1\}$ such that $x \notin f^{-1}(D_{j(x)}) \cup f(D_{j(x)})$. Because $f^{-1}(D_{j(x)})$ and $f(D_{j(x)})$ are closed in X, there is an open neighborhood W_x of x in X such that $W_x \subset B_r \backslash (f^{-1}(D_{j(x)}) \cup f(D_{j(x)}))$.

Put $\mathcal{W} = \{W_x : x \in D_r\}$. By paracompactness of D_r there is a locally finite closed refinement $\mathcal{K} = \{K_s : s \in S\}$ of $\mathcal{W}$, where S is an index set. Define $\psi : S \to \{1, ..., r-1\}$ as it satisfies that $K_s \subset B_r \backslash (f^{-1}(D_{\psi(s)}) \cup f(D_{\psi(s)}))$. Put $E_j = \bigcup \{K_s : j = \psi(s)\}$ and $F_j = D_j \cup E_j$ for $j = 1, ..., r-1$. Then $\mathcal{F} = \{F_j : j = 1, ..., r-1\}$ is a coloring of f consisting of $r-1$ colors. In fact, since $\mathcal{K}$ is locally finite, E_j is closed in X and so F_j is closed in X. To show that F_j is a color of f for each $j = 1, ..., r-1$ we check the followings:

$$\begin{aligned} D_j \cap f(D_j) &= \emptyset, \\ D_j \cap f(E_j) &= \emptyset, \\ E_j \cap f(D_j) &= \emptyset, \\ E_j \cap f(E_j) &= \emptyset \end{aligned}$$

for $j = 1, ..., r-1$. The second can be replaced by $E_j \cap f^{-1}(D_j) = \emptyset$. Therefore, all hold since $\mathcal{D}$ is the coloring of f and $E_j \subset B_r \backslash (f^{-1}(D_j) \cup f(D_j))$.

We have reduced the number of colors by one, under the assumption that this number is greater than $kn + k + l + 1$. Inductively, the coloring of f can be reduced to a coloring of f with $kn + k + l + 1$ colors. $\square$

When X is compact, Theorem 2.10 implies the following corollary by the same way as Corollary 2.6.

Corollary 2.11. *Let X be a compact metrizable space with $\dim X \le n$ and let $f : X \to \mathcal{F}_k(X)$ be a fixed-point free upper semi-continuous map such that $l = \sup\{|f^{-1}(x)| : x \in X\} < \infty$. Suppose that $\dim f^{-1}(F) = \dim F = \dim f(F)$ for any closed subset F of X. Then f is colorable with at most $kn + k + l + 1$ colors.*

We would like to finish the paper by mentioning a relation between colorability and the Stone-Čech compactification. Let X be a normal space. Then the Stone-Čech compactification βX of X is equivalent to the Wallman compactification of X with respect to the Wallman base consisting of all closed subsets of X. Hence $\overline{F}^{\beta X} \cap \overline{G}^{\beta X} = \emptyset$ for any pair (F, G) of disjoint closed subsets of X. Also if F is closed in X, $\beta F = \overline{F}^{\beta X}$. So we may assume that $\beta F \subset \beta X$.

The following is a generalization of Proposition 1.1.

Proposition 2.12. *Let X be a closed subspace of a normal space Y and let $f : X \to \mathcal{K}(Y)$ be a fixed-point free continuous map. Then, the following are equivalent:*

(1) *f is brightly colorable.*

(2) *The Stone-Čech extension $\beta f : \beta X \to 2^{\beta Y}$ of f is fixed-point free.*

Proof. We will show that (1) implies (2). Since $2^{\beta Y}$ is compact and $\mathcal{K}(Y) \subset 2^{\beta Y}$, there is a continuous extension $\beta f : \beta X \to 2^{\beta Y}$ of f. Take $z \in \beta X$ to show that βf is fixed-point free . By (1) there is a bright coloring $\mathcal{C}$ of f. Then $\overline{\mathcal{C}}^{\beta Y}$ is a finite cover of βX and hence there is a $C \in \mathcal{C}$ such that $z \in \overline{C}^{\beta Y}$. Because C is a bright color of f, $C \cap \overline{f(C)}^{Y} = \emptyset$. By the property of the Stone-Čech compactification $\overline{C}^{\beta Y} \cap \overline{f(C)}^{\beta Y} = \emptyset$. By continuity of f

$$\beta f(z) \subset \beta f(\overline{C}^{\beta Y}) \subset \overline{\beta f(C)}^{\beta Y} \subset \overline{f(C)}^{\beta Y}.$$

Thus, $z \notin \beta f(z)$.

Next, we will show that (2) implies (1). Since βf is fixed-point free continuous and βX is compact, βf is colorable from Proposition 2.1. So there is a coloring $\mathcal{C}$ of βf. Then the restriction of $\mathcal{C}$ to X is as required. $\square$

This shows that colorability for set-valued continuous maps with compact values is similar to that for single-valued continuous maps.

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