# BIOPERATIONS ON $\alpha$-SEPARATIONS AXIOMS IN TOPOLOGICAL SPACES

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**Abstract**

In this paper, we consider the class of $\alpha_{[\gamma,\gamma']}$-generalized closed set in topological spaces and investigate some of their properties. We also present and study new separation axioms by using the notions of $\alpha$-open and $\alpha$-bioperations. Also, we analyze the relations with some well known separation axioms.

# 1 Introduction

The study of $\alpha$-open sets was initiated by Njȧstad [3]. Maheshwari et al. [8] and Maki et al. [9] introduced and studied a new separation axiom called $\alpha$-separation axiom. Kasahara [2] defined the concept of an operation on topological spaces and introduced $\alpha$-closed graphs of an operation. Ogata [4] called the operation $\alpha$ as $\gamma$ operation and introduced the notion of $\gamma$-open sets and used it to investigate some new separation axioms. For two operations on $\tau$ some bioperation-separation axioms were defined [7], [5]. Moreover, H.Z.Ibrahim [6] defined the concept of an operation on $\alpha O(X,\tau)$ and introduced $\alpha_\gamma$-open sets and $\alpha_\gamma$-$T_i$ $(i = 0, \frac{1}{2}, 1, 2)$ in topological spaces. In this paper, in Section 3, we introduce the concept of $\alpha_{[\gamma,\gamma']}$-generalized closed sets and investigate some of its important properties. The notion of new bioperation $\alpha$-separation axioms is introduced in Section 4. We compare these separation axioms with the separation axioms in [10], [4], [6], [7] and [5].

# 2 Preliminaries

Throughout this paper, $(X,\tau)$ and $(Y,\sigma)$ represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $(X,\tau)$ be a topological space and $A$ be a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset $A$ of a topological space $(X,\tau)$ is said to be $\alpha$-open [3] if $A \subseteq Int(Cl(Int(A)))$. The complement of an $\alpha$-open set is said to be $\alpha$-closed. The intersection of all $\alpha$-closed sets containing $A$ is called the $\alpha$-closure of $A$ and is denoted by $\alpha Cl(A)$. The family of all $\alpha$-open (resp. $\alpha$-closed) sets in a topological space $(X,\tau)$ is denoted by $\alpha O(X,\tau)$ (resp. $\alpha C(X,\tau)$). An operation $\gamma$ [2] on a topology $\tau$ is a

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mapping from $\tau$ in to the power set $P(X)$ of $(X,\tau)$ such that $V \subseteq V^{\gamma}$ for each $V \in \tau$, where $V^{\gamma}$ denotes the value of $\gamma$ at $V$. A subset $A$ of $(X,\tau)$ with an operation $\gamma$ on $\tau$ is called $\gamma$-open [4] if for each $x \in A$, there exists an open set $U$ such that $x \in U$ and $U^{\gamma} \subseteq A$. An operation $\gamma : \alpha O(X,\tau) \to P(X)$ [6] is a mapping satisfying the following property, $V \subseteq V^{\gamma}$ for each $V \in \alpha O(X,\tau)$. We call the mapping $\gamma$ an operation on $\alpha O(X,\tau)$. A subset $A$ of $(X,\tau)$ is called an $\alpha_{\gamma}$-open set [6] if for each point $x \in A$, there exists an $\alpha$-open set $U$ of $(X,\tau)$ containing $x$ such that $U^{\gamma} \subseteq A$. We denote the set of all $\alpha_{\gamma}$-open sets of $(X,\tau)$ by $\alpha O(X,\tau)_{\gamma}$. An operation $\gamma$ on $\alpha O(X,\tau)$ is said to be $\alpha$-regular [6] if for each point $x \in X$ and every $\alpha$-open sets $U$ and $V$ containing $x$, there exists an $\alpha$-open set $W$ of $(X,\tau)$ containing $x$ such that $W^{\gamma} \subseteq U^{\gamma} \cap V^{\gamma}$. An operation $\gamma$ on $\alpha O(X,\tau)$ is said to be $\alpha$-open [6] if for every $\alpha$-open set $U$ of each $x \in X$, there exists an $\alpha_{\gamma}$-open set $V$ such that $x \in V$ and $V \subseteq U^{\gamma}$. Let $\gamma$ and $\gamma^{'}$ be two operations on $\alpha O(X,\tau)$. A subset $A$ of $(X,\tau)$ is said to be $\alpha_{[\gamma,\gamma']}$-open [1; Definition 3.1] if for each $x \in A$ there exist $\alpha$-open sets $U$ and $V$ of $(X,\tau)$ containing $x$ such that $U^{\gamma} \cap V^{\gamma^{'}} \subseteq A$. The set of all $\alpha_{[\gamma,\gamma']}$-open sets of $(X,\tau)$ is denoted by $\alpha O(X,\tau)_{[\gamma,\gamma']}$. A subset $F$ of $(X,\tau)$ is said to be $\alpha_{[\gamma,\gamma']}$-closed if its complement $X \setminus F$ is $\alpha_{[\gamma,\gamma']}$-open. The intersection of all $\alpha_{[\gamma,\gamma']}$-closed sets containing $A$ is called the $\alpha_{[\gamma,\gamma']}$-closure of $A$ and denoted by $\alpha_{[\gamma,\gamma']}$-$Cl(A)$. The union of all $\alpha_{[\gamma,\gamma']}$-open sets contained in $A$ is called the $\alpha_{[\gamma,\gamma']}$-interior of $A$ and denoted by $\alpha_{[\gamma,\gamma']}$-$Int(A)$. A point $x \in X$ is in $\alpha Cl_{[\gamma,\gamma']}(A)$ [1; Definition 3.33], if $(U^{\gamma} \cap W^{\gamma^{'}}) \cap A \neq \phi$ for each $\alpha$-open sets $U$ and $W$ containing $x$.

**Proposition 2.1** ([1; Proposition 3.45]) *Let $A$ be any subset of a topological space $(X,\tau)$. Then, $X \setminus \alpha_{[\gamma,\gamma']}$-$Int(A) = \alpha_{[\gamma,\gamma']}$-$Cl(X \setminus A)$.*

**Theorem 2.2** ([1; Theorem 3.38]) *If $\gamma$ and $\gamma^{'}$ are $\alpha$-open operations and $A$ a subset of $(X,\tau)$. Then, we have $\alpha Cl_{[\gamma,\gamma']}(\alpha Cl_{[\gamma,\gamma']}(A)) = \alpha Cl_{[\gamma,\gamma']}(A)$.*

**Proposition 2.3** ([1; Proposition 3.14]) *Let $A$ be any subset of a topological space $(X,\tau)$. If $A$ is $[\gamma,\gamma^{'}]$-open [5], then $A$ is $\alpha_{[\gamma,\gamma']}$-open.*

**Remark 2.4** ([1; Remark 3.5]) *If $\gamma$ and $\gamma^{'}$ are $\alpha$-regular operations, then $\alpha O(X,\tau)_{[\gamma,\gamma']}$ forms a topology on $X$.*

**Proposition 2.5** ([1; Proposition 3.17]) *Let $A$ and $B$ be any subsets of a topological space $(X,\tau)$. If $A$ is $\alpha_{\gamma}$-open and $B$ is $\alpha_{\gamma'}$-open, then $A \cap B$ is $\alpha_{[\gamma,\gamma']}$-open.*

**Definition 2.6** ([1; Definition 4.1]) *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous if for each point $x \in X$ and each $\alpha$-open sets $W$ and $S$ of $(Y,\sigma)$ containing $f(x)$ there exist $\alpha$-open sets $U$ and $V$ of $(X,\tau)$ containing $x$ such that $f(U^{\gamma} \cap V^{\gamma^{'}}) \subseteq W^{\beta} \cap S^{\beta'}$.*

**Definition 2.7** ([1; Definition 4.11]) *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-closed if for each $\alpha_{[\gamma,\gamma']}$-closed set $A$ of $(X,\tau)$, $f(A)$ is $\alpha_{[\beta,\beta']}$-closed in $(Y,\sigma)$.*

## 3 $\alpha_{[\gamma,\gamma']}$-g.Closed Sets

In this section, we define and study some properties of $\alpha_{[\gamma,\gamma']}$-g.closed sets.

**Definition 3.1** *A subset $A$ of $(X,\tau)$ is said to be an $\alpha_{[\gamma,\gamma']}$-generalized closed (briefly, $\alpha_{[\gamma,\gamma']}$-g.closed) set if $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an $\alpha_{[\gamma,\gamma']}$-open set in $(X,\tau)$.*

**Remark 3.2** *It is clear that every $\alpha_{[\gamma,\gamma']}$-closed set is $\alpha_{[\gamma,\gamma']}$-g.closed. But the converse is not true in general as it is shown in the following example.*

**Example 3.3** *Let $X = \{a,b,c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\}$. For each $A \in \alpha O(X,\tau)$, we define two operations $\gamma$ and $\gamma^{'}$, respectively, by $A^{\gamma} = A^{\gamma^{'}} = A$ if $A \in \boldsymbol{B}$; $A^{\gamma} = A^{\gamma^{'}} = X$ if $A \notin \boldsymbol{B}$, where $\boldsymbol{B} = \{\{b\}, \{a,c\}\}$. Now, if we let $A = \{a\}$, since the only $\alpha_{[\gamma,\gamma']}$-open supersets of $A$ are $\{a,c\}$ and $X$, then $A$ is $\alpha_{[\gamma,\gamma']}$-g.closed. But it is easy to see that $A$ is not $\alpha_{[\gamma,\gamma']}$-closed.*

**Proposition 3.4** *If $A$ is $\gamma$-open and $\alpha_{[\gamma,\gamma']}$-g.closed then $A$ is $\alpha_{[\gamma,\gamma']}$-closed.*

Proof. As every $\gamma$-open set is $\alpha_{[\gamma,\gamma']}$-open and $A \subseteq A$, we have $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq A$, also $A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A)$, therefore $\alpha_{[\gamma,\gamma']}$-$Cl(A) = A$. That is, $A$ is $\alpha_{[\gamma,\gamma']}$-closed.

**Remark 3.5** *If $A$ is $\alpha_{[\gamma,\gamma']}$-open and $\alpha_{[\gamma,\gamma']}$-g.closed then $A$ is $\alpha_{[\gamma,\gamma']}$-closed.*

**Proposition 3.6** *The intersection of an $\alpha_{[\gamma,\gamma']}$-g.closed set and an $\alpha_{[\gamma,\gamma']}$-closed set is always $\alpha_{[\gamma,\gamma']}$-g.closed.*

Proof. Let $A$ be an $\alpha_{[\gamma,\gamma']}$-g.closed set and $F$ be an $\alpha_{[\gamma,\gamma']}$-closed set. Assume that $U$ is an $\alpha_{[\gamma,\gamma']}$-open set such that $A \cap F \subseteq U$. Set $G = X \setminus F$. Then we have $A \subseteq U \cup G$, since $G$ is $\alpha_{[\gamma,\gamma']}$-open, then $U \cup G$ is $\alpha_{[\gamma,\gamma']}$-open and since $A$ is $\alpha_{[\gamma,\gamma']}$-g.closed, then $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U \cup G$. Now, $\alpha_{[\gamma,\gamma']}$-$Cl(A \cap F) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A) \cap \alpha_{[\gamma,\gamma']}$-$Cl(F) = \alpha_{[\gamma,\gamma']}$-$Cl(A) \cap F \subseteq (U \cup G) \cap F = (U \cap F) \cup \phi \subseteq U$ (cf. [1; Proposition 3.32]).

**Remark 3.7** *The intersection of two $\alpha_{[\gamma,\gamma']}$-g.closed sets need not be $\alpha_{[\gamma,\gamma']}$-g.closed in general. It is shown by the following example.*

**Example 3.8** *Let $X = \{a,b,c\}$ and $\tau$ be a discrete topology on $X$. For each $A \in \alpha O(X,\tau)$ we define two operations $\gamma$ and $\gamma^{'}$, respectively, by $A^{\gamma} = A^{\gamma^{'}} = A$ if $A = \{a\}$; $A^{\gamma} = A^{\gamma^{'}} = X$ if $A \neq \{a\}$. Set $A = \{a,b\}$ and $B = \{a,c\}$. Clearly, $A$ and $B$ are $\alpha_{[\gamma,\gamma']}$-g.closed sets, since $X$ is their only $\alpha_{[\gamma,\gamma']}$-open superset. But $C = \{a\} = A \cap B$ is not $\alpha_{[\gamma,\gamma']}$-g.closed, since $C \subseteq \{a\} \in \alpha O(X,\tau)_{[\gamma,\gamma']}$ and $\alpha_{[\gamma,\gamma']}$-$Cl(C) = X \nsubseteq \{a\}$.*

**Proposition 3.9** *If $\gamma$ ana $\gamma^{'}$ are $\alpha$-regular operations on $\alpha O(X,\tau)$. Then the finite union of $\alpha_{[\gamma,\gamma']}$-g.closed sets is always an $\alpha_{[\gamma,\gamma']}$-g.closed set.*

Proof. Let $A$ and $B$ be two $\alpha_{[\gamma,\gamma']}$-g.closed sets, and let $A \cup B \subseteq U$, where $U$ is $\alpha_{[\gamma,\gamma']}$-open. Since $A$ and $B$ are $\alpha_{[\gamma,\gamma']}$-g.closed sets, we have $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U$ and $\alpha_{[\gamma,\gamma']}$-$Cl(B) \subseteq U$; and so $\alpha_{[\gamma,\gamma']}$-$Cl(A) \cup \alpha_{[\gamma,\gamma']}$-$Cl(B) \subseteq U$. But, we have $\alpha_{[\gamma,\gamma']}$-$Cl(A) \cup \alpha_{[\gamma,\gamma']}$-$Cl(B) = \alpha_{[\gamma,\gamma']}$-$Cl(A \cup B)$ (cf. [1; Proposition 3.32]). Therefore $\alpha_{[\gamma,\gamma']}$-$Cl(A \cup B) \subseteq U$ and so $A \cup B$ is an $\alpha_{[\gamma,\gamma']}$-g.closed set.

**Remark 3.10** *The union of two $\alpha_{[\gamma,\gamma']}$-g.closed sets need not be $\alpha_{[\gamma,\gamma']}$-g.closed in general. It is shown by the following example.*

**Example 3.11** *Let $X = \{a, b, c\}$ and $\tau$ be a discrete topology on $X$. For each $A \in \alpha O(X, \tau)$ we define two operations $\gamma$ and $\gamma'$, respectively, by $A^\gamma = A$ if $A \in \boldsymbol{B}$; $A^\gamma = X$ if $A \notin \boldsymbol{B}$, where $\boldsymbol{B} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$; and $A^{\gamma'} = X$. Let $A = \{a\}$ and $B = \{b\}$. Here $A$ and $B$ are $\alpha_{[\gamma,\gamma']}$-g.closed but $A \cup B = \{a, b\}$ is not $\alpha_{[\gamma,\gamma']}$-g.closed, since $\{a, b\}$ is $\alpha_{[\gamma,\gamma']}$-open and $\alpha_{[\gamma,\gamma']}$-$Cl(\{a, b\}) = X$.*

**Proposition 3.12** *If a subset $A$ of $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-g.closed and $A \subseteq B \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A)$, then $B$ is an $\alpha_{[\gamma,\gamma']}$-g.closed set in $(X, \tau)$.*

Proof. Let $U$ be an $\alpha_{[\gamma,\gamma']}$-open set of $(X, \tau)$ such that $B \subseteq U$. Since A is $\alpha_{[\gamma,\gamma']}$-g.closed, we have $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U$. Now, by [1; Proposition 3.32] and assumptions, it is shown that $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(B) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl[\alpha_{[\gamma,\gamma']}$-$Cl(A)] = \alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq U$; and so $\alpha_{[\gamma,\gamma']}$-$Cl(B) \subseteq U$. Therefore, $B$ is an $\alpha_{[\gamma,\gamma']}$-g.closed set of $(X, \tau)$.

**Proposition 3.13** *For each $x \in X$, $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed or $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-g.closed in $(X, \tau)$.*

Proof. Suppose that $\{x\}$ is not $\alpha_{[\gamma,\gamma']}$-closed. Then $X \setminus \{x\}$ is not $\alpha_{[\gamma,\gamma']}$-open. Let $U$ be any $\alpha_{[\gamma,\gamma']}$-open set such that $X \setminus \{x\} \subseteq U$. Then this implies $U = X$; and so $\alpha_{[\gamma,\gamma']}$-$Cl(X \setminus \{x\}) \subseteq U$. Hence $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-g.closed.

**Proposition 3.14** *The following statements (1), (2) and (3) are equivalent for a subset $A$ of $(X, \tau)$.*

1. *$A$ is $\alpha_{[\gamma,\gamma']}$-g.closed in $(X, \tau)$.*
2. *$\alpha_{[\gamma,\gamma']}$-$Cl(\{x\}) \cap A \neq \phi$ holds for every $x \in \alpha_{[\gamma,\gamma']}$-$Cl(A)$.*
3. *$\alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A$ does not contain any non-empty $\alpha_{[\gamma,\gamma']}$-closed set.*

Proof. (1) $\Rightarrow$ (2). Suppose that there exists a point $x \in \alpha_{[\gamma,\gamma']}$-$Cl(A)$ such that $\alpha_{[\gamma,\gamma']}$-$Cl(\{x\}) \cap A = \phi$. Since $\alpha_{[\gamma,\gamma']}$-$Cl(\{x\})$ is $\alpha_{[\gamma,\gamma']}$-closed (cf. [1; Proposition 3.32]), $X \setminus \alpha_{[\gamma,\gamma']}$-$Cl(\{x\})$ is an $\alpha_{[\gamma,\gamma']}$-open set of $(X, \tau)$. Since $A \subseteq X \setminus (\alpha_{[\gamma,\gamma']}$-$Cl(\{x\}))$ and $A$ is $\alpha_{[\gamma,\gamma']}$-g.closed, this implies $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq X \setminus \alpha_{[\gamma,\gamma']}$-$Cl(\{x\})$ holds; and hence $x \notin \alpha_{[\gamma,\gamma']}$-$Cl(A)$. This is a contradiction.

Therefore, we conclude that $\alpha_{[\gamma,\gamma']}$-$Cl(\{x\}) \cap A \neq \phi$ holds for every $x \in \alpha_{[\gamma,\gamma']}$-$Cl(A)$.
(2) $\Rightarrow$ (3). Suppose that there exists a non-empty $\alpha_{[\gamma,\gamma']}$-closed set $F$ such that $F \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A$; and so $A \cap F = \phi$. Let $y \in F$. Then, $y \in \alpha_{[\gamma,\gamma']}$-$Cl(A)$ and $y \notin A$. By (2), it is obtained that $\phi \neq \alpha_{[\gamma,\gamma']}$-$Cl(\{y\}) \cap A \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(F) \cap A = F \cap A$; and so $F \cap A \neq \phi$. This is a contradiction; and so (3) is claimed.
(3) $\Rightarrow$ (1). Let $A \subseteq U$, where $U$ is $\alpha_{[\gamma,\gamma']}$-open in $(X,\tau)$. If $\alpha_{[\gamma,\gamma']}$-$Cl(A)$ is not contained in $U$, then $\alpha_{[\gamma,\gamma']}$-$Cl(A) \cap (X \setminus U) \neq \phi$. Now, since $\alpha_{[\gamma,\gamma']}$-$Cl(A) \cap (X \setminus U) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A$ and $\alpha_{[\gamma,\gamma']}$-$Cl(A) \cap (X \setminus U)$ is a non-empty $\alpha_{[\gamma,\gamma']}$-closed set, we obtain a contradiction and therefore $A$ is $\alpha_{[\gamma,\gamma']}$-g.closed.

**Proposition 3.15** *If $A$ is an $\alpha_{[\gamma,\gamma']}$-g.closed set of a space $X$, then the following are equivalent:*

1. *$A$ is $\alpha_{[\gamma,\gamma']}$-closed.*
2. *$\alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A$ is $\alpha_{[\gamma,\gamma']}$-closed.*

Proof. (1) $\Rightarrow$ (2). Since $A$ is $\alpha_{[\gamma,\gamma']}$-closed, then $\alpha_{[\gamma,\gamma']}$-$Cl(A) = A$ holds (cf. [1; Proposition 3.32 (3)]); and so $\alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A = \phi$ and the set $\phi$ is $\alpha_{[\gamma,\gamma']}$-closed.
(2) $\Rightarrow$ (1). Since $A$ is $\alpha_{[\gamma,\gamma']}$-g.closed, $\alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A$ does not contain any non-empty $\alpha_{[\gamma,\gamma']}$-closed subset (cf. Proposition 3.14); and so $\alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A = \phi$. This shows that $A$ is $\alpha_{[\gamma,\gamma']}$-closed.

**Proposition 3.16** *For a space $(X,\tau)$, the following are equivalent:*

1. *Every subset of $X$ is $\alpha_{[\gamma,\gamma']}$-g.closed.*
2. *$\alpha O(X,\tau)_{[\gamma,\gamma']} = \alpha C(X,\tau)_{[\gamma,\gamma']}$.*

Proof. (1) $\Rightarrow$ (2). Let $U \in \alpha O(X,\tau)_{[\gamma,\gamma']}$. Then, by hypothesis, $U$ is $\alpha_{[\gamma,\gamma']}$-g.closed which implies that $\alpha_{[\gamma,\gamma']}$-$Cl(U) \subseteq U$, so, $\alpha_{[\gamma,\gamma']}$-$Cl(U) = U$. Thus, we have $U \in \alpha C(X,\tau)_{[\gamma,\gamma']}$; and so $\alpha O(X,\tau)_{[\gamma,\gamma']} \subseteq \alpha C(X,\tau)_{[\gamma,\gamma']}$. Conversely, let $V \in \alpha C(X,\tau)_{[\gamma,\gamma']}$. Then, $X \setminus V \in \alpha O(X,\tau)_{[\gamma,\gamma']}$. By using the above result, it is shown that $V \in \alpha O(X,\tau)_{[\gamma,\gamma']}$; and so $\alpha C(X,\tau)_{[\gamma,\gamma']} \subseteq \alpha O(X,\tau)_{[\gamma,\gamma']}$. Therefore, we have the proof of (2).
(2) $\Rightarrow$ (1). If $A$ is a subset of a space $(X,\tau)$ such that $A \subseteq U$ where $U \in \alpha O(X,\tau)_{[\gamma,\gamma']}$, then $U \in \alpha C(X,\tau)_{[\gamma,\gamma']}$. Therefore $\alpha_{[\gamma,\gamma']}$-$Cl(A) \subseteq \alpha_{[\gamma,\gamma']}$-$Cl(U) = U$ which shows that $A$ is $\alpha_{[\gamma,\gamma']}$-g.closed.

**Definition 3.17** *A subset $A$ of $X$ is $\alpha_{[\gamma,\gamma']}$-g.open if its complement $X \setminus A$ is $\alpha_{[\gamma,\gamma']}$-g.closed in $(X,\tau)$.*

**Remark 3.18** *It is clear that every $\alpha_{[\gamma,\gamma']}$-open set is $\alpha_{[\gamma,\gamma']}$-g.open. But the converse is not true in general as it is shown in the following example.*

**Example 3.19** *Consider Example 3.3, if* $A = \{b, c\}$ *then* $A$ *is* $\alpha_{[\gamma,\gamma']}$*-g.open but not* $\alpha_{[\gamma,\gamma']}$*-open.*

**Proposition 3.20** *A subset* $A$ *of* $(X, \tau)$ *is* $\alpha_{[\gamma,\gamma']}$*-g.open if and only if* $F \subseteq \alpha_{[\gamma,\gamma']}$*-Int*$(A)$ *whenever* $F \subseteq A$ *and* $F$ *is* $\alpha_{[\gamma,\gamma']}$*-closed in* $(X, \tau)$.

Proof. By Definition 3.17 and Proposition 2.1, the proof is obtained.

**Remark 3.21** *The union of two* $\alpha_{[\gamma,\gamma']}$*-g.open sets need not be* $\alpha_{[\gamma,\gamma']}$*-g.open in general. It is shown by the following example.*

**Example 3.22** *Consider Example 3.8, if* $A = \{b\}$ *and* $B = \{c\}$ *then* $A$ *and* $B$ *are* $\alpha_{[\gamma,\gamma']}$*-g.open sets in* $X$*, but* $A \cup B = \{b, c\}$ *is not an* $\alpha_{[\gamma,\gamma']}$*-g.open set in* $X$.

**Proposition 3.23** *Let* $\gamma$ *and* $\gamma^{'}$ *be an* $\alpha$*-regular operations on* $\alpha O(X, \tau)$*, and let* $A$ *and* $B$ *be two* $\alpha_{[\gamma,\gamma']}$*-g.open sets in a space* $(X, \tau)$*. Then* $A \cap B$ *is also* $\alpha_{[\gamma,\gamma']}$*-g.open.*

Proof. By Definition 3.17 and Proposition 3.9, it is proved.

**Proposition 3.24** *Every singleton point set in a space* $(X, \tau)$ *is either* $\alpha_{[\gamma,\gamma']}$*-g.open or* $\alpha_{[\gamma,\gamma']}$*-closed.*

Proof. By Definition 3.17 and Proposition 3.13, it is proved.

**Proposition 3.25** *If* $\alpha_{[\gamma,\gamma']}$*-Int*$(A) \subseteq B \subseteq A$ *and* $A$ *is* $\alpha_{[\gamma,\gamma']}$*-g.open, then* $B$ *is* $\alpha_{[\gamma,\gamma']}$*-g.open.*

Proof. By Definition 3.17 and Propositions 2.1, 3.12, the proof is obtained.

# 4 $\alpha_{[\gamma,\gamma']}$-Separations Spaces

In this section we introduce $\alpha_{[\gamma,\gamma']}$-$T_i$ spaces ($i = 0, \frac{1}{2}, 1, 2$) and investigate relations among these spaces.

**Definition 4.1** *A topological space* $(X, \tau)$ *is said to be* $\alpha_{[\gamma,\gamma']}$*-*$T_{\frac{1}{2}}$ *if every* $\alpha_{[\gamma,\gamma']}$*-g.closed set is* $\alpha_{[\gamma,\gamma']}$*-closed.*

**Remark 4.2** *It follows from Remark 3.2 that* $(X, \tau)$ *is* $\alpha_{[\gamma,\gamma']}$*-*$T_{\frac{1}{2}}$ *if and only if the* $\alpha_{[\gamma,\gamma']}$*-g.closedness coincides with the* $\alpha_{[\gamma,\gamma']}$*-closedness.*

**Definition 4.3** *A topological space* $(X, \tau)$ *is said to be* $\alpha_{[\gamma,\gamma']}$*-*$T_0$ *if for each pair of distinct points* $x, y$ *in* $X$*, there exist* $\alpha$*-open sets* $U$ *and* $V$ *such that* $x \in U \cap V$ *and* $y \notin U^{\gamma} \cap V^{\gamma^{'}}$*, or* $y \in U \cap V$ *and* $x \notin U^{\gamma} \cap V^{\gamma^{'}}$.

**Definition 4.4** *A topological space $(X, \tau)$ is said to be $\alpha_{[\gamma,\gamma']}$-$T_1$ if for each pair of distinct points $x, y$ in $X$, there exist $\alpha$-open sets $U$ and $V$ containing $x$ and $\alpha$-open sets $W$ and $S$ containing $y$ such that $y \notin U^{\gamma} \cap V^{\gamma^{'}}$ and $x \notin W^{\gamma} \cap S^{\gamma^{'}}$.*

**Definition 4.5** *A topological space $(X, \tau)$ is said to be $\alpha_{[\gamma,\gamma']}$-$T_2$ if for each pair of distinct points $x, y$ in $X$, there exist $\alpha$-open sets $U$ and $V$ containing $x$ and $\alpha$-open sets $W$ and $S$ containing $y$ such that $(U^{\gamma} \cap V^{\gamma^{'}}) \cap (W^{\gamma} \cap S^{\gamma^{'}}) = \phi$.*

**Remark 4.6** *For given two distinct points $x$ and $y$, the $\alpha_{[\gamma,\gamma']}$-$T_0$-axiom requires that there exist $\alpha$-open sets $U$, $V$, $W$ and $S$ satisfying one of conditions $(1), (2), (3)$ and $(4)$:*

1. $x \in U \cap V$, $y \in W \cap S$, $y \notin U^{\gamma} \cap V^{\gamma^{'}}$ *and* $x \notin W^{\gamma} \cap S^{\gamma^{'}}$.
2. $x \in U \cap V$, $x \in W \cap S$, $y \notin U^{\gamma} \cap V^{\gamma^{'}}$ *and* $y \notin W^{\gamma} \cap S^{\gamma^{'}}$.
3. $y \in U \cap V$, $y \in W \cap S$, $x \notin U^{\gamma} \cap V^{\gamma^{'}}$ *and* $x \notin W^{\gamma} \cap S^{\gamma^{'}}$.
4. $y \in U \cap V$, $x \in W \cap S$, $x \notin U^{\gamma} \cap V^{\gamma^{'}}$ *and* $y \notin W^{\gamma} \cap S^{\gamma^{'}}$.

**Remark 4.7** *(1) A topological space $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_0$ if and only if (2) for each distinct points $x, y$ in $X$, there exists an $\alpha$-open set $W$ such that $x \in W$ and $y \notin W^{\gamma} \cap W^{\gamma^{'}}$, or $y \in W$ and $x \notin W^{\gamma} \cap W^{\gamma^{'}}$.*

**Proposition 4.8** *A topological space $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$ if and only if for each $x \in X$, $\{x\}$ is either $\alpha_{[\gamma,\gamma']}$-closed or $\alpha_{[\gamma,\gamma']}$-open.*

Proof. (Necessity) Suppose $\{x\}$ is not $\alpha_{[\gamma,\gamma']}$-closed. Then by Proposition 3.13, $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-g.closed. Since $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$, $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed, that is $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-open.
(Sufficiency) Let $A$ be any $\alpha_{[\gamma,\gamma']}$-g.closed set in $(X, \tau)$ and $x \in \alpha_{[\gamma,\gamma']}$-$Cl(A)$. It suffices to prove that $x \in A$ for the following two cases:

Case 1. $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed: for this case, by Proposition 3.14, it is shown that $\{x\} \not\subseteq \alpha_{[\gamma,\gamma']}$-$Cl(A) \setminus A$; and so $x \in A$.

Case 2. $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-open: for this case, we have that $\{x\} \cap A \neq \phi$ (cf. [1; Proposition 3.31]); and so $x \in A$.

Hence, $A$ is $\alpha_{[\gamma,\gamma']}$-closed; and so $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$.

**Proposition 4.9** *Let $\gamma$ and $\gamma^{'}$ be $\alpha$-open operations. Then, a topological space $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_0$ if and only if for each pair of distinct points $x, y$ of $X$, $\alpha Cl_{[\gamma,\gamma']}(\{x\}) \neq \alpha Cl_{[\gamma,\gamma']}(\{y\})$.*

Proof. (Necessity) Let $x, y$ be any two distinct points of $X$. Then, there exist $\alpha$-open sets $U$ and $V$ such that $x \in U \cap V$ and $y \notin U^{\gamma} \cap V^{\gamma^{'}}$, or $x \notin U^{\gamma} \cap V^{\gamma^{'}}$ and $y \in U \cap V$. For the first case, we have that $(U^{\gamma} \cap V^{\gamma^{'}}) \cap \{y\} = \phi$ and this implies that $x \notin \alpha Cl_{[\gamma,\gamma']}(\{y\})$. For the final case, we have that $y \notin \alpha Cl_{[\gamma,\gamma']}(\{x\})$.

Consequently, for the distinct points $x$ and $y$, $\alpha Cl_{[\gamma,\gamma']}(\{x\}) \neq \alpha Cl_{[\gamma,\gamma']}(\{y\})$.
(Sufficiency) By hypothesis, there exists a point $z \in X$ satisfying the following two cases:

Case 1. $z \in \alpha Cl_{[\gamma,\gamma']}(\{x\})$ but $z \notin \alpha Cl_{[\gamma,\gamma']}(\{y\})$: we claim that $x \notin \alpha Cl_{[\gamma,\gamma']}(\{y\})$. Indeed, suppose that $x \in \alpha Cl_{[\gamma,\gamma']}(\{y\})$ holds; then, by Theorem 2.2, it is shown that $\alpha Cl_{[\gamma,\gamma']}(\{x\}) \subseteq \alpha Cl_{[\gamma,\gamma']}(\{y\})$; this contradicts the fact that $z \notin \alpha Cl_{[\gamma,\gamma']}(\{y\})$. Thus, $x \notin \alpha Cl_{[\gamma,\gamma']}(\{y\})$; and so there exist $\alpha$-open sets $U$ and $V$ such that $x \in U \cap V$ and $(U^{\gamma} \cap V^{\gamma'}) \cap \{y\} = \phi$.

Case 2. $z \in \alpha Cl_{[\gamma,\gamma']}(\{y\})$ but $z \notin \alpha Cl_{[\gamma,\gamma']}(\{x\})$: for this case, by similar way to Case 1 above, it is shown that $y \notin \alpha Cl_{[\gamma,\gamma']}(\{x\})$; and so there exist $U'$ and $V'$ such that $y \in U' \cap V'$ and $(U'^{\gamma} \cap V'^{\gamma'}) \cap \{x\} = \phi$.

Therefore, by Case 1 and Case 2, it is proved that $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_0$.

**Proposition 4.10** *A topological space $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_1$ if and only if for each $x \in X$, $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed.*

Proof. (Necessity) Let $x$ be a point of $X$. Suppose $y \in X \setminus \{x\}$. Then, there exist $\alpha$-open sets $W$ and $S$ containing $y$ and $x \notin W^{\gamma} \cap S^{\gamma'}$. Consequently $y \in W^{\gamma} \cap S^{\gamma'} \subseteq X \setminus \{x\}$, that is $X \setminus \{x\}$ is $\alpha_{[\gamma,\gamma']}$-open.
(Sufficiency) Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X \setminus \{x\}$ and $x \in X \setminus \{y\}$. Hence $X \setminus \{y\}$ is an $\alpha_{[\gamma,\gamma']}$-open set containing $x$, so there exist $\alpha$-open sets $U$ and $V$ containing $x$ such that $U^{\gamma} \cap V^{\gamma'} \subseteq X \setminus \{y\}$. Similarly $X \setminus \{x\}$ is an $\alpha_{[\gamma,\gamma']}$-open set containing $y$, so there exist $\alpha$-open sets $W$ and $S$ containing $y$ such that $W^{\gamma} \cap S^{\gamma'} \subseteq X \setminus \{x\}$. Accordingly $X$ is an $\alpha_{[\gamma,\gamma']}$-$T_1$ space.

**Proposition 4.11** *The following statements are equivalent for a topological space $(X,\tau)$ with an operations $\gamma$ and $\gamma'$ on $\alpha O(X,\tau)$.*

1. *$(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_2$.*
2. *Let $x \in X$. For each $y \neq x$, there exist $\alpha$-open sets $U$ and $V$ containing $x$ such that $y \notin \alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma'})$.*
3. *For each $x \in X$, $\cap\{\alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma'}) : U, V \in \alpha O(X,\tau)$ and $x \in U \cap V\} = \{x\}$.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$. For each $y \neq x$, it follows from (1) that there exist $\alpha$-open sets $U$ and $V$ containing $x$ and $\alpha$-open sets $W$ and $S$ containing $y$ such that $(U^{\gamma} \cap V^{\gamma'}) \cap (W^{\gamma} \cap S^{\gamma'}) = \phi$. This implies that $y \notin \alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma'})$.
(2) $\Rightarrow$ (3). Set $B(z) = \cap\{\alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma'}) : U, V \in \alpha O(X,\tau)$ and $z \in U \cap V\}$, where $z \in X$. Let $x \in X$. We claim that $B(x) = \{x\}$. Indeed, $y$ be any point of $X$ with $x \neq y$. It follows from (2) that there exist $\alpha$-open sets $U$ and $V$ such

that $x \in U \cap V$ and $y \notin \alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma'})$. Thus, we have that $y \notin B(x)$ and so $\{x\} = B(x)$, because $\{x\} \subseteq B(x) \subseteq \alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma'})$ hold.
(3) $\Rightarrow$ (1). Let $x, y \in X$ with $x \neq y$. By (3), it is assumed that $B(x) = \{x\}$ where $B(x)$ is defined in the proof of (2) $\Rightarrow$ (3) above. Then, there exist $\alpha$-open sets $U$ and $V$ such that $y \notin \alpha Cl_{[\gamma,\gamma']}(U^{\gamma} \cap V^{\gamma'})$; and hence $(U^{\gamma} \cap V^{\gamma'}) \cap (W^{\gamma} \cap S^{\gamma'}) = \phi$ for some $\alpha$-open sets $W$ and $S$ containing $y$. Therefore, $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_2$.

**Proposition 4.12** (i) *If $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_2$, then it is $\alpha_{[\gamma,\gamma']}$-$T_1$.*
(ii) *If $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_1$, then it is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$.*
(iii) *If $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$, then it is $\alpha_{[\gamma,\gamma']}$-$T_0$.*

Proof. (i) The proof is straightforward from Definitions 4.4 and 4.5.
(ii) The proof is obvious by Propositions 4.8 and 4.10.
(iii) Let $x$ and $y$ be any two distinct points of $(X, \tau)$. By Proposition 4.8, the singleton $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed or $\alpha_{[\gamma,\gamma']}$-open.
Case 1. $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed: for this case, $X \setminus \{x\}$ is an $\alpha_{[\gamma,\gamma']}$-open set containing $y$; and so there exist $\alpha$-open sets $W$ and $S$ containing $y$ such that $W^{\gamma} \cap S^{\gamma'} \subseteq X \setminus \{x\}$. Thus we have that $y \in W \cap S$ and $x \notin W^{\gamma} \cap S^{\gamma'}$.
Case 2. $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-open: for this case, there exist $\alpha$-open sets $U$ and $V$ containing $x$ such that $U^{\gamma} \cap V^{\gamma'} \subseteq \{x\}$. This implies that $x \in U \cap V$ and $y \notin U^{\gamma} \cap V^{\gamma'}$.
Therefore, we have $X$ is $\alpha_{[\gamma,\gamma']}$-$T_0$.

**Remark 4.13** *The following series of examples show that all converses of Proposition 4.12 can not be reserved.*

**Example 4.14** *Let $(X, \tau)$, $\gamma$ and $\gamma'$ be the same space and the same operations as in Example 3.11. Then, it is shown directly that each singleton is $\alpha_{[\gamma,\gamma']}$-closed in $(X, \tau)$. By Proposition 4.10, $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_1$. But, we can show that $(U^{\gamma} \cap V^{\gamma'}) \cap (W^{\gamma} \cap S^{\gamma'}) \neq \phi$ holds for any $\alpha$-open sets $U$, $V$, $W$ and $S$. This implies $(X, \tau)$ is not $\alpha_{[\gamma,\gamma']}$-$T_2$*

**Example 4.15** *Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ be a topology on $X$. For each $A \in \alpha O(X, \tau)$ we define two operations $\gamma$ and $\gamma'$, respectively, by $A^{\gamma} = A^{\gamma'} = A$. Then, it is shown directly that each singleton is $\alpha_{[\gamma,\gamma']}$-closed or $\alpha_{[\gamma,\gamma']}$-open in $(X, \tau)$. By Proposition 4.8, $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$. However, by Proposition 4.10, $(X, \tau)$ is not $\alpha_{[\gamma,\gamma']}$-$T_1$, in fact, a singleton $\{a\}$ is not $\alpha_{[\gamma,\gamma']}$-closed.*

**Example 4.16** *Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ be a topology on $X$. For each $A \in \alpha O(X, \tau)$ we define two operations $\gamma$ and $\gamma'$, respectively, by $A^{\gamma} = A^{\gamma'} = A$ if $b \notin A$; $A^{\gamma} = A^{\gamma'} = X$ if $b \in A$. Then, $(X, \tau)$ is not $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$ because a singleton $\{c\}$ is neither $\alpha_{[\gamma,\gamma']}$-open nor $\alpha_{[\gamma,\gamma']}$-closed. It is shown directly that $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_0$.*

**Remark 4.17** *From Proposition 4.12 and Examples 4.14, 4.15 and 4.16, the following implications hold and none of the implications is reversible:*

$$\alpha_{[\gamma,\gamma']}\text{-}T_2 \longrightarrow \alpha_{[\gamma,\gamma']}\text{-}T_1 \longrightarrow \alpha_{[\gamma,\gamma']}\text{-}T_{\frac{1}{2}} \longrightarrow \alpha_{[\gamma,\gamma']}\text{-}T_0,$$

where $A \to B$ represents that $A$ implies $B$.

**Proposition 4.18** *If $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_i$, then it is $\alpha$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.*

Proof. The proofs for $i = 0, 2$ follow from definitions. The proof for $i = 1$ (resp. $i = \frac{1}{2}$) follows from [1; Proposition 3.7] and Proposition 4.10 (resp. Proposition 4.8).

**Remark 4.19** *The following example shows that all converses of Proposition 4.18 can not be reserved.*

**Example 4.20** *Let $X = \{a, b, c\}$ and $\tau$ be a discrete topology on $X$. For each $A \in \alpha O(X,\tau)$ we define two operations $\gamma$ and $\gamma^{'}$, respectively, by $A^{\gamma} = A^{\gamma^{'}} = X$. Then, $(X,\tau)$ is $\alpha$-$T_i$ but it is not $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.*

**Proposition 4.21** *If $(X,\tau)$ is $\alpha_{\gamma}$-$T_i$, then it is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.*

Proof. The proofs for $i = 0, 1, 2$ follow from Definitions 4.3, 4.4, 4.5 and [6; Definition 3.6]. The proof for $i = \frac{1}{2}$ is obtained as follows: Let $x \in X$. Then,$\{x\}$ is $\alpha_{\gamma}$-open or $\alpha_{\gamma}$-closed by [6; Theorem 3.2]. So, $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-open or $\alpha_{[\gamma,\gamma']}$-closed because every $\alpha_{\gamma}$-open is $\alpha_{[\gamma,\gamma']}$-open (cf. [1; Proposition 3.18]). The proof is completed from Proposition 4.8.

**Remark 4.22** *The following series of examples show that all converses of Proposition 4.21 can not be reserved.*

**Example 4.23** *Let $X = \{a, b, c\}$ and $\tau$ be a discrete topology on $X$.*

*(i) For each $A \in \alpha O(X,\tau)$ we define two operations $\gamma$ and $\gamma^{'}$, respectively, by $A^{\gamma} = A^{\gamma^{'}} = A$ if $A \in \boldsymbol{B}$; $A^{\gamma} = A^{\gamma^{'}} = X$ if $A \notin \boldsymbol{B}$, where $\boldsymbol{B} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. Then, $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_2$ but not $\alpha_{\gamma}$-$T_2$.*

*(ii) For each $A \in \alpha O(X,\tau)$ we define two operations $\gamma$ and $\gamma^{'}$, respectively, by $A^{\gamma} = A$ if $A \in \boldsymbol{B}$; $A^{\gamma} = X$ if $A \notin \boldsymbol{B}$, where $\boldsymbol{B} = \{\{a, b\}, \{a, c\}\}$; and $A^{\gamma^{'}} = A$ if $A = \{b, c\}$; $A^{\gamma^{'}} = X$ if $A \neq \{b, c\}$. Then, $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_i$ but not $\alpha_{\gamma}$-$T_i$, where $i = \frac{1}{2}, 1$.*

*(iii) For each $A \in \alpha O(X,\tau)$ we define two operations $\gamma$ and $\gamma^{'}$, respectively, by $A^{\gamma} = A$ if $A = \{a\}$; $A^{\gamma} = X$ if $A \neq \{a\}$; and $A^{\gamma^{'}} = A$ if $A = \{b\}$; $A^{\gamma^{'}} = X$ if $A \neq \{b\}$. Then, $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_0$ but not $\alpha_{\gamma}$-$T_0$.*

**Proposition 4.24** *If $(X,\tau)$ is $[\gamma,\gamma^{'}]$-$T_i$, then it is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.*

Proof. The proofs for $i = 0, 2$ follow from Proposition 2.3, Definitions 4.3, 4.5 and [5; Definitions 5.2, 5.4]. The proof for $i = 1$ (resp. $i = \frac{1}{2}$) follows from [5; Proposition 5.8] (resp. [5; Proposition 5.7]) and Proposition 2.3.

**Remark 4.25** *The following example show that the converses of Proposition 4.24 can not be reserved, for $i = 0, \frac{1}{2}$. We propose the following two questions since we could not find counter examples:*

*Are the spaces $\alpha_{[\gamma,\gamma']}$-$T_1$ and $[\gamma, \gamma^{'}]$-$T_1$ equivalent or not?*

*What about $\alpha_{[\gamma,\gamma']}$-$T_2$ and $[\gamma, \gamma^{'}]$-$T_2$?*

**Example 4.26** *Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}$ be a topology on $X$. For each $A \in \alpha O(X, \tau)$ we define two operations $\gamma$ and $\gamma^{'}$, respectively, by $A^{\gamma} = A^{\gamma^{'}} = A$. Then, $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_i$ but not $[\gamma, \gamma^{'}]$-$T_i$, where $i = 0, \frac{1}{2}$.*

**Proposition 4.27** *If $(X, \tau)$ is $(\gamma, \gamma^{'})$-$T_i$, then it is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.*

Proof. The proofs follow from [5; Proposition 6.12] and Proposition 4.24.

**Remark 4.28** *The converse of Proposition 4.27 can not reversible by [[5]; Remark 6.13, Examples 6.14 and 6.15] and Proposition 4.24.*

**Proposition 4.29** *If $(X, \tau)$ is $\gamma$-$T_i$, then it is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.*

Proof. The proofs follow from [5; Proposition 6.1] and Proposition 4.24.

**Remark 4.30** *The converse of Proposition 4.29 can not reversible by [5; Remark 6.2] and Proposition 4.24.*

**Remark 4.31** *From Propositions 4.12, 4.18, 4.21, 4.24, 4.27, 4.29, [10; Remark 2.1], and [4; p.180], for distinct operations $\gamma$ and $\gamma^{'}$ we have the following diagram. We note that some implications in the following diagram are not reversible by Remarks 4.13, 4.19, 4.22, 4.25, 4.28 and 4.30:*

$$
\begin{tikzcd}
 & & \gamma\text{-}T_{\frac{1}{2}} \arrow[r] \arrow[ddr] & \gamma\text{-}T_0 \arrow[ddr] & \\
\gamma\text{-}T_1 \arrow[urr] \arrow[drr] & (\gamma,\gamma^{'})\text{-}T_2 \arrow[r] \arrow[d] & (\gamma,\gamma^{'})\text{-}T_1 \arrow[r] \arrow[d] & (\gamma,\gamma^{'})\text{-}T_{\frac{1}{2}} \arrow[r] \arrow[d] & (\gamma,\gamma^{'})\text{-}T_0 \arrow[d] \\
\gamma\text{-}T_2 \arrow[u] \arrow[r] & {[\gamma,\gamma^{'}]\text{-}T_2} \arrow[r] \arrow[d] & {[\gamma,\gamma^{'}]\text{-}T_1} \arrow[r] \arrow[d] & {[\gamma,\gamma^{'}]\text{-}T_{\frac{1}{2}}} \arrow[r] \arrow[d] & {[\gamma,\gamma^{'}]\text{-}T_0} \arrow[d] \\
\alpha_\gamma\text{-}T_2 \arrow[r] \arrow[d] & \alpha_{[\gamma,\gamma']}\text{-}T_2 \arrow[r] \arrow[d] & \alpha_{[\gamma,\gamma']}\text{-}T_1 \arrow[r] \arrow[d] & \alpha_{[\gamma,\gamma']}\text{-}T_{\frac{1}{2}} \arrow[r] \arrow[d] & \alpha_{[\gamma,\gamma']}\text{-}T_0 \arrow[d] \\
\alpha_\gamma\text{-}T_1 \arrow[urr] \arrow[drr] & \alpha\text{-}T_2 \arrow[r] & \alpha\text{-}T_1 \arrow[r] & \alpha\text{-}T_{\frac{1}{2}} \arrow[r] & \alpha\text{-}T_0 \\
 & & \alpha_\gamma\text{-}T_{\frac{1}{2}} \arrow[r] \arrow[uur] & \alpha_\gamma\text{-}T_0, \arrow[uur] &
\end{tikzcd}
$$

where $A \to B$ represents that $A$ implies $B$.

**Proposition 4.32** *Suppose that $\gamma$ and $\gamma^{'}$ are $\alpha$-regular operations on $\alpha O(X,\tau)$. A space $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_i$ if and only if an associated space $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$ is $T_i$, where $i = 1, 1/2$.*

Proof. It follows from Remark 2.4 that a subset $A$ is $\alpha_{[\gamma,\gamma']}$-open in $(X,\tau)$ if and only if $A$ is open in $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$. Therefore, the proof for $i = \frac{1}{2}$ (resp. $i = 1$) follows from Propositions 4.8 (resp. Proposition 4.10).

**Proposition 4.33** *Let $\gamma$ and $\gamma^{'}$ be $\alpha$-regular operations on $\alpha O(X,\tau)$.*
(i) *If $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$ is $T_i$, then $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, 2$.*
(ii) *Moreover, suppose that $\gamma$ and $\gamma^{'}$ are $\alpha$-open operations on $\alpha O(X,\tau)$. If $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_i$, then $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$ is $T_i$, where $i = 0, 2$.*

Proof. (i) The proof for $i = 0$ (resp. $i = 2$) follows from the $T_0$-separation property (resp. Hausdorffness) of $(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$, the concept of $\alpha_{[\gamma,\gamma']}$-open sets (cf. [1; Definition 3.1]) and Definton 4.3 (resp. Definition 4.5).

(ii) Let $x$ and $y$ be distinct points of $X$. For $i = 2$, since $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_2$, there exist $\alpha$-open sets $U_1$ and $V_1$ containing $x$ and $\alpha$-open sets $W_1$ and $S_1$ containing $y$ such that $(U_1^\gamma \cap V_1^{\gamma^{'}}) \cap (W_1^\gamma \cap S_1^{\gamma^{'}}) = \phi$. Since $\gamma$ and $\gamma^{'}$ are $\alpha$-open operations on $\alpha O(X,\tau)$, so there exist $\alpha_\gamma$-open sets $U, W$ and $\alpha_{\gamma'}$-open sets $V, S$ such that $x \in U \cap V \subseteq U_1^\gamma \cap V_1^{\gamma^{'}}$, $y \in W \cap S \subseteq W_1^\gamma \cap S_1^{\gamma^{'}}$ and $(U \cap V) \cap (W \cap S) \subseteq (U_1^\gamma \cap V_1^{\gamma^{'}}) \cap (W_1^\gamma \cap S_1^{\gamma^{'}}) = \phi$. It follows from Proposition 2.5, that $U \cap V \in \alpha O(X,\tau)_{[\gamma,\gamma']}$ and $W \cap S \in \alpha O(X,\tau)_{[\gamma,\gamma']}$. This implies that

$(X, \alpha O(X,\tau)_{[\gamma,\gamma']})$ is $T_2$. The proof for $i = 0$ follows from Definition 4.3, and Proposition 2.5.

**Proposition 4.34** *If $f : (X,\tau) \to (Y,\sigma)$ is $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\beta,\beta']})$-continuous and $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\beta,\beta']})$-closed, then*
(i) *$f(A)$ is $\alpha_{[\beta,\beta']}$-g.closed for every $\alpha_{[\gamma,\gamma']}$-g.closed set $A$ of $(X,\tau)$; and*
(ii) *$f^{-1}(B)$ is $\alpha_{[\gamma,\gamma']}$-g.closed for every $\alpha_{[\beta,\beta']}$-g.closed set $B$ of $(Y,\sigma)$.*

Proof. (i) Let $V$ be an $\alpha_{[\beta,\beta']}$-open set containing $f(A)$. Then, $f^{-1}(V)$ is an $\alpha_{[\gamma,\gamma']}$-open set containing $A$ (cf. [1; Theorem 4.2]) and so $\alpha_{[\gamma,\gamma']}\text{-}Cl(A) \subseteq f^{-1}(V)$. It follows that $f(\alpha_{[\gamma,\gamma']}\text{-}Cl(A))$ is an $\alpha_{[\beta,\beta']}$-closed set (cf. Definition 2.7) and hence $\alpha_{[\beta,\beta']}\text{-}Cl(f(A)) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(f(\alpha_{[\gamma,\gamma']}\text{-}Cl(A))) = f(\alpha_{[\gamma,\gamma']}\text{-}Cl(A)) \subseteq V$. This implies that $f(A)$ is $\alpha_{[\beta,\beta']}$-g.closed.

(ii) Let $U$ be any $\alpha_{[\gamma,\gamma']}$-open set such that $f^{-1}(B) \subseteq U$. Let $F = \alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(B)) \cap (X \setminus U)$, then $F$ is $\alpha_{[\gamma,\gamma']}$-closed in $(X,\tau)$. This implies $f(F)$ is $\alpha_{[\beta,\beta']}$-closed set in $(Y,\sigma)$ (cf. Definition 2.7). Since $f(F) = f(\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(B)) \cap (X \setminus U)) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(B) \cap f(X \setminus U) \subseteq \alpha_{[\beta,\beta']}\text{-}Cl(B) \cap (Y \setminus B)$ (cf. [1; Theorem 4.2]), it is shown that $\alpha_{[\beta,\beta']}\text{-}Cl(B) \setminus B$ contains an $\alpha_{[\beta,\beta']}$-closed set $f(F)$. It follows from Proposition 3.14 that $f(F) = \phi$ and hence $F = \phi$. Therefore $\alpha_{[\gamma,\gamma']}\text{-}Cl(f^{-1}(B)) \subseteq U$. This shows that $f^{-1}(B)$ is $\alpha_{[\gamma,\gamma']}$-g.closed.

**Theorem 4.35** *Suppose that there exists an $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\beta,\beta']})$-continuous and $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\beta,\beta']})$-closed function, say $f : (X,\tau) \to (Y,\sigma)$.*
(i) *If $f$ is injective and $(Y,\sigma)$ is $\alpha_{[\beta,\beta']}$-$T_{\frac{1}{2}}$, then $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$.*
(ii) *If $f$ is surjective and $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$, then $(Y,\sigma)$ is $\alpha_{[\beta,\beta']}$-$T_{\frac{1}{2}}$.*

Proof. (i) Let $A$ be an $\alpha_{[\gamma,\gamma']}$-g.closed set of $(X,\tau)$. We claim that $A$ is $\alpha_{[\gamma,\gamma']}$-closed in $(X,\tau)$. By Proposition 4.34 (i), $f(A)$ is $\alpha_{[\beta,\beta']}$-g.closed. Since $(Y,\sigma)$ is $\alpha_{[\beta,\beta']}$-$T_{\frac{1}{2}}$, this implies that $f(A)$ is $\alpha_{[\beta,\beta']}$-closed. Since $f$ is $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous and injective, then, we have $A = f^{-1}(f(A))$ is $\alpha_{[\gamma,\gamma']}$-closed (cf. [1; Theorem 4.2]). Hence $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_{\frac{1}{2}}$.

(ii) Let $B$ be an $\alpha_{[\beta,\beta']}$-g.closed set in $(Y,\sigma)$. By Proposition 4.34 (ii) and assumptions, it is shown that $B = f(f^{-1}(B))$ is $\alpha_{[\gamma,\gamma']}$-closed; and hence $(Y,\sigma)$ is $\alpha_{[\beta,\beta']}$-$T_{\frac{1}{2}}$.

**Theorem 4.36** *Suppose that there exists an $(\alpha_{[\gamma,\gamma']},\ \alpha_{[\beta,\beta']})$-continuous injection. If $(Y,\sigma)$ is $\alpha_{[\beta,\beta']}$-$T_i$, then $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_i$, where $i = 0, 1, 2$.*

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be the $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous injection. The proof for $i = 1$ is as follows: Let $x \in X$. Then, by Proposition 4.10, $\{f(x)\}$ is $\alpha_{[\beta,\beta']}$-closed in $(Y,\sigma)$. By [1; Theorem 4.2] and Proposition 4.10, $\{x\}$ is $\alpha_{[\gamma,\gamma']}$-closed and hence $(X,\tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_1$. The proofs for $i = 0, 2$ follow from Definitions 4.3, 4.5 and [1; Theorem 4.2].

**Definition 4.37** *A function $f : (X, \tau) \to (Y, \sigma)$ is called an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-homeomorphism if $f$ is an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-continuous bijection and $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $(\alpha_{[\beta,\beta']}, \alpha_{[\gamma,\gamma']})$-continuous. The collection of all $(\alpha_{[\gamma,\gamma']}, \alpha_{[\gamma,\gamma']})$-homeomorphisms from $(X, \tau)$ onto itself is denoted by $\alpha_{[\gamma,\gamma']}$-$h(X, \tau)$.*

**Theorem 4.38** (i) *Suppose that there exists an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-homeomorphism between topological spaces $(X, \tau)$ and $(Y, \sigma)$. Then, $(X, \tau)$ is $\alpha_{[\gamma,\gamma']}$-$T_i$ if and only if $(Y, \sigma)$ is $\alpha_{[\beta,\beta']}$-$T_i$, where $i = 0, \frac{1}{2}, 1, 2$.*

(ii) *For each topological space $(X, \tau)$, the collection $\alpha_{[\gamma,\gamma']}$-$h(X, \tau)$ forms a group under the composition of functions.*

(iii) *For an $(\alpha_{[\gamma,\gamma']}, \alpha_{[\beta,\beta']})$-homeomorphism $f : (X, \tau) \to (Y, \sigma)$, there exists a group isomorphism, say $f_* : \alpha_{[\gamma,\gamma']}$-$h(X, \tau) \to \alpha_{[\beta,\beta']}$-$h(Y, \sigma)$.*

Proof. Put $H_X = \alpha_{[\gamma,\gamma']}$-$h(X, \tau)$.

(i) This follows from Theorems 4.35, 4.36 and Definition 4.37.

(ii) First we prove that: if $a \in H_X$ and $b \in H_X$, then $b \circ a \in H_X$. Indeed, since $a$ and $b$ (resp. $a^{-1}$ and $b^{-1}$) are $\alpha_{[\gamma,\gamma']}$-continuous bijectons, $b \circ a$ (resp. $a^{-1} \circ b^{-1} = (b \circ a)^{-1}$) is also an $\alpha_{[\gamma,\gamma']}$-continuous bijection (cf. [1; Theorem 4.10]); and so $b \circ a \in H_X$, where $b \circ a : X \to X$ is the composite functions of $a : X \to X$ and $b : X \to X$ such that $(b \circ a)(x) = b(a(x))$ for every point $x \in X$. Thus, the following binary operation $\eta_X : H_X \times H_X \to H_X$ is well defined by $\eta_X(a, b) = b \circ a$. Putting $a \cdot b = \eta_X(a, b)$, we have the following properties:

1. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds for every elements $a, b, c \in H_X$;
2. for all element $a \in H_X$, there exists an element $e \in H_X$ such that $a \cdot e = e \cdot a = a$ hold in $H_X$;
3. for each element $a \in H_X$, there exists an element $a_1 \in H_X$ such that $a \cdot a_1 = a_1 \cdot a = e$ hold in $H_X$.

Indeed, (1) is obtained obviously; (2) is obtained by taking $e = 1_X$ and using the fact that $1_X \in H_X$, where $1_X : X \to X$ is the identity function; (3) is obtained by taking $a_1 = a^{-1}$ for each $a \in H_X$. Then, by definition of groups, the pair $(H_X, \eta_X)$ forms a group under the composition of functions.

(iii) The required group isomorphism $f_* : H_X \to H_Y$ is well defined by $f_*(a) = f \circ (a \circ f^{-1})$ for every element $a \in H_X$. Indeed, $f_*(a) \in H_Y$ holds for every $a \in H_X$ (cf. [1; Theorem 4.10]); $f_*(a \cdot b) = f \circ (b \circ a) \circ f^{-1} = f \circ b \circ f^{-1} \circ f \circ a \circ f^{-1} = (f_*(b)) \circ (f_*(a)) = f_*(a) \cdot f_*(b)$ hold for every elements $a, b \in H_X$ and so $f_* : H_X \to H_Y$ is a homomorphism. Hence $f_* : H_X \to H_Y$ is the required isomorphism.

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