SOME KINDS OF TOPOLOGICAL HYPER K-ALGEBRAS

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[1]

Abstract

In this paper we introduce and characterise the concepts of pseudo topological hyper K-algebras, strong pseudo topological hyper K-algebras and topological hyper K-algebras. Then we find some properties of this structures. Also we define a topology on a hyper K-algebra, which makes a pseudo topological hyper K-algebra.

1 Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [4] in 1966, as a generalization of the concept of set-theoretic difference and propositional calculi. Since then a great deal of literature has been produced on the theory of BCK-algebras. The hyper structure theory (called also multi algebras) was introduced in 1934 by F. Marty [6] at the 8th congress of Scandinavian Mathematicians. Hyper structures have many applications to several sectors of both pure and applied sciences. In [5] Y. B. Jun, X. L. Xin and d. S. Lee defined topological BCI-algebras. In [2] R. A. Borzooei et al. applied the hyper structures to BCK-algebras and introduced the notion of a hyper K-algebra which is a generalization of BCK-algebra and investigated some related properties. In [1] R. Ameri introduced and studied the notion of topological (transposition) hypergroups. The aim of this paper is study of topological hyper K-algebras. In this regard we introduce various kinds of topological hyper K-algebras such as strong, pseudo topological hyper K-algebras and investigate its basic properties. Now, in this paper, we follow the references and we obtain some results as mentioned in the abstract.

2 Preliminaries

Definition 2.1. Let H be a nonempty set and " $\circ$ " be a hyperoperation on H, that is " $\circ$ " is a function from $H \times\ H$ to $P^*(H) = P(H) \setminus \{0\}$. Then H is called a *hyper K-algebra*, if it contains a constant 0 and satisfies the following axioms :

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(I_1) $(x \circ z) \circ (y \circ z) < x \circ y$,

(I_2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(I_3) $x < x$,

(I_4) $x < y$ and $y < x$ implies $x = y$,

(I_5) $0 < x$,

for all $x, y, z \in H$ where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$. Note that if $A, B \subseteq H$ then by $A \circ B$ we mean the subset $\bigcup\limits_{a \in A, b \in B} a \circ b$ of H. For non-empty subset A, B of H by $A \approx B$, we mean $A \cap B \neq \emptyset$. Let H be a hyper K-algebra. By $S(H)$ we mean

$$S(H) = \{x \in H |\ x \circ x = \{0\}\}.$$

In a hyper K-algebra the following hold (for more details see [2]).

(I_6) $(A \circ B) \circ C = (A \circ C) \circ B$,

(I_7) $x \circ (x \circ y) < y$,

(I_8) $x \circ y < z$ if and only if $x \circ z < y$,

(I_9) $A \circ B < C$ if and only if $A \circ C < B$,

(I_{10}) $(x \circ z) \circ (x \circ y) < y \circ z$,

(I_{11}) $(A \circ C) \circ (B \circ C) < A \circ B$,

(I_{12}) $A \circ (A \circ B) < B$,

(I_{13}) $A < A$,

(I_{14}) $A \subseteq B$ implies $A < B$,

(I_{15}) $x \circ 0 < x$,

(I_{16}) $x \in x \circ 0$,

for every $x, y, z \in H$ and for all nonempty subsets A, B and C of H.

Definition 2.2. [3] Let I be a nonempty subset of a hyper K-algebra $(H, \circ, 0)$ and $0 \in I$. Then

(i) I is called a *weak hyper K-ideal* of H if $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$,

(ii) I is called a *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$,

(iii) I is called a *strong hyper K-ideal* of H if $x \circ y \approx I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Theorem 2.3. *[3] Any hyper K-ideal of a hyper K-algebra is a strong hyper K-ideal and any hyper K-ideal of a hyper K-algebra is a weak hyper K-ideal.*

Definition 2.4. [2] The subset A of a hyper K-algebra H is said subalgebra if $x \circ y \subseteq A$, for all $x, y \in A$.

Definition 2.5. [5] A topology τ on a BCI-algebra X is said to be a BCI-topology and X furnished with τ, is said a topological *BCI-algebra* if $(x, y) \mapsto x * y$ is continuous from $X \times X$, furnished with the cartesian product topology defined by τ, to X.

3 Some Properties Of hyper K-algebras

Lemma 3.1. *Let H be a hyper K-algebra and A be a subset of H. Then $A < \{0\}$ implies $0 \in A$.*

Proof. Let $A < \{0\}$. Then there exists $a \in A$ such that $a < 0$. By (I_5), $a = 0$. Therefore, $0 \in A$. □

Theorem 3.2. *In every hyper K-algebra, H, $\{0\}$ and H are hyper K-ideals, strong hyper K-ideals and weak hyper K-ideals.*

Proof. Let H be a hyper K-algebra. It is easy to see that H is a hyper K-ideal, strong hyper K-ideal and weak hyper I-ideal.

Let $x \circ y < \{0\}$ and $y = 0$. Then $x \circ 0 < \{0\}$. By Theorem 3.1, $0 \in x \circ 0$ and so $x < 0$. By (I_5), $x = 0$. Thus $\{0\}$ is a hyper I-ideal. The proof of other statements are similar. □

Definition 3.3. The subset A of a hyper K-algebra H is said to be closed if $0 \circ x \subseteq A$ and $x \circ 0 \subseteq A$, for all $x \in A$.

Theorem 3.4. *If $\{A_i | i \in \Lambda\}$ is a family of closed subsets, then $\bigcap_{i \in \Lambda} A_i$ is also a closed subset.*

Proof. The proof is routine and we are omitted. □

4 topological hyper K-algebras

Definition 4.1. Let $(H, \circ, 0)$ be a hyper K-algebra, τ be a topology on H and τ_* be a topology on $P^*(H)$. Then $\circ$ is said:

(i) strong pseudo continuous (for short s.p-continuous), if for every open set U (member of τ), the set

$$U^* = \{(x, y) | x \circ y \approx U\}$$

is open in $H \times H$, with respect to the product topology on $H \times H$.

(ii) pseudo continuous (for short p-continuous), if for every open set U (member of τ) the set

$$U_* = \{(x, y) | x \circ y \subseteq U\}$$

is open in $H \times H$, with respect to the product topology on $H \times H$.

(iii) continuous if for every open set U of $P^*(P^*(H))$ (the member of τ_*) the set

$$\{(x, y) | x \circ y \in U\}$$

is open in $H \times H$.

A hyper K-algebra $(H, \circ)$ is called a

(i) strong pseudo topological hyper K-algebra (briefly s.p.t-hyper K-algebra) if " $\circ$ " is strong pseudo continuous,

(ii) pseudo topological hyper K-algebra (briefly p.t-hyper K-algebra) if "$\circ$" is pseudo continuous,

(iii) topological hyper K-algebra (briefly t-hyper K-algebra) if " $\circ$ " is continuous.

Remark 4.2. For an arbitrary open subset U of H, $U_* \subseteq U^*$.

Note that if $(H, *, 0)$ is a BCI-algebra and we set $x \circ y = \{x * y\}$, for each $x, y \in H$, then $(H, \circ, 0)$ is a hyper K-algebra. We assert that if $(H, *, 0)$ is a topological BCI-algebra, then $(H, \circ, 0)$ is a t-hyper K-algebra, p.t-hyper K-algebra and s.p.t-hyper K-algebra.

In the next example we will show that really these three types of topological hyper K-algebra are different.

Example 4.3. (i) A hyper K-algebra with the discrete topology and the indiscreet topology is a s.p.t-hyper K-algebra and a p.t-hyper K-algebra.

(ii) Let $H = \{0, a, b\}$ be a set and the hyperoperation " $\circ_3$ " is defined as follows:

$\circ_1$	0	a	b
0	$\{0\}$	$\{0, a\}$	$\{0, a\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Then $(H, \circ_1)$ is a hyper K-algebra. If we define $\tau = \{\emptyset, H, \{0\}\}$, then τ is a topology on H. By routine calculation one shows that H furnished with τ is a p.t-hyper K-algebra but H is not a s.p.t-hyper K-algebra, because $\{0\}^* = \{(0, 0), (0, a), (0, b), (a, b), (b, b)\}$ is not open in $H \times H$.

(iii) Let $H = \{0, a, b\}$ with Cayley‘s table as follows:

$\circ_2$	0	a	b
0	$\{0, a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a, b\}$
b	$\{a, b\}$	$\{0, a\}$	$\{0, a, b\}$

Then $\tau = \{\emptyset, H, \{a, b\}\}$, then "$\circ_2$" is strong pseudo continuous. Clearly $\{a, b\}_* = \{(a, 0), (a, b), (b, 0)\}$ is not in $\tau \times \tau$, then " $\circ_2$ " is not pseudo continuous.

(iv) Consider hyper K-algebra $H = \{0, a, b\}$ with Cayley‘s table as follows:

$\circ_3$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{a, b\}$	$\{0\}$	$\{0, a\}$

equipped with $\tau = \{\emptyset, H, \{b\}\}$ and $\tau^* = \{\emptyset, P_*(P_*(H)), \{\{0, a\}, \{b\}\}\}$. It is easy to see that $\circ_3$ is continuous on H. Therefore, $(H, \circ_3, \tau, \tau^*)$ is a t-hyper K-algebra.

5 (Pseudo) topological hyper K-algebras

Lemma 5.1. *Let H be a hyper K-algebra and τ be a topology on H. If $(x \circ y)_*$ is open in $H \times H$, for every $x, y \in H$, then there are neighborhoods U of x and V of y such that $U \circ V = x \circ y$.*

Proof. Let $(x \circ y)_*$ be open in $H \times H$, for $x, y \in H$. Thus

$$(x \circ y)_* = \{(z,t) \ \mid \ z \circ t \subseteq x \circ y\} = \bigcup_{i \in I} (U_i \times V_i),$$

for all $i \in I$ and for all $U_i, V_i \in \tau$. It is obvious that $(x, y) \in (x \circ y)_*$. Hence, there exist U_i and V_i in τ, such that $x \in U_i$ and $y \in V_i$. We rename U_i by U and V_i by V. We have

$$U \times V \subseteq \bigcup_{i \in I} (U_i \times V_i) = (x \circ y)_*.$$

Hence

$$U \circ V = \bigcup_{z \in U, t \in V} z \circ t = \bigcup_{(z,t) \in U \times V} z \circ t \subseteq \bigcup_{(z,t) \in (x \circ y)_*} z \circ t = \bigcup_{z \circ t \subseteq x \circ y} z \circ t \subseteq x \circ y.$$

Thus $U \circ V \subseteq x \circ y$. On the other hands $x \in U$ and $y \in V$, so $x \circ y \subseteq U \circ V$. Therefore, $x \circ y = U \circ V$. □

Proposition 5.2. *Let $(H, \circ, 0)$ be a hyper K-algebra and τ be a topology on H such that the intersection of every open set is open in H. Then $(x \circ y)_*$ is open, for each $x, y \in H$, if and only if τ is discreet topology.*

Proof. Assume that $(H, \circ, 0)$ be a hyper K-algebra. It is obvious that if τ is the discreet topology then $(x \circ y)_*$ is open, for each $x, y \in H$.

Conversely, let $(x \circ y)_*$ be open, for each $x, y \in H$ and $\Theta = \bigcap\{U \ \mid \ U \text{ is a neighborhood of } x\}$.

Note that H a neighborhood of x and hence $\Theta \neq \emptyset$. By hypothesis Θ is open. If $\Theta = \{x\}$, then $\{x\}$ is open, for each $x \in H$ and so τ is the discreet topology. Otherwise, if $y \in \Theta$ and $x \neq y$, then $0 \notin x \circ y$ or $0 \notin y \circ x$. Without lose of generality let $0 \notin x \circ y$. By Lemma 5.1, there exist neighborhoods of x and y respectively such that $A \circ B = x \circ y$. Thus $0 \notin A \circ B$. On the other hands A is a neighborhood of x, so $\Theta \subseteq A$ and hence $y \in A$. Since $y \in A$ and $y \in B$, then $0 \in y \circ y \subseteq A \circ B$, which is a contradiction. So $\Theta = \{x\}$ and τ is discreet topology. □

Theorem 5.3. *Let H be a hyper K-algebra and τ be a topology on H. H equipped with τ is a p.t-hyper K-algebra if and only if for each x and y in H and for each open set W such that $x \circ y \subseteq W$, there exist neighborhoods U and V respectively of x and y such that $U \circ V \subseteq W$.*

Proof. The proof is similar to Lemma 5.1 and we are omitted. □

Remark 5.4. Let H be a hyper K-algebra and τ be a topology on H. It is clear that

$$\tau_* = \{\{V\} | \emptyset \neq V \in \ \tau\} \cup \{\emptyset, P_*(H)\},$$

is a topology on $P_*(H)$. It is easily proven that $(H, \circ, 0, \tau)$ is a p.t-hyper K-algebra if and only if $(H, \circ, 0, \tau, \tau_*)$ is a t-hyper K-algebra.

In the following we show that Remark 5.4 is not correct about strong pseudo continuous in general.

Example 5.5. Let $H = \{0, a, b\}$. Then according to Example 4.3 (iii), the hyper K-algebra $(H, \circ_4, 0)$ with topology $\tau = \{\emptyset, H, \{a, b\}\}$ is a s.p.t-hyper K-algebra. Letting $\tau_* = \{\emptyset, \{H\}, \{\{a, b\}\}, P_*(H)\}$, then $\circ_4 : H \times H \longrightarrow P_*(H)$ is not continuous, because $\circ^{-1}(\{\{a, b\}\}) = \{(x, y) \in H \times H \mid x \circ y \in \{\{a, b\}\}\} = \{(a, b), (b, 0)\}$, which is not open in $H \times H$.

Corollary 5.6. *Let H be a p.t-hyper K-algebra. If $\{0\}$ is open in H, then $\{x\}$ is open, for each $x \in S(H)$.*

Proof. Assume that H be a p.t-hyper K-algebra. If $x \in S(H)$, then $x \circ x \subseteq \{0\}$. By Theorem 5.3, there are neighborhoods U and V of x such that $U \circ V \subseteq \{0\}$ and so $U \circ V = \{0\}$. We note that $W = U \cap V$ is open and W is a neighborhood of x. Thus $W \circ W \subseteq U \circ V = \{0\}$. Therefore, $W = \{x\}$ and $\{x\}$ is open. □

In the next example we show that the converse of Corrolary 5.6 is not correct in generality.

Example 5.7. Let $H = \{0, a, b\}$. Then according the following Cayley's table $(H, \circ_4)$ is a hyper K-algebra.

$\circ_4$	0	a	b
0	H	H	H
a	$\{a\}$	H	$\{a, b\}$
b	$\{a, b\}$	H	$\{0\}$

If we define $\tau = \{\emptyset, H, \{b\}\}$, then H is a p.t-hyper K-algebra. We can see that $S(H) = \{b\}$ and b is open. but 0 is not open.

The next result immediately follows from Corollary 5.6.

Corollary 5.8. *Let H be a p.t-hyper K-algebra, which $\{0\}$ is open and $S(H) = H$. Then the topology on H is the discreet topology.*

Corollary 5.9. *Let $(H, \circ, 0, \tau, \tau_*)$ be a topological hyper K-algebra. If $\{\{0\}\}$ is open in $P_*(H)$, then $\{x\}$ is open in H, for every $x \in S(H)$.*

Proof. By Corollary 5.6, the proof is easy. □

Theorem 5.10. *Let H is a p.t-hyper K-algebra. Then $\{0\}$ is closed in H if and only if H is Hausdorff.*

Proof. Assume that $\{0\}$ is closed, $x, y \in H$ and $x \neq y$. Then $0 \notin x \circ y$ or $0 \notin y \circ x$. We may assume that $0 \notin x \circ y$ without lose of generality. Hence $x \circ y \subseteq H \backslash \{0\}$. By Theorem 5.3, there exist neighborhoods U and V of x and y respectively such that $U \circ V \subseteq H \backslash \{0\}$. Since $0 \notin U \circ V$, then $U \cap V = \emptyset$ and H is Hausdorff.

Conversely, suppose that H is Hausdorff. Then every finite set is close. Therefore, $\{0\}$ is closed. □

In the next example we show that the Theorem 5.10 is not correct for s.p.t-hyper K-algebra and T-hyper K-algebra in general.

Example 5.11. (i) Let $H = \{0, a, b\}$. According Example 4.3 (iii), $(H, \circ_3)$ is a s.p.t-hyper K-algebra and $\{0\}$ is close. Since there are not two open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$, then H is not Hausdorff. Moreover H is not T_1 either.

(ii) Let $H = \{0, a, b\}$. According Example 4.3 (iii), $(H, \circ_4)$ is a hyper K-algebra. If we define

$$\tau_* = \{\emptyset, p_*(H), \{\{a,b\},\{0,b\},\{0,a\},\{a\},\{b\},\{0,a,b\}\}\} \text{ and } \tau = \{\emptyset, H\}.$$

Then $(H, \circ, 0, \tau, \tau_*)$ is a t-hyper K-algebra and $\{\{0\}\}$ is close in $P_*(H)$. Since there are not two open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$, then H is not Hausdorff.

Proposition 5.12. *Let H be a p.t-hyper K-algebra and A be a weak hyper K-ideal. If $x \circ x$ is an interior subset of A, for each $x \in A$, then A is open.*

Proof. Let $x \in A$. Then there exists open set U such that $x \circ x \subseteq U \subseteq A$. By Proposition 5.3, there are neighborhoods G and H of x such that $G \circ H \subseteq U \subseteq A$. Since $x \in A$ and $y \circ x \subseteq G \circ H \subseteq$, for every $y \in G$, we imply that $y \in A$. Therefore, $G \subseteq A$ and A is open. □

In the next example we show that the converse of Proposition 5.12 is not correct in generality.

Example 5.13. In Example 4.3(iii) if we define $\tau = \{\emptyset, H, \{0,b\}\}$, then $(H, \circ_2)$ is a p.t-hyper K-algebra and $\{0,b\}$ is a weak hyper K-ideal of H. We note that A is open and $0 \in A$ but $H = 0 \circ_2 0$ is not an interior subset of A.

Proposition 5.14. *Let $(H, \circ, 0, \tau, \tau_*)$ be a t-hyper K-algebra and A be a weak hyper K-ideal. If $p_*(A)$ is an open set in τ_* and $x \circ x \subseteq A$, for each $x \in A$, then A is open in τ.*

Proof. The proof is similar to the proof of Proposition 5.12 and we are omitted. □

Theorem 5.15. *Let H be a p.t-hyper K-algebra and A be an open weak hyper K-ideal of H. If $x \circ x \subseteq A$, for each $x \in H \backslash A$, then A is closed.*

Proof. We prove that $H \backslash A$ is open. Let $x \in H \backslash A$. By hypothesis, $x \circ x \subseteq A$. By Theorem 5.3, there are neighborhoods U and V of x such that $U \circ V \subseteq A$. We claim that $V \subseteq H \backslash A$. On the contrary if $V \nsubseteq H \backslash A$, then there exists $y \in V \cap A$. Note that $z \circ y \subseteq U \circ V \subseteq A$, for all $z \in U$. Since $y \in A$ and A is a weak hyper K-ideal, $z \in A$. Hence $U \subseteq A$ and so $x \in A$. This is a contradiction. Therefore, $x \in V \subseteq H \backslash A$ and $H \backslash A$ is close. □

In the next example we show that the converse of Theorem 5.15 is not correct in generality.

Example 5.16. Let $H = \{0, a, b\}$ and define the hyper operation $\circ_6$ as follows:

$\circ_6$	0	a	b	c
0	$\{0\}$	$\{0\}$	H	H
a	$\{a\}$	$\{0\}$	$\{a,b\}$	$\{a,b\}$
b	$\{b\}$	$\{a,b,c\}$	$\{0,a,b\}$	$\{0.b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	H

Then according the following Cayley's table $(H, \circ_6)$ is a hyper K-algebra. If we define $\tau = \{\emptyset, H, \{0,b\}, \{a\}\}$, then H is a p.t-hyper K-algebra. We note that $A = \{0,b\}$ is a clopen set $a \in H \setminus A$ but $a \circ_6 a \nsubseteq A$.

Theorem 5.17. *Let $(H, \circ, 0, \tau, \tau_*)$ be a t-hyper K-algebra and A be an open weak hyper K-ideal of H. If $P^*(A)$ is an open set in $P^*(H)$ and $x \circ x \subseteq A$, for each $x \in H \backslash A$, then A is close.*

Proof. The proof is similar to the proof of Theorem 5.15 and we are omitted. □

Theorem 5.18. *Let H be a p.t-hyper K-algebra and A be an open subalgebra of H. Then A is also a p.t-hyper K-algebra.*

Proof. By Theorem 5.3, it suffices to show that if W is an open set in A, such that $x \circ y \subseteq W$, where $x, y \in A$, then there are neighborhoods S and T respectively of x and y such that $S \circ T \subseteq W$. We note that W is an open set in A, so there exists open set U such that $W = U \cap A$. Thus $x \circ y \subseteq U$ and $x \circ y \subseteq A$. By Theorem 5.3, there exist neighborhoods U_1 and U_2 of x and V_1 and V_2 of y such that $U_1 \circ V_1 \subseteq U$ and $U_2 \circ V_2 \subseteq A$. Let $S = U_1 \cap U_2 \cap A$ and $T = V_1 \cap V_2 \cap A$. Then S and T are open in A. Also they are neighborhoods respectively of x and y. Thus

$$S \circ T \subseteq U_1 \circ V_1 \subseteq U \text{ and } S \circ T \subseteq U_2 \circ V_2 \subseteq A.$$

This implies that $S \circ T \subseteq U \cap A = W$ and hence A is a p.t-hyper K-algebra. □

6 strong hyper K-algebras

Theorem 6.1. *Let H equipped with τ be a s.p.t-hyper K-algebra and W be an open set. Then for each x and y in H such that $x \circ y \approx W$, there exist neighborhoods U and V of x and y respectively such that $u \circ v \approx W$ for every $u \in U$ and $v \in V$.*

Proof. The proof is similar to Lemma 5.1 and we are omitted. □

Corollary 6.2. *Let $(H, \circ, 0, \tau)$ be a s.p.t-hyper K-algebra. $\{0\}$ is open in H if and only if τ is discreet topology.*

Proof. Assume that $(H, \circ, 0, \tau)$ be a s.p.t-hyper K-algebra and $\{0\}$ be open. By (I_3), $x \circ x \approx \{0\}$, for an arbitrary $x \in H$. By Theorem 6.1, there are neighborhoods U and V of x such that for all $u \in U$ and $v \in V$, $u \circ v \approx \{0\}$. Let $W = U \cap V$. Thus W is open, a neighborhood of x and

$$w_1 \circ w_2 \approx \{0\} \text{ and } w_2 \circ w_1 \approx \{0\} \quad (1)$$

for all w_1 and $w_2 \in W$. We claim that $W = \{x\}$. Let $y \in W$. Then by (1), $x \circ y \approx \{0\}$ and $y \circ x \approx \{0\}$. By (I_4), we have $x = y$. This means $\{x\}$ is open, for an arbitrary $x \in H$ and τ is discreet topology. □

In the next example we show that the Corollary 6.2 is not correct about p.t-hyper K-algebra in general.

Example 6.3. Let $H = \{0, a, b\}$. Then according Example 4.3 (ii), $(H, \circ_1)$ equipped with τ is a p.t-hyper K-algebra and $\{0\}$ is open but τ is not the discreet topology.

Proposition 6.4. *Let H be a s.p.t-hyper K-algebra and A be a strong hyper K-ideal. Then A is open if and only if there exists an open subset U of A such that $x \circ x \approx U$, for each $x \in A$.*

Proof. The proof is similar to the proof of Proposition 5.12. □

In the following we show that the Proposition 6.4 is not correct for p.t-hyper K-algebra in general.

Example 6.5. Let $H = \{0, a, b, c\}$. Then according the following Cayley‘s table $(H, \circ_6)$ is a hyper K-algebra.

$\circ_6$	0	a	b	c
0	$\{0\}$	$\{0\}$	H	H
a	$\{a\}$	$\{0\}$	$\{a, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b, c\}$	$\{0, a, b\}$	$\{0, b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	H

It is easy to see that $(H, \circ_6, 0)$ with $\tau = \{\emptyset, H, \{0, a\}\}$ is a p.t-hyper K-algebra. $A = \{0, a, b\}$ is a strong hyper K-ideal and $x \circ x \approx \{0, a\}$, for all $x \in A$ also $\{0, a\} \subseteq A$, but A is not open.

Corollary 6.6. *Let H be a s.p.t-hyper K-algebra and A be an open strong hyper K-ideal of H. Then A is a clopen set in H.*

Proof. We prove that $H \backslash A$ is open. Assume that $x \in H \backslash A$. By (I_3), $0 \in x \circ x$ and so $x \circ x \approx A$. By Theorem 6.1, there are neighborhoods U and V of x such that for all $u \in U$ and $v \in V$, $u \circ v \approx A$.

We claim that $V \subseteq H \backslash A$. On the contrary if $V \nsubseteq H \backslash A$, then there exist $y \in V \cap A$. Let $u \in U$. By Theorem 6.1, $u \circ y \approx A$. Since $y \in A$ and A is a strong hyper K-ideal, $u \in A$. It means that $U \subseteq A$ and so $x \in A$ which is a contradiction. Therefore, $V \subseteq H \backslash A$, $H \backslash A$ is open and A is close. $\square$

In the following we show that the Corollary 6.6 is not correct for p.t-hyper K-algebra in general.

Example 6.7. Let $H = \{0, a, b, c\}$. According to Example 6.5, $(H, \circ_3)$ with $\tau = \{\emptyset, H, \{0, a\}\}$ is a p.t-hyper K-algebra. We can see that $A = \{0, a\}$ is a strong hyper K-ideal and A is open but A is not close.

Theorem 6.8. *Let H be a s.p.t-hyper K-algebra and A be an open subalgebra of H. Then A is also a s.p.t-hyper K-algebra.*

Proof. Let W be an open set in A. Thus $W = U \cap A$, where U is an open set in H. We have $W^* = U^* \cap (A \times A)$ and hence W^* is open. Therefore, A is a s.p.t-hyper K-algebra. $\square$

Theorem 6.9. *Let H be a hyper K-algebra and $A = \bigcap_{x,y \in H} x \circ y$ and τ be a topology on H. If $A \neq \emptyset$ and $A \subseteq O$, for every $O \in \tau$, then H with τ is a s.p.t-hyper K-algebra.*

Proof. Let H be a hyper K-algebra and $A = \bigcap_{x,y \in H} x \circ y$. To prove H with τ is a s.p.t-hyper K-algebra it is sufficient to show that O^* is open in $H \times H$, for each $O \in \tau$. By Definition 4.1,

$$O^* = \{(x, y) \mid x \circ y \approx O\} \supseteq \{(x, y) \mid x \circ y \approx A\} = H \times H.$$

Thus H with τ is a s.p.t-hyper K-algebra. $\square$

Example 6.10. In the Example 4.3(iii), $\bigcap_{x,y \in H} x \circ y = \{a\}$ and we can see that $(H, \circ_4)$ with $\tau = \{\emptyset, H, \{a\}, \{a, b\}\}$ is a s.p.t-hyper K-algebra.

7 Some topology on hyper K-algebras

Theorem 7.1. *Let H be a hyper K-algebra, ϑ be a family of close subsets of H which closed under intersection and $\bigcap_{I \in \vartheta} I \neq \emptyset$. If we define*

$$U_I = \{(x, y) | x \circ y \subseteq I \text{ and } y \circ x \subseteq I\},$$

$$\eta = \{U \subseteq H \times H | U_I \subseteq U \text{ for some } I \in \vartheta\} \cup \{\emptyset\},$$

then η is a topology on $H \times H$.

Proof. By definition of η, $\emptyset \in \eta$ and $H \times H \in \eta$.

Let $U, V \in \eta$. Then there exist $I, J \in \vartheta$ such that $U_I \subseteq U$ and $U_J \subseteq V$. By hypothesis $I \cap J \neq \emptyset$. It is easy to prove that $U_{I\cap J} \subseteq U_I$ and $U_{I\cap J} \subseteq U_J$. Thus $U_{I\cap J} \subseteq U$ and $U_{I\cap J} \subseteq V$. Hence $U_{I\cap J} \subseteq U \cap V$ and so $U \cap V \in \eta$.

Let $\{U_\alpha\}$ be a family of members of η. Then there exist $I_\alpha \in \vartheta$ such that $U_{I_\alpha} \subseteq U_\alpha$, for every α. By $U_{I_\alpha} \subseteq U_\alpha \subseteq \cup U_\alpha$, $\cup U_\alpha \in \eta$. Therefore, η is a topology. □

Theorem 7.2. *With condition Theorem 7.1, if we define*

$$U_I(x) = \{y | (x, y) \in U_I \text{ for some } I \in \vartheta\} \text{ and}$$

$$\tau = \{G \subseteq H \mid \forall x \in G\ \exists I \in \vartheta \text{ such that } U_I(x) \neq \emptyset \text{ and } U_I(x) \subseteq G\} \cup \{H\},$$

then τ is a topology on H.

Proof. It is clear that $\emptyset, H \in \tau$. Let $A, B \in \tau$ and $x \in A \cap B$. Then there exist $I, J \in \vartheta$ such that $U_I(x) \subseteq A$ and $U_J(x) \subseteq B$, for all $x \in A \cap B$. Since $U_{I\cap J} \subseteq U_I$ and $U_{I\cap J} \subseteq U_J$, $U_{I\cap J}(x) \subseteq A$ and $U_{I\cap J}(x) \subseteq B$. Thus $U_{I\cap J}(x) \subseteq A \cap B$. Because $I \cap J \in \vartheta$, $A \cap B \in \tau$.

Let $\{A_\alpha\}$ be a family of members of τ and $x \in \cup A_\alpha$. Then $x \in A_\alpha$, for some α. Hence there exist $I \in \vartheta$ such that $U_I(x) \neq \emptyset$ and $U_I(x) \subseteq A_\alpha \subseteq \cup A_\alpha$. Therefore, $\cup A_\alpha \in \tau$ and τ is a topology on H. □

Lemma 7.3. *With condition Theorem 7.2, $\eta \subseteq \tau \times \tau$.*

Proof. Let $A, B \subseteq H$ and $A \times B \in \eta$. Then $U_I \subseteq A \times B$, for some $I \in \vartheta$. By definition of U_I, $(x, y) \in U_I$ if and only if $(y, x) \in U_I$. So $(x, y) \in U_I \subseteq A \times B$ implies $x \in A$ and $y \in B$. Also

$$(x, y) \in U_I \Rightarrow (y, x) \in U_I \subseteq A \times B \text{ implies } x \in B \text{ and } y \in A.$$

We imply that if $(x, y) \in U_I$, then $x, y \in A \cap B$. Let $x \in A$. We will claim that $U_I(x) \subseteq A$. If $y \in U_I(x)$, then $(x, y) \in U_I$. Hence $y \in A \cap B \subseteq A$ and so $U_I(x) \subseteq A$. Thus $A \in \tau$. By the similar way $B \in \tau$, which implies $A \times B \in \tau \times \tau$. Therefore, $\eta \subseteq \tau \times \tau$. □

Theorem 7.4. *Let H be a hyper K-algebra and τ is defined in Theorem 7.2. If $0 \in G$, for every $G \in \tau$, then H equipped with τ is a p.t-hyper K-algebra.*

Proof. Let $G \in \tau$ and $x \in G$. Then there exist $I_x \in \vartheta$ such that $U_{I_x}(x) \subseteq G$. Let $I = \bigcap_{x\in G} I_x$. So we have $U_I(x) \subseteq G$, for all $x \in G$. We will claim that $U_I \subseteq G_*$. Let $(z, t) \in U_I$. Hence $z \circ t \subseteq I$ and $t \circ z \subseteq I$. Let $a \in z \circ t$. By $z \circ t \subseteq I$, $a \in I$. Since I is close, $a \circ 0 \subseteq I$ and $0 \circ a \subseteq I$, for every $a \in z \circ t$. Thus $a \in U_I(0)$. By $U_I(x) \subseteq G$ and $0 \in G$, $a \in G$. It means that $z \circ t \subseteq G$. Thus $(z, t) \in G_*$. Hence $U_I \subseteq G_*$. Thus we prove that $G_* \in \eta$. By Lemma 7.3, $G_* \in \tau \times \tau$ and H is a p.t-hyper K-algebra. □

In the next example we show that the condition $0 \in G$ is neccesarry.

Example 7.5. (i). Let $H = \{0, a, b, c\}$, $\vartheta = \{I = \{0, a\}\}$. Then according Example 6.5, $(H, \circ_6)$ is a hyper K-algebra. Consider Theorem 7.2,

$$U_I = \{(0, 0)(0, a)(a, 0)(a, a)\}, U_I(0) = U_I(a) = \{0, a\}, \quad U_I(b) = U_I(c) = \emptyset,$$

so $\tau = \{\emptyset, \{0, a, b\}, \{0, a\}\}$. We can see that H with τ is a p.t-hyper K-algebra.

(ii). Let $H = \{0, a, b, c\}$. According Example 6.5, $(H, \circ_6)$ is a hyper K-algebra. If we define $\vartheta = \{I = \{a, b, c\}\}$, then we get $U_I = \{(a, b)(b, a)(a, c)(c, a)\}$, $U_I(a) = \{a, b, c\}$, $U_I(b) = \{a, b\}$, $U_I(c) = \{a, c\}$ and $U_I(0) = \emptyset$. Thus $\tau = \{\emptyset, H, \{a, b, c\}\}$. We note that

$$\{a, b, c\}_* = \{(a, 0)(a, b)(a, c)(b, 0)(b, a)(c, 0)(c, a)(c, b)\},$$

which is not open on $H \times H$. Hence H eqquipped τ is not a p.t-hyper K-algebra.

Theorem 7.6. *Under conditions of the Theorem 7.2, if* $\{0\} \in \vartheta$, *then* τ *is the discreet topology.*

Proof. Let H satisfies the conditions of Theorem 7.2 and x be an arbitrary element in H. We will show that $\{x\}$ is open. Let $(x, y) \in U_0$. Hence we have $x \circ y \subseteq 0$ and $y \circ x \subseteq 0$. By (I_4), $y = x$. Thus $U_0 = \{(x, x)|x \in H\}$ and so $U_0(x) = \{x\}$. Thus $\{x\}$ is open and τ_1 is the discreet topology. $\square$

Definition 7.7. Let H be a hyper K-algebra. We define

$$V(A) := \{X \subseteq H|A \circ X < V \text{ and } X \circ A < V\},$$

For every $A, V \subseteq H$. If $A = \{a\}$ we denote $V(A)$ by $V(a)$.

Theorem 7.8. *Let* H *be a hyper* K*-algebra,* Ω *be a set which is closed under intersection and*

(C_1) $0 \circ P \subseteq V$,

(C_2) $0 \in (X \circ P) \circ Q$ *implies* $X \subseteq V$,

for every $V \in \Omega$, $P < V$ *and* $Q < V$. *Then* $\theta = \{O \subseteq H|\forall a \in O\ \exists V \in \Omega \text{ st } V(a) \subseteq O\}$ *is a topology on* H *which satisfies the following conditions*

- (i) $0 \in V$, *for every* $V \in \Omega$,
- (ii) $V \in \theta$, *for every* $V \in \Omega$,
- (iii) $V(x)$ *is a neighborhood of* x, *for each* $x \in H$ *and each* $V \in \Omega$,
- (iv) Ω *is a fundamental system of neighborhoods of* 0,
- (v) *if* $W \in \theta$ *and* $A, B \subseteq W$, *then* $A \circ B \subseteq W$,
- (vi) *if* $V \in \Omega$ *and* $x \circ y < V$ *then* $V(x \circ y) \subseteq V$,
- (vii) *every* $V \in \Omega$ *is a hyper* K*-ideal.*

Proof. It is easy to prove that θ is a topology on H.

(i) Let $P \subseteq V$ and $V \in \Omega$. By (C_1), $0 \circ P \subseteq V$ and by (I_3), $0 \in (0 \circ P) \circ (0 \circ P)$. By (C_2) we imply that $0 \in V$.

(ii) Let $p \in V$ and $V \in \Omega$. We claim that $V(p) \subseteq V$. Let $x \in V(p)$. Then $x \circ p < V$ and $p \circ x < V$. By $0 \in (x \circ p) \circ (x \circ p)$ and (C_2), $x \in V$. Therefore, $V(p) \subseteq V$, for all $p \in V$ and $V \in \theta$.

(iii) Suppose that $x \in H$ and $V \in \Omega$. If $y \in V(x)$, then $x \circ y < V$ and $y \circ x < V$.

We prove that $V(y) \subseteq V(x)$. Assume that $z \in V(y)$. Thus $z \circ y < V$ and $y \circ z < V$. By (I_1) and (C_2),

$$0 \in ((x \circ z) \circ (x \circ y)) \circ (y \circ z) \text{ implies } x \circ z \subseteq V \text{ implies } x \circ z < V,$$

$$0 \in ((z \circ x) \circ (y \circ x)) \circ (z \circ y) \text{ implies } z \circ x \subseteq V \text{ implies } z \circ x < V.$$

Hence $z \in V(x)$. Thus $V(y) \subseteq V(x)$, for all $y \in V(x)$ and so $V(x) \in \theta$. By (i) and (I_3), $x \in V(x)$, for each $x \in H$.

(iv) Let V is a neighborhood of 0. Then there is $U \in \Omega$ such that $U(0) \subseteq V$. If $x \in U$, by (C_1), $0 \circ x \subseteq U$. Thus $0 \circ x < U$. On the other hands by (I_{16}), $x \in x \circ 0$, so $x \in U(0)$. Therefore, $U \subseteq U(0) \subseteq V$.

(v) Suppose that $W \in \Omega$ and $A, B \subseteq W$. By $0 \in ((A \circ B) \circ A) \circ (0 \circ B)$ and (C_1) and (C_2), $A \circ B \subseteq W$.

(vi) Let $V \in \Omega$ and $x \circ y < V$. If $z \in V(x \circ y)$, then $z \circ (x \circ y) < V$. Hence $0 \in (z \circ (x \circ y)) \circ V$. By $x \circ y < V$ and (C_2), $z \in V$. Therefore, $V(x \circ y) \subseteq V$.

(vii) Let $V \in \Omega$. By (i), $0 \in V$. Let $x \circ y < V$ and $y \in V$. Thus $y < V$. By $0 \in (x \circ (x \circ y)) \circ y$ and (C_2), $x \in V$. Hence V is a hyper K-ideal. □

Theorem 7.9. *Let H be a hyper K-algebra and Ω be a set which is closed under intersection, satisfies the Theorem 7.8 and*

$$x \circ y \subseteq O \quad \textit{implies} \quad \textit{there exists } V \in \Omega \textit{ such that } V(x) \subseteq O \textit{ and } V(y) \subseteq O,$$

for all $O \in \theta$. Then H equipped with θ is a p.t-hyper K-algebra.

Proof. Let $O \in \theta$ and $x \circ y \subseteq O$. By hypothesis, there exists $V \in \Omega$ such that $V(x) \subseteq O$ and $V(y) \subseteq O$. By Theorem 7.8 (iii),(v), $V(x)$ and $V(y)$ are respectively neighborhoods of x and y also $V(x) \circ V(y) \subseteq O$. Hence by Theorem 5.3, H is a p.t-hyper K-algebra. □

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