ON CONJUGATIONS FOR BANACH SPACES

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Abstract

In this paper we introduce a conjugation C on a complex Banach space $\mathcal{X}$ and define complex symmetric operators. We show some spectral properties of complex symmetric operators.

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An antilinear operator C is said to be *conjugation* if $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. T. Takagi in [14] studied antilinear eigenvalue problem. V.I. Godic and I.E. Lucenko in [9] showed that U is unitary if and only if there exist conjugations C, J such that $U = CJ$. S.R. Garcia and M. Putinar showed that, for conjugations C, J on a Hilbert space, CJ is both C-symmetric and J-symmetric. See Lemma 1 of [7]. Now we have many research about conjugations of Hilbert spaces. For examples, see [6], [7], [10] and [8]. In this paper we introduce a conjugation on a Banach space and show some properties concerning with conjugations.

2 Conjugations on Banach spaces

Let $\mathcal{X}$ be a complex Banach space, $\| \cdot \|$ be the norm of $\mathcal{X}$ and $B(\mathcal{X})$ be the set of all bounded linear operators on $\mathcal{X}$. For an operator $T \in B(\mathcal{X})$, the spectrum, the point spectrum, the approximate point spectrum and the surjective spectrum of T are denoted by $\sigma(T), \sigma_p(T), \sigma_a(T), \sigma_s(T)$, respectively. It holds $\sigma(T) = \sigma_a(T) \bigcup \sigma_s(T)$, $\sigma_s(T) = \sigma_a(T^*)$ and $\sigma_a(T) = \sigma_s(T^*)$, where T^* is the dual operator T on the dual space $\mathcal{X}^*$. See [1] for details. $\ker(T)$ and $R(T)$ denote the kernel and the range of T, respectively. For a subset M of $\mathbb{C}$, $M^* = \{\overline{z} : z \in M\}$. For an operator C on $\mathcal{X}$, we define a conjugation as follows.

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Definition 2.1 *Let $\mathcal{X}$ be a complex Banach space. An operator $C : \mathcal{X} \to \mathcal{X}$ is said to be a conjugation if C satisfies*

$$(1) \qquad C^2 = I,\ \|C\| \le 1,\ C(x+y) = Cx + Cy,\ C(\lambda x) = \overline{\lambda} Cx \quad (\forall x, y \in \mathcal{X}, \lambda \in \mathbb{C}),$$

where I is the identity operator on $\mathcal{X}$ and $\|C\| = \sup_{\|x\| \le 1} \{\|Cx\| : \ x \in \mathcal{X}\}$.

Next theorem shows that if the space $\mathcal{X}$ is a Hilbert space and C satisfies condition (1), then C is a conjugation as follows.

Theorem 2.2 *If C satisfies condition (1) on a complex Hilbert space $\mathcal{H}$, then $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$, i.e., C is a conjugation on $\mathcal{H}$.*

Proof. Let $x, y \in \mathcal{H}, \alpha \in \mathbb{R}$ and let $Cy = z$. Since

$$\begin{aligned} \|Cx + \alpha z\| &= \|C(x + \alpha Cz)\| \\ &\le \|x + \alpha Cz\| = \|C(Cx + \alpha z)\| \le \|Cx + \alpha z\|, \end{aligned}$$

we have

$$\|Cx + \alpha z\| = \|x + \alpha Cz\|.$$

By taking square, we have

$$\begin{aligned} &\|Cx\|^2 + \alpha\left(\langle Cx, z\rangle + \langle z, Cx\rangle\right) + \alpha^2\|z\|^2 \\ &= \|x\|^2 + \alpha\left(\langle x, Cz\rangle + \langle Cz, x\rangle\right) + \alpha^2\|Cz\|^2. \end{aligned}$$

Hence $\|Cx\| = \|x\|$ and

$$\begin{aligned} \mathrm{Re}\ \langle Cx, Cy\rangle &= \mathrm{Re}\ \langle Cx, z\rangle \\ &= \mathrm{Re}\ \langle Cz, x\rangle = \mathrm{Re}\ \langle C^2 y, x\rangle = \mathrm{Re}\ \langle y, x\rangle. \end{aligned}$$

By taking ix instead of x, we have

$$\begin{aligned} \mathrm{Re}\{-i\langle Cx, Cy\rangle\} &= \mathrm{Re}\ \langle Cix, Cy\rangle \\ &= \mathrm{Re}\ \langle y, ix\rangle = \mathrm{Re}\{-i\langle y, x\rangle\}. \end{aligned}$$

Hence

$$\mathrm{Im}\ \langle Cx, Cy\rangle = \mathrm{Im}\ \langle y, x\rangle.$$

Thus $\langle Cx, Cy\rangle = \langle y, x\rangle$. □

Theorem 2.3 *Let C be a conjugation on a complex Banach space $\mathcal{X}$. Then $\|Cx\| = \|x\|$ for all $x \in \mathcal{X}$.*

Proof. Since $\|C\| \le 1$, it holds $\|x\| = \|C^2 x\| \le \|C\|\,\|Cx\| \le \|Cx\|$. Hence $\|x\| \le \|Cx\|$. Therefore $\|Cx\| \le \|C^2 x\| = \|x\|$ and $\|Cx\| = \|x\|$. □

Example 2.4 *For a complex Hilbert space $\mathcal{H}$, let $\mathcal{X} = B(\mathcal{H})$ and C, J be conjugations on $\mathcal{H}$. Let M_{CJ} is defined by*

$$M_{CJ}(T) = CTJ.$$

Then M_{CJ} is a conjugation on $\mathcal{X}$.

Proof. It is clear that $M_{CJ} : B(\mathcal{H}) \longrightarrow B(\mathcal{H})$. Since $C^2 = J^2 = I$, it holds $M_{CJ}^2(T) = T$ for all $T \in B(\mathcal{X})$. Next it holds

$$M_{CJ}(\lambda T) = C(\lambda T)J = \overline{\lambda} CTJ = \overline{\lambda} M_{CJ}(T).$$

Since $\|M_{CJ}(T)\| \leq \|T\|$ for all $T \in B(\mathcal{X})$, $\|M_{CJ}\| \leq 1$. Since CJ is in $B(\mathcal{X})$ and $\|CJ\| = 1$, we have $M_{CJ}(CJ) = I$. Hence, $\|M_{CJ}\| = 1$ and M_{CJ} is a conjugation on $\mathcal{X}$. □

For a complex Banach space $\mathcal{X}$, let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$ and the dual operator of $T \in B(\mathcal{X})$ is denoted by T^*.

Definition 2.5 *For a conjugation C on a Banach space $\mathcal{X}$, the dual operator $C^* : \mathcal{X}^* \rightarrow \mathcal{X}^*$ of C is defined by*

$$(C^*(f))(x) = \overline{f(Cx)} \quad (x \in \mathcal{X}, f \in \mathcal{X}^*).$$

Then we have following result.

Theorem 2.6 *Let C be a conjugation on a complex Banach space $\mathcal{X}$. Then C^* is a conjugation on $\mathcal{X}^*$.*

Proof. It is clear that $C^{*2} = I^*$, $C^*(f+g) = C^*(f) + C^*(g)$ $(\forall f, g \in \mathcal{X}^*)$. For $\lambda \in \mathbb{C}, x \in \mathcal{X}$, it holds

$$(C^*(\lambda f))(x) = \overline{\lambda}\ \overline{f(Cx)} = \overline{\lambda}\ (C^* f)(x).$$

Hence $C^*(\lambda f) = \overline{\lambda}\ C^*(f)$. Let $f \in \mathcal{X}^*$. Then $|(C^* f)(x)| = |\overline{f(Cx)}| \leq \|f\| \|Cx\| \leq \|f\| \|x\|$. Hence $\|C^* f\| \leq \|f\|$ and $\|C^*\| \leq 1$. □

Hence we say C^* *the dual conjugation* of C.

First we show spectral properties of complex symmetric operators.

Theorem 2.7 *Let C be a conjugation on a complex Banach space $\mathcal{X}$. Then $\sigma_a(CTC) = \sigma_a(T)^*, \sigma_p(CTC) = \sigma_p(T)^*, \sigma_s(CTC) = \sigma_s(T)^*$ and $\sigma(CTC) = \sigma(T)^*$.*

Proof. Let $z \in \sigma_a(CTC)$ and $\{x_n\}$ be a sequence of unit vectors such that $(CTC - z)x_n \to 0$. Then since $C(T - \overline{z})Cx_n \to 0$ and $\|Cx_n\| = 1$, we have $\overline{z} \in \sigma_a(T)$. Hence $\sigma_a(CTC) \subset \sigma_a(T)^*$. Therefore, $\sigma_a(T) = \sigma_a(C^2TC^2) \subset \sigma_a(CTC)^*$ and $\sigma_a(CTC) = \sigma_a(T)^*$. Similarly, we have $\sigma_p(CTC) = \sigma_p(T)^*$. Let $z \notin \sigma_s(CTC)$ and $x \in \mathcal{X}$. Then there exists $y \in \mathcal{X}$ such that $(CTC - z)y = Cx$. Hence $(T - \overline{z})Cy = C(CTC - z)y = C^2x = x$. Hence $\sigma_s(CTC) \subset \sigma_s(T)^*$ The converse is similar. Hence $\sigma_s(CTC) = \sigma_s(T)^*$. Also, $\sigma(CTC) = \sigma_a(CTC) \cup \sigma_s(CTC) = \sigma_a(T)^* \cup \sigma_s(T)^* = \sigma(T)^*$. □

Next we introduce numerical range of Banach space operator.

Definition 2.8 *Let* Π *be the set*

$$\Pi := \{(x, f) \in \mathcal{X} \times \mathcal{X}^* \,:\, \|f\| = f(x) = \|x\| = 1\}.$$

For an operator $T \in B(\mathcal{X})$*, the numerical range* $V(T)$ *of* T *is given by*

$$V(T) = \{f(Tx) \,:\, (x, f) \in \Pi\}.$$

Hence, *normal* and *hyponormal* are defined as follows.

(1) T is called *Hermitian* and *positive* (denoted by $T \geq 0$) if $V(T) \subset \mathbb{R}$ and $V(T) \subset [0, \infty)$, respectively.

(2) T is called *normal* if there exist Hermitian operators H, K such that $HK = KH$, $T = H + iK$.

(3) T is called *hyponormal* if there exist Hermitian operators H, K such that $T = H + iK,\ i(HK - KH) \geq 0$.

Let $T \in B(\mathcal{X})$. If $T = H + iK$ for some Hermitian H and K, then H and K are unique. Hence we denote $H - iK$ by $\overline{T}$. Let $T^* \in B(\mathcal{X}^*)$ be the dual operator of T. Hence if $T = H + iK$, then $T^* = H^* + iK^*$.

Definition 2.9 *Let* $T \in B(\mathcal{X})$ *be* $T = H + iK$ *for some Hermitian* H *and* K*. Let* C *be a conjugation on* $\mathcal{X}$*. Then* T *is said to be* C*-symmetric if* $C\overline{T}C = T$*.*

Theorem 2.10 *Let* $T = H + iK$ *for some Hermitian* H *and* K*. Let* C *be a conjugation on* $\mathcal{X}$ *and* T *is* C*-symmetric. Then* T *is invertible if and only if* $\overline{T}$ *is invertible.*

Proof. Let, for a conjugation C, $C\overline{T}C = T$ and T be invertible. Then we have $\overline{T}CT^{-1}C = CTC\,CT^{-1}C = CTT^{-1}C = C^2 = I = CT^{-1}CT$. Hence $\overline{T}$ is invertible. Converse is clear. □

Theorem 2.11 *Let* $\mathcal{X}$ *be a complex Banach space. If an operator* $T = H + iK$ *is complex symmetric, then* $\sigma_p(T) = \sigma_p(\overline{T})^*$*,* $\sigma_a(T) = \sigma_a(\overline{T})^*, \sigma_s(T) = \sigma_s(\overline{T})^*$ *and* $\sigma(T) = \sigma(\overline{T})^*$*.*

Proof. By Theorem 2.7, we have $\sigma_p(C\overline{T}C) = \sigma_p(\overline{T})^*$. Since $C\overline{T}C = T$ for some conjugation, we have $\sigma_p(T) = \sigma_p(T)^*$. Others are similar. □

Definition 2.12 *An operator $T = H + iK \in B(\mathcal{X})$ is said to be an ExB-operator if there exists $M > 0$ such that*

$$\|e^{z\overline{T}} \cdot e^{-\overline{z}T}\| \le M \quad \textit{for all } z \in \mathbb{C}.$$

We have $\|e^{z\overline{T}} \cdot e^{-\overline{z}T}\| \le M$ $(\forall z \in \mathbb{C})$ if and only if $\|e^{z\overline{T}}x\| \le M\|e^{\overline{z}T}x\|$ $(\forall x \in \mathcal{X}, \forall z \in \mathbb{C})$. For $T = H + iK \in B(\mathcal{H})$ (Hilbert space operator case), T is an ExB-operator if and only if $\|(e^{zT})^*x\| \le M\|e^{zT}x\|$ $(\forall x \in \mathcal{H}, \forall z \in \mathbb{C})$. It is easy to see that if T is an ExB-operator, then so is $aT + b$ for all $a, b \in \mathbb{C}$. When $M = 1$, K. Mattila in [13] called **-hyponormal.*

Proposition 2.13 (Lemma 2 of [5]).
If $T = H + iK$ is an ExB-operator and $Tx = 0$, then $\overline{T}x = 0$.

Since $(e^{z\overline{T}} \cdot e^{-\overline{z}T})^* = e^{-\overline{z}T^*} \cdot e^{z\overline{T}^*}$, if T is an ExB-operator, then so is $\overline{T}^*$.

Theorem 2.14 *Let $\mathcal{X}$ be a complex Banach space and C be a conjugation on $\mathcal{X}$. If T is an ExB-operator on $\mathcal{X}$ and C-symmetric, then* $\ker(T - \lambda) = C\ker(T - \lambda)$ *for all $\lambda \in \mathbb{C}$.*

Proof. Let $Tx = \lambda x$. Since $aT + b$ is an ExB-operator for all $a, b \in \mathbb{C}$ and $(T - \lambda)x = 0$, by Proposition 2.13 it holds $\overline{T}x = \overline{\lambda}x$. Hence $\overline{\lambda}x = \overline{T}x = (CTC)x = C(TCx)$ and it holds $T(Cx) = C^2T(Cx) = C(\overline{\lambda}x) = \lambda Cx$. Hence $C\ker(T - \lambda) \subset \ker(T - \lambda)$. Also, we have $\ker(T - \lambda) = C^2\ker(T - \lambda) \subset C\ker(T - \lambda)$. Hence $\ker(T - \lambda) = C\ker(T - \lambda)$. □

For a study of properties of a complex symmetric ExB-operator, we recall from [2] and [3] the construction of a larger space $\mathcal{X}^\circ$ from a given Banach space $\mathcal{X}$. Then the mapping $T \to T^\circ$ is an isometric isomorphism of $B(\mathcal{X})$ onto a closed subalgebra of $B(\mathcal{X}^\circ)$ as follows: Let Lim be fixed Banach limit on the space of all bounded sequences of complex numbers with the norm $\|\{\lambda_n\}\| = \sup\{|\lambda_n| : n \in \mathbb{N}\}$. Let $\tilde{\mathcal{X}}$ be the space of all bounded sequences $\{x_n\}$ of $\mathcal{X}$. Let N be the subspace of $\tilde{\mathcal{X}}$ consisting of all bounded sequences $\{x_n\}$ with $\mathrm{Lim}\,\|x_n\|^2 = 0$. The space $\mathcal{X}^\circ$ is defined as the completion of the quotient space $\tilde{\mathcal{X}}/N$ with respect to the norm $\|\{x_n\} + N\| = (\mathrm{Lim}\,\|x_n\|^2)^{\frac{1}{2}}$. Operator T' is defined by $T'(\{x_n\} + N) = \{Tx_n\} + N$ on $\tilde{\mathcal{X}}/N$. The operator T° is defined by the unique extension of T' on $\mathcal{X}^\circ$. Then the following results hold:

$$\sigma(T) = \sigma(T^\circ),\ \sigma_a(T) = \sigma_a(T^\circ) = \sigma_p(T^\circ) \ \text{ and } \ \overline{\mathrm{co}}\,V(T) = V(T^\circ),$$

where $\overline{\mathrm{co}}\,V(T)$ is the closed convex hull of $V(T)$. See [2] and [3] for details. Therefore, if T is Hermitian, normal or hyponormal, then so is T°, respectively. Since the mapping $T \to T^\circ$ is an isometric isomorphism of $B(\mathcal{X})$ onto a closed subalgebra of $B(\mathcal{X}^\circ)$, if T is an ExB-operator, then so is T°.

Let C be a conjugation on $\mathcal{X}$. The operator C' is defined by $C'(\{x_n\} + N) = \{Cx_n\} + N$ on $\tilde{\mathcal{X}}/N$ and we define C° as the unique extension of C' on $\mathcal{X}^\circ$. Then it is easy to see that

$$C^{\circ 2} = I^\circ,\ \|C^\circ\| = 1,\ C^\circ(x^\circ + y^\circ) = C^\circ x^\circ + C^\circ y^\circ,\ C^\circ(\lambda x^\circ) = \overline{\lambda}\, C^\circ x^\circ \ \ (\forall x^\circ, y^\circ \in \mathcal{X}^\circ, \lambda \in \mathbb{C}).$$

Theorem 2.15 *With the above assertion, if C is a conjugation, then so is C° on $\mathcal{X}^\circ$.*

Since $(CTC)'(\{x_n\} + N) = \{CTCx_n\} + N = C'(\{TCx_n\} + N) = C'T'C'(\{x_n\} + N)$, it holds $(CTC)^\circ = C^\circ T^\circ C^\circ$. Hence the following result holds.

Theorem 2.16 *With the above assertion, if T is C-symmetric on $\mathcal{X}$, then T° is C°-symmetric on $\mathcal{X}^\circ$.*

For the final result, we introduce *orthogonality* of Banach space as follows.

Definition 2.17 *Let M be a subspace of $\mathcal{X}$. A vector x is orthogonal to M if*

$$\|m\| \leq \|m + x\| \quad (\textit{for all } m \in M).$$

Then we denote by $M \perp x$.

For a subspace $M \subset \mathcal{X}$, let $M^\perp$ be the set $\{x \in \mathcal{X} : M \perp x\}$. Let M, N be subspaces of $\mathcal{X}$. If, for all $n \in N$, $M \perp n$, then we denote $M \perp N$.

Theorem 2.18 *If a subspace M is invariant for a conjugation C, then $M^\perp$ is invariant for C.*

Proof. Let $x \in M^\perp$ and $m \in M$ be arbitrary. Then

$$\|m\| = \|Cm\| \leq \|Cm + x\| = \|C(Cm + x)\| = \|m + Cx\|.$$

Hence $Cx \in M^\perp$. $\square$

Hence we have following corollary.

Corollary 2.19 *Let $T = H + iK$ be an ExB-operator. If T is C-symmetric, then $\ker(T)^\perp$ is invariant for C.*

Proposition 2.20 (Theorem 20.7, [4]).
Let $\mathcal{X}$ be a reflexive Banach space and $x \in \mathcal{X}$ be $\|x\| = 1$. Then there exists $f \in \mathcal{X}^*$ such that $\|f\| = f(x) = 1$ and $H^* f = 0$ for all Hermitian operators H for which $Hx = 0$.

Proposition 2.21 (Lemma 20.3, [4]).
It holds that $\ker(T) \perp R(T)$ if and only if for any unit vector $x \in \ker(T)$ there exists $f \in \mathcal{X}^*$ such that $\|f\| = f(x) = 1$ and $T^* f = 0$.

Theorem 2.22 *Let Let $\mathcal{X}$ be a reflexive Banach space and $T = H + iK$ be an ExB-operator. Then $\ker(T) \perp R(T)$.*

Proof. Let $x \in \ker(T)$ be a unit vector. Since $T = H + iK$ is an ExB-operator, by Proposition 2.13 we have $Hx = Kx = 0$. Since $\mathcal{X}$ is reflexive, by Proposition 2.20 there exists $f \in \mathcal{X}^*$ such that $\|f\| = f(x) = 1$ and $T^* f = 0$. Hence by Proposition 2.21 it holds $\ker(T) \perp R(T)$. □

In [11], K. Mattila proved the following result for a normal operator.

Proposition 2.23 (Propositions 3.7 and 3.9, Corollary 3.8, [11]).
Let T be normal and $\lambda, \mu \in \mathbb{C}$ be $\lambda \neq \mu$.
(1) *If $\{x_n\}$ is a sequence of unit vectors such that $Tx_n \to 0$, then $1 \leq \liminf \|x_n + Ty_n\|$, for any bounded sequence $\{y_n\}$ of $\mathcal{X}$.*
(2) *If $\{x_n\}$ is a sequence of unit vectors such that $(T-\lambda)x_n \to 0$, then $1 \leq \liminf \|x_n + y_n\|$, for any bounded sequence $\{y_n\}$ such that $(T-\mu)y_n \to 0$.*
In particular, $\ker(T-\lambda) \perp \ker(T-\mu)$.
(3) *For a bounded sequence $\{x_n\}$, if $T^2 x_n \to 0$, then $Tx_n \to 0$.*

We show the similar result. For the completeness, we give a proof. We need the following definition and result.

Definition 2.24 *A Banach space $\mathcal{X}$ is said to be uniformly convex if, for every $\epsilon > 0$, there exists a number $\delta > 0$ such that, for all $x, y \in \mathcal{X}$, the conditions*

$$\|x\| = \|y\| = 1, \quad \|x - y\| \ \geq \ \epsilon \quad \textit{imply} \quad \|\frac{x+y}{2}\| \ \leq \ 1 - \delta.$$

It's well known that $\mathcal{X}$ is uniformly convex, then $\mathcal{X}$ is reflexive, i.e., $\mathcal{X}^{**} = \mathcal{X}$. Then we have the following result.

Proposition 2.25 (Theorem 4, [3]).
$\mathcal{X}$ is uniformly convex if and only if $\mathcal{X}^\circ$ is uniformly convex.

Theorem 2.26 *Let T be an ExB-operator on a uniformly convex Banach space $\mathcal{X}$ and $\lambda, \mu \in \mathbb{C}$ be $\lambda \neq \mu$.*
(1) *If $\{x_n\}$ is a sequence of unit vectors such that $Tx_n \to 0$, then $1 \leq \liminf \|x_n + Ty_n\|$, for any bounded sequence $\{y_n\}$ of $\mathcal{X}$.*
(2) *If $\{x_n\}$ is a sequence of unit vectors such that $(T-\lambda)x_n \to 0$, then $1 \leq \liminf \|x_n + y_n\|$, for any bounded sequence $\{y_n\}$ such that $(T-\mu)y_n \to 0$.*
In particular, $\ker(T-\lambda) \perp \ker(T-\mu)$.
(3) *For a bounded sequence $\{x_n\}$, if $T^2 x_n \to 0$, then $Tx_n \to 0$.*

Proof. Let $\mathcal{X}^\circ$ be the larger Banach space of $\mathcal{X}$ and T° be the extension of T on $\mathcal{X}^\circ$ as a previous way. Then $\mathcal{X}^\circ$ is uniformly convex and T° is an ExB-operator. And by Theorem 2.22 it holds $\ker(T^\circ) \perp R(T^\circ)$. We may assume that all vectors x_n and y_n of (1), (2) and (3) are unit. Put $x^\circ = \{x_n\} + N$. Then $\|x^\circ\| = 1$ and $x^\circ \in \ker(T^\circ)$.
(1) Since it holds $\ker(T^\circ) \perp R(T^\circ)$ by Theorem 2.22, we have $1 = \|x^\circ\| \leq \|x^\circ + T^\circ y^\circ\|$.

Assume that $\liminf \|x_n + Ty_n\| < 1$. Then there exist subsequences $\{x_{n_j}\}, \{y_{n_j}\}$ such that $\lim_{j\to\infty} \|x_{n_j} + Ty_{n_j}\| = \alpha < 1$. Let $x_1^\circ = \{x_{n_j}\} + N$, $y_1^\circ = \{y_{n_j}\} + N$. Then we have $\|x_1^\circ\| = 1$, $T^\circ(x_1^\circ) = 0$ and $\|x_1^\circ + T^\circ y_1^\circ\| = \alpha < 1 = \|x_1^\circ\|$. It's a contradiction. Hence $\liminf \|x_n + Ty_n\| \geq 1$ and it completes (1).
(2) Since $\lambda \neq \mu$, $\{(\mu - \lambda)^{-1} y_n\}$ is a bounded sequence, by (1) we have

$$1 \leq \liminf \|x_n + (\mu - \lambda)^{-1}(T - \lambda)y_n\|$$

$$= \liminf \|x_n + y_n + (\mu - \lambda)^{-1}(T - \mu)y_n\| = \liminf \|x_n + y_n\|.$$

By previous result, it is easy to see $\ker(T - \lambda) \perp \ker(T - \mu)$ and it completes (2).
(3) Let $T^2 x_n \to 0$. Assume $Tx_n \not\to 0$. Then there exist $\epsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|Tx_{n_k}\| \geq \epsilon$ $(k \in \mathbb{N})$. Let $w_k = \|Tx_{n_k}\|^{-1} Tx_{n_k}$. Then $\|Tw_k\| \leq \epsilon^{-1}\|T^2 x_{n_k}\|$. Hence $Tw_k \to 0$ and by (1) it holds $1 \leq \liminf \|w_k + Tu_k\|$ for any bounded sequence $\{u_k\}$. Taking $u_k = -\|Tx_{n_k}\|^{-1} x_{n_k}$, it's a contradiction. Hence we have $Tx_n \to 0$ and it completes (3). □

Remark 2.27 *In the case of conjugations C, J on a Hilbert space $\mathcal{H}$, $U = CJ$ is a unitary operator. Hence $U^* = U^{-1}$ and it holds $CUC = JC = U^{-1} = U^* = JUJ$. In the case of conjugations C, J on a Banach space $\mathcal{X}$, we have only $CUC = JC = U^{-1} = JUJ$.*

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