ON CONJUGATIONS FOR BANACH SPACES

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Abstract

In this paper we introduce a conjugation $C$ on a complex Banach space $X$ and define complex symmetric operators. We show some spectral properties of complex symmetric operators.

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An antilinear operator $C$ is said to be conjugation if $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. T. Takagi in [14] studied antilinear eigenvalue problem. V.I. Godic and I.E. Lucenko in [9] showed that $U$ is unitary if and only if there exist conjugations $C, J$ such that $U = CJ$. S.R. Garcia and M. Putinar showed that, for conjugations $C, J$ on a Hilbert space, $CJ$ is both $C$-symmetric and $J$-symmetric. See Lemma 1 of [7]. Now we have many research about conjugations of Hilbert spaces. For examples, see [6], [7], [10] and [8]. In this paper we introduce a conjugation on a Banach space and show some properties concerning with conjugations.

2 Conjugations on Banach spaces

Let $\mathcal{X}$ be a complex Banach space, $\| \cdot \|$ be the norm of $\mathcal{X}$ and $B(\mathcal{X})$ be the set of all bounded linear operators on $\mathcal{X}$. For an operator $T \in B(\mathcal{X})$, the spectrum, the point spectrum, the approximate point spectrum and the surjective spectrum of $T$ are denoted by $\sigma(T), \sigma_p(T), \sigma_a(T), \sigma_s(T)$, respectively. It holds $\sigma(T) = \sigma_a(T) \cup \sigma_s(T)$, $\sigma_a(T) = \sigma_a(T^*)$ and $\sigma_a(T) = \sigma_s(T^*)$, where $T^*$ is the dual operator $T$ on the dual space $\mathcal{X}^*$. See [1] for details. ker$(T)$ and R$(T)$ denote the kernel and the range of $T$, respectively. For a subset $M$ of $\mathbb{C}$, $M^* = \{ z \in M \}$. For an operator $C$ on $\mathcal{X}$, we define a conjugation as follows.
Definition 2.1 Let $\mathcal{X}$ be a complex Banach space. An operator $C : \mathcal{X} \to \mathcal{X}$ is said to be a conjugation if $C$ satisfies

\[(1) \quad C^2 = I, \quad \|C\| \leq 1, \quad C(x + y) = Cx + Cy, \quad C(\lambda x) = \overline{x} Cx \quad (\forall x, y \in \mathcal{X}, \lambda \in \mathbb{C}),\]

where $I$ is the identity operator on $\mathcal{X}$ and $\|C\| = \sup \{ \|Cx\| : x \in \mathcal{X} \}$.

Next theorem shows that if the space $\mathcal{X}$ is a Hilbert space and $C$ satisfies condition (1), then $C$ is a conjugation as follows.

Theorem 2.2 If $C$ satisfies condition (1) on a complex Hilbert space $\mathcal{H}$, then $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$, i.e., $C$ is a conjugation on $\mathcal{H}$.

Proof. Let $x, y \in \mathcal{H}, \alpha \in \mathbb{R}$ and let $Cy = z$. Since

$$\|Cx + \alpha z\| = \|C(x + \alpha Cz)\| \leq \|x + \alpha Cz\| = \|C(x + \alpha z)\| \leq \|Cx + \alpha z\|,$$

we have

$$\|Cx + \alpha z\| = \|x + \alpha Cz\|.$$ 

By taking square, we have

$$\|Cx\|^2 + \alpha (\langle Cx, z \rangle + \langle z, Cx \rangle) + \alpha^2 \|z\|^2 = \|x\|^2 + \alpha (\langle x, Cz \rangle + \langle Cz, x \rangle) + \alpha^2 \|Cz\|^2.$$

Hence $\|Cx\| = \|x\|$ and

$$\text{Re} \langle Cx, Cy \rangle = \text{Re} \langle Cx, z \rangle = \text{Re} \langle Cz, x \rangle = \text{Re} \langle C^2 y, x \rangle = \text{Re} \langle y, x \rangle.$$ 

By taking $ix$ instead of $x$, we have

$$\text{Re}\{-i\langle Cx, Cy \rangle\} = \text{Re} \langle Cix, Cy \rangle = \text{Re} \langle y, ix \rangle = \text{Re}\{-i\langle y, x \rangle\}.$$ 

Hence

$$\text{Im} \langle Cx, Cy \rangle = \text{Im} \langle y, x \rangle.$$ 

Thus $\langle Cx, Cy \rangle = \langle y, x \rangle$. □

Theorem 2.3 Let $C$ be a conjugation on a complex Banach space $\mathcal{X}$. Then $\|Cx\| = \|x\|$ for all $x \in \mathcal{X}$.

Proof. Since $\|C\| \leq 1$, it holds $\|x\| = \|C^2 x\| \leq \|C\| \|Cx\| \leq \|Cx\|$. Hence $\|x\| \leq \|Cx\|$. Therefore $\|Cx\| \leq \|C^2 x\| = \|x\|$ and $\|Cx\| = \|x\|$. □
Example 2.4 For a complex Hilbert space $\mathcal{H}$, let $\mathcal{X} = B(\mathcal{H})$ and $C, J$ be conjugations on $\mathcal{H}$. Let $M_{CJ}$ be defined by

$$M_{CJ}(T) = CTJ.$$ 

Then $M_{CJ}$ is a conjugation on $\mathcal{X}$.

Proof. It is clear that $M_{CJ} : B(\mathcal{H}) \longrightarrow B(\mathcal{H})$. Since $C^2 = J^2 = I$, it holds $M_{CJ}^2(T) = T$ for all $T \in B(\mathcal{X})$. Next it holds

$$M_{CJ}(\lambda T) = C(\lambda T)J = \bar{\lambda}CTJ = \bar{\lambda}M_{CJ}(T).$$

Since $||M_{CJ}(T)|| \leq ||T||$ for all $T \in B(\mathcal{X})$, $||M_{CJ}|| \leq 1$. Since $CJ$ is in $B(\mathcal{X})$ and $||CJ|| = 1$, we have $M_{CJ}(CJ) = I$. Hence, $||M_{CJ}|| = 1$ and $M_{CJ}$ is a conjugation on $\mathcal{X}$. $\square$

For a complex Banach space $\mathcal{X}$, let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$ and the dual operator of $T \in B(\mathcal{X})$ is denoted by $T^*$.

Definition 2.5 For a conjugation $C$ on a Banach space $\mathcal{X}$, the dual operator $C^* : \mathcal{X}^* \longrightarrow \mathcal{X}^*$ of $C$ is defined by

$$(C^*(f))(x) = \overline{f(Cx)} \quad (x \in \mathcal{X}, f \in \mathcal{X}^*).$$

Then we have following result.

Theorem 2.6 Let $C$ be a conjugation on a complex Banach space $\mathcal{X}$. Then $C^*$ is a conjugation on $\mathcal{X}^*$.

Proof. It is clear that $C^{*2} = I^*$, $C^*(f + g) = C^*(f) + C^*(g) \quad (\forall f, g \in \mathcal{X}^*)$. For $\lambda \in \mathbb{C}, x \in \mathcal{X}$, it holds

$$(C^*(\lambda f))(x) = \overline{\lambda f(Cx)} = \overline{\lambda} \overline{(C^*f)(x)}. $$

Hence $C^*(\lambda f) = \overline{\lambda} C^*(f)$. Let $f \in \mathcal{X}^*$. Then $||(C^*f)(x)|| = ||\overline{f(Cx)}|| \leq ||f||||Cx|| \leq ||f||||x||$. Hence $||C^*f|| \leq ||f||$ and $||C^*|| \leq 1$. $\square$

Hence we say $C^*$ the dual conjugation of $C$.

First we show spectral properties of complex symmetric operators.

Theorem 2.7 Let $C$ be a conjugation on a complex Banach space $\mathcal{X}$. Then $\sigma_a(CTC) = \sigma_a(T)^*, \sigma_p(CTC) = \sigma_p(T)^*, \sigma_s(CTC) = \sigma_s(T)^*$ and $\sigma(CTC) = \sigma(T)^*$. 

Proof. Let \( z \in \sigma_a(CTC) \) and \( \{x_n\} \) be a sequence of unit vectors such that \((CTC - z)x_n \to 0\). Then since \( C(T - \bar{z})Cx_n \to 0 \) and \( \|Cx_n\| = 1 \), we have \( z \in \sigma_a(T) \). Hence \( \sigma_a(CTC) \subseteq \sigma_a(T)^* \). Therefore, \( \sigma_a(T) = \sigma_a(C^2TC^2) \subseteq \sigma_a(CTC)^* \) and \( \sigma_a(CTC) = \sigma_a(T)^* \). Similarly, we have \( \sigma_p(CTC) = \sigma_p(T)^* \). Let \( z \notin \sigma_a(CTC) \) and \( x \in \mathcal{X} \). Then there exists \( y \in \mathcal{X} \) such that \((CTC - z)y = Cx\). Hence \((T - \bar{z})Cy = C(CTC - z)y = C^2x = x\). Hence \( \sigma_a(CTC) \subseteq \sigma_a(T)^* \). The converse is similar. Hence \( \sigma_a(CTC) = \sigma_a(T)^* \). Also, \( \sigma(CTC) = \sigma_a(CTC) \cup \sigma_s(CTC) = \sigma_a(T)^* \cup \sigma_s(T)^* = \sigma(T)^* \). \( \square \)

Next we introduce numerical range of Banach space operator.

**Definition 2.8** Let \( \Pi \) be the set

\[
\Pi := \{(x, f) \in \mathcal{X} \times \mathcal{X}^* : \|f\| = f(x) = \|x\| = 1\}.
\]

For an operator \( T \in B(\mathcal{X}) \), the numerical range \( V(T) \) of \( T \) is given by

\[
V(T) = \{f(Tx) : (x, f) \in \Pi\}.
\]

Hence, normal and hyponormal are defined as follows.

1. \( T \) is called Hermitian and positive (denoted by \( T \geq 0 \)) if \( V(T) \subset \mathbb{R} \) and \( V(T) \subset [0, \infty) \), respectively.
2. \( T \) is called normal if there exist Hermitian operators \( H, K \) such that \( HK = KH \), \( T = H + iK \).
3. \( T \) is called hyponormal if there exist Hermitian operators \( H, K \) such that \( T = H + iK, \ i(HK - KH) \geq 0 \).

Let \( T \in B(\mathcal{X}) \). If \( T = H + iK \) for some Hermitian \( H \) and \( K \), then \( H \) and \( K \) are unique. Hence we denote \( H - iK \) by \( \overline{T} \). Let \( T^* \in B(\mathcal{X}^*) \) be the dual operator of \( T \). Hence if \( T = H + iK \), then \( T^* = H^* + iK^* \).

**Definition 2.9** Let \( T \in B(\mathcal{X}) \) be \( T = H + iK \) for some Hermitian \( H \) and \( K \). Let \( C \) be a conjugation on \( \mathcal{X} \). Then \( T \) is said to be \( C \)-symmetric if \( CTC = T \).

**Theorem 2.10** Let \( T = H + iK \) for some Hermitian \( H \) and \( K \). Let \( C \) be a conjugation on \( \mathcal{X} \) and \( T \) is \( C \)-symmetric. Then \( T \) is invertible if and only if \( T \) is invertible.

**Proof.** Let, for a conjugation \( C \), \( CTC = T \) and \( \overline{T} \) be invertible. Then we have \( \overline{TCT^{-1}C} = CTCCT^{-1}C = CTT^{-1}C = C^2 = I = CT^{-1}CT \). Hence \( \overline{T} \) is invertible. Converse is clear. \( \square \)

**Theorem 2.11** Let \( \mathcal{X} \) be a complex Banach space. If an operator \( T = H + iK \) is complex symmetric, then \( \sigma_p(T) = \sigma_p(\overline{T})^* \), \( \sigma_a(T) = \sigma_a(\overline{T})^* \), \( \sigma_s(T) = \sigma_s(\overline{T})^* \), and \( \sigma(T) = \sigma(\overline{T})^* \).

**Proof.** By Theorem 2.7, we have \( \sigma_p(CTC) = \sigma_p(\overline{T})^* \). Since \( CTC = T \) for some conjugation, we have \( \sigma_p(T) = \sigma_p(\overline{T})^* \). Others are similar. \( \square \)
Definition 2.12 An operator $T = H + iK \in B(\mathcal{X})$ is said to be an ExB-operator if there exists $M > 0$ such that
\[ \|e^{zT} \cdot e^{-zT}\| \leq M \quad \text{for all } z \in \mathbb{C}. \]

We have $\|e^{zT} \cdot e^{-zT}\| \leq M$ (\forall z \in \mathbb{C}) if and only if $\|e^{zT}x\| \leq M\|e^{zT}x\|$ (\forall x \in \mathcal{X}, \forall z \in \mathbb{C}).

For $T = H + iK \in B(\mathcal{H})$ (Hilbert space operator case), $T$ is an ExB-operator if and only if $\|(e^{zT})^*x\| \leq M\|e^{zT}x\|$ (\forall x \in \mathcal{H}, \forall z \in \mathbb{C}). It is easy to see that if $T$ is an ExB-operator, then so is $aT + b$ for all $a, b \in \mathbb{C}$. When $M = 1$, K. Mattila in [13] called $*$-hyponormal.

Proposition 2.13 (Lemma 2 of [5]).
If $T = H + iK$ is an ExB-operator and $Tx = 0$, then $Tx = 0$.

Since $(e^{zT} \cdot e^{-zT})^* = e^{-zT^*} \cdot e^{zT}$, if $T$ is an ExB-operator, then so is $T^*$.

Theorem 2.14 Let $\mathcal{X}$ be a complex Banach space and $C$ be a conjugation on $\mathcal{X}$. If $T$ is an ExB-operator on $\mathcal{X}$ and $C$-symmetric, then $\ker(T - \lambda) = C\ker(T - \lambda)$ for all $\lambda \in \mathbb{C}$.

Proof. Let $Tx = \lambda x$. Since $aT + b$ is an ExB-operator for all $a, b \in \mathbb{C}$ and $(T - \lambda)x = 0$, by Proposition 2.13 it holds $\overline{T}x = \lambda x$. Hence $\lambda x = \overline{T}x = (CTC)x = C(TCx)$ and it holds $T(Cx) = C^2Tx(Cx) = C(\lambda x) = \lambda Cx$. Hence $C\ker(T - \lambda) \subset \ker(T - \lambda)$. Also, we have $\ker(T - \lambda) = C^2\ker(T - \lambda) \subset C\ker(T - \lambda)$. Hence $\ker(T - \lambda) = C\ker(T - \lambda)$. □

For a study of properties of a complex symmetric ExB-operator, we recall from [2] and [3] the construction of a larger space $\mathcal{X}^o$ from a given Banach space $\mathcal{X}$. Then the mapping $T \rightarrow T^o$ is an isometric isomorphism of $B(\mathcal{X})$ onto a closed subalgebra of $B(\mathcal{X}^o)$ as follows: Let $\text{Lim}$ be fixed Banach limit on the space of all bounded sequences of complex numbers with the norm $\|\{\lambda_n\}\| = \sup\{|\lambda_n| : n \in \mathbb{N}\}$. Let $\mathcal{X}$ be the space of all bounded sequences $\{x_n\}$ of $\mathcal{X}$. Let $N$ be the subspace of $\mathcal{X}$ consisting of all bounded sequences $\{x_n\}$ with $\text{Lim} \|x_n\|^2 = 0$. The space $\mathcal{X}^o$ is defined as the completion of the quotient space $\mathcal{X}/N$ with respect to the norm $\|\{x_n\} + N\| = (\text{Lim} \|x_n\|^2)^{\frac{1}{2}}$. Operator $T'$ is defined by $T'(\{x_n\} + N) = \{Tx_n\} + N$ on $\mathcal{X}/N$. The operator $T^o$ is defined by the unique extension of $T'$ on $\mathcal{X}^o$. Then the following results hold:
\[ \sigma(T) = \sigma(T^o), \quad \sigma_a(T) = \sigma_a(T^o) = \sigma_p(T^o) \quad \text{and} \quad \overline{\sigma}V(T) = V(T^o), \]
where $\overline{\sigma}V(T)$ is the closed convex hull of $V(T)$. See [2] and [3] for details. Therefore, if $T$ is Hermitian, normal or hyponormal, then so is $T^o$, respectively. Since the mapping $T \rightarrow T^o$ is an isometric isomorphism of $B(\mathcal{X})$ onto a closed subalgebra of $B(\mathcal{X}^o)$, if $T$ is an ExB-operator, then so is $T^o$.

Let $C$ be a conjugation on $\mathcal{X}$. The operator $C'$ is defined by $C'(\{x_n\} + N) = \{Cx_n\} + N$ on $\mathcal{X}/N$ and we define $C^o$ as the unique extension of $C'$ on $\mathcal{X}^o$. Then it is easy to see that
\[ C^{o^2} = I^o, \quad \|C^o\| = 1, \quad C^o(x^o + y^o) = C^o x^o + C^o y^o, \quad C^o(\lambda x^o) = \overline{\lambda} C^o x^o \quad (\forall x^o, y^o \in \mathcal{X}^o, \lambda \in \mathbb{C}). \]
Theorem 2.15 With the above assertion, if \( C \) is a conjugation, then so is \( C^\circ \) on \( X^\circ \).

Since \( (CTC)'(\{x_n\} + N) = \{TCx_n\} + N = C'(\{TCx_n\} + N) = C'T'C(\{x_n\} + N) \), it holds \( (CTC)^\circ = C^\circ T^\circ C^\circ \). Hence the following result holds.

Theorem 2.16 With the above assertion, if \( T \) is \( C \)-symmetric on \( X \), then \( T^\circ \) is \( C^\circ \)-symmetric on \( X^\circ \).

For the final result, we introduce orthogonality of Banach space as follows.

Definition 2.17 Let \( M \) be a subspace of \( X \). A vector \( x \) is orthogonal to \( M \) if
\[
\|m\| \leq \|m + x\| \quad (\text{for all } m \in M).
\]
Then we denote by \( M \perp x \).

For a subspace \( M \subset X \), let \( M^\perp \) be the set \( \{x \in X : M \perp x\} \). Let \( M, N \) be subspaces of \( X \). If, for all \( n \in N \), \( M \perp n \), then we denote \( M \perp N \).

Theorem 2.18 If a subspace \( M \) is invariant for a conjugation \( C \), then \( M^\perp \) is invariant for \( C \).

Proof. Let \( x \in M^\perp \) and \( m \in M \) be arbitrary. Then
\[
\|m\| = \|Cm\| \leq \|Cm + x\| = \|C(Cm + x)\| = \|m + Cx\|.
\]
Hence \( Cx \in M^\perp \). \( \square \)

Hence we have following corollary.

Corollary 2.19 Let \( T = H + iK \) be an ExB-operator. If \( T \) is \( C \)-symmetric, then \( \ker(T)^\perp \) is invariant for \( C \).

Proposition 2.20 (Theorem 20.7, [4]).
Let \( X \) be a reflexive Banach space and \( x \in X \) be \( \|x\| = 1 \). Then there exists \( f \in X^* \) such that \( \|f\| = f(x) = 1 \) and \( H^*f = 0 \) for all Hermitian operators \( H \) for which \( Hx = 0 \).

Proposition 2.21 (Lemma 20.3, [4]).
It holds that \( \ker(T) \perp \mathbb{R}(T) \) if and only if for any unit vector \( x \in \ker(T) \) there exists \( f \in X^* \) such that \( \|f\| = f(x) = 1 \) and \( T^*f = 0 \).

Theorem 2.22 Let \( X \) be a reflexive Banach space and \( T = H + iK \) be an ExB-operator. Then \( \ker(T) \perp \mathbb{R}(T) \).
Proof. Let \( x \in \ker(T) \) be a unit vector. Since \( T = H + iK \) is an ExB-operator, by Proposition 2.13 we have \( Hx = Kx = 0 \). Since \( \mathcal{X} \) is reflexive, by Proposition 2.20 there exists \( f \in \mathcal{X}^* \) such that \( \|f\| = f(x) = 1 \) and \( T^* f = 0 \). Hence by Proposition 2.21 it holds \( \ker(T) \perp R(T) \). \( \square \)

In [11], K. Mattila proved the following result for a normal operator.

Proposition 2.23 (Propositions 3.7 and 3.9, Corollary 3.8, [11]).

Let \( T \) be normal and \( \lambda, \mu \in \mathbb{C} \) be \( \lambda \neq \mu \).

(1) If \( \{x_n\} \) is a sequence of unit vectors such that \( Tx_n \to 0 \), then \( 1 \leq \lim \inf \|x_n + Ty_n\| \), for any bounded sequence \( \{y_n\} \) of \( \mathcal{X} \).

(2) If \( \{x_n\} \) is a sequence of unit vectors such that \( (T - \lambda)x_n \to 0 \), then \( 1 \leq \lim \inf \|x_n + y_n\| \), for any bounded sequence \( \{y_n\} \) such that \( (T - \mu)y_n \to 0 \).

In particular, \( \ker(T - \lambda) \perp \ker(T - \mu) \).

(3) For a bounded sequence \( \{x_n\} \), if \( T^2 x_n \to 0 \), then \( Tx_n \to 0 \).

We show the similar result. For the completeness, we give a proof. We need the following definition and result.

Definition 2.24 A Banach space \( \mathcal{X} \) is said to be uniformly convex if, for every \( \epsilon > 0 \), there exists a number \( \delta > 0 \) such that, for all \( x, y \in \mathcal{X} \), the conditions

\[
\|x\| = \|y\| = 1, \quad \|x - y\| \geq \epsilon \quad \text{imply} \quad \|\frac{x + y}{2}\| \leq 1 - \delta.
\]

It’s well known that \( \mathcal{X} \) is uniformly convex, then \( \mathcal{X} \) is reflexive, i.e., \( \mathcal{X}^{**} = \mathcal{X} \). Then we have the following result.

Proposition 2.25 (Theorem 4, [3]).

\( \mathcal{X} \) is uniformly convex if and only if \( \mathcal{X}^\circ \) is uniformly convex.

Theorem 2.26 Let \( T \) be an ExB-operator on a uniformly convex Banach space \( \mathcal{X} \) and \( \lambda, \mu \in \mathbb{C} \) be \( \lambda \neq \mu \).

(1) If \( \{x_n\} \) is a sequence of unit vectors such that \( Tx_n \to 0 \), then \( 1 \leq \lim \inf \|x_n + Ty_n\| \), for any bounded sequence \( \{y_n\} \) of \( \mathcal{X} \).

(2) If \( \{x_n\} \) is a sequence of unit vectors such that \( (T - \lambda)x_n \to 0 \), then \( 1 \leq \lim \inf \|x_n + y_n\| \), for any bounded sequence \( \{y_n\} \) such that \( (T - \mu)y_n \to 0 \).

In particular, \( \ker(T - \lambda) \perp \ker(T - \mu) \).

(3) For a bounded sequence \( \{x_n\} \), if \( T^2 x_n \to 0 \), then \( Tx_n \to 0 \).

Proof. Let \( \mathcal{X}^\circ \) be the larger Banach space of \( \mathcal{X} \) and \( T^\circ \) be the extension of \( T \) on \( \mathcal{X}^\circ \) as a previous way. Then \( \mathcal{X}^\circ \) is uniformly convex and \( T^\circ \) is an ExB-operator. And by Theorem 2.22 it holds \( \ker(T^\circ) \perp R(T^\circ) \). We may assume that all vectors \( x_n \) and \( y_n \) of (1), (2) and (3) are unit. Put \( x^\circ = \{x_n\} + N \). Then \( \|x^\circ\| = 1 \) and \( x^\circ \in \ker(T^\circ) \).

(1) Since it holds \( \ker(T^\circ) \perp R(T^\circ) \) by Theorem 2.22, we have \( 1 = \|x^\circ\| \leq \|x^\circ + T^\circ y^\circ\| \).
Assume that \( \liminf_{k} \|x_n + Ty_n\| < 1 \). Then there exist subsequences \( \{x_{n_j}\}, \{y_{n_j}\} \) such that \( \lim_{j \to \infty} \|x_{n_j} + Ty_{n_j}\| = \alpha < 1 \). Let \( x^0_i = \{x_{n_j}\} + N, y^0_i = \{y_{n_j}\} + N \). Then we have \( \|x^0_i\| = 1, T^0(x^0_i) = 0 \) and \( \|x^0_i + T^0y^0_i\| = \alpha < 1 = \|x^0_i\| \). It's a contradiction. Hence \( \liminf \|x_n + Ty_n\| \geq 1 \) and it completes (1).

(2) Since \( \lambda \neq \mu \), \( \{(\mu - \lambda)^{-1}y_n\} \) is a bounded sequence, by (1) we have

\[
1 \leq \liminf \|x_n + (\mu - \lambda)^{-1}(T - \lambda)y_n\| = \liminf \|x_n + y_n + (\mu - \lambda)^{-1}(T - \mu)y_n\| = \liminf \|x_n + y_n\|.
\]

By previous result, it is easy to see \( \ker(T - \lambda) \perp \ker(T - \mu) \) and it completes (2).

(3) Let \( T^2x_n \to 0 \). Assume \( Tx_n \not\to 0 \). Then there exist \( \epsilon > 0 \) and a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \|Tx_{n_k}\| \geq \epsilon \) \((k \in \mathbb{N})\). Let \( w_k = \|Tx_{n_k}\|^{-1}Tx_{n_k} \). Then \( \|Tw_k\| \leq \epsilon^{-1}\|T^2x_{n_k}\| \). Hence \( Tw_k \to 0 \) and by (1) it holds \( 1 \leq \liminf \|w_k + Tu_k\| \) for any bounded sequence \( \{u_k\} \). Taking \( u_k = -\|Tx_{n_k}\|^{-1}x_{n_k} \), it's a contradiction. Hence we have \( Tx_n \to 0 \) and it completes (3). \( \square \)

**Remark 2.27** In the case of conjugations \( C, J \) on a Hilbert space \( \mathcal{H} \), \( U = CJ \) is a unitary operator. Hence \( U^* = U^{-1} \) and it holds \( CUC = JC = U^{-1} = U^* = JUJ \). In the case of conjugations \( C, J \) on a Banach space \( X \), we have only \( CUC = JC = U^{-1} = JUJ \).

**References**


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