The K-theory for the group and subgroup C^*-algebras of the special or general linear groups over integers

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Abstract. We consider the K-theory of the group and subgroup C^*-algebras of the special or general linear groups over the ring of integers and of their canonical subgroups. We furhter consider the K-theory of the associated, crossed product C^*-algebras.

1 Introduction In this paper, as the main purpose, we consider the K-theory of the (full) group and subgroup C^*-algebras of the special or general linear groups over integers, i.e., $SL_n(\mathbb{Z})$ or $GL_n(\mathbb{Z})$ with their canonical subgroups, of higher interest in the literature (for instance, see [3], [4], [6], [9], [11], [12]) and of still being rather mysterious. We furhter consider the K-theory of the associated, (full) crossed product C^*-algebras by actions of $SL_n(\mathbb{Z})$, involved in this case.

As results, we obtain several C^*-algebra homomorphisms and their induced K-theory group homomorphisms, involved in that case, and several consequences of some interest. The results are as somewhat expected as our goal, but unfortunately, it turns out that they seems to be not enough to compute completely the K-theory groups of the group C^*-algebras of $SL_n(\mathbb{Z})$ or $GL_n(\mathbb{Z})$ targeted. Our idea for these is to consider a reduction of computing the K-theory groups for the group C^*-algebras to doing that for the subgroup C^*-algebras.

After this introduction, there are two sections as follows: 2 The group and subgroup C^*-algebras; 3 The crossed product C^*-algebras.

We begin with several fundamental definitions and notations for convenience to the readers, which may be skipped if not needed.

Let G be a discrete group. Let $\mathbb{C}[G]$ be the group $*$-algebra of all finitely supported, complex-valued functions on G with convolution and involution, and $l^1(G)$ be the Banach $*$-algebra of all summable, $\mathbb{C}$-valued functions on G with convolution and involution, defined by

$$f * g(t) = \sum_{s\in G} f(s)g(s^{-1}t) \quad \text{and} \quad f^*(t) = \overline{f(t^{-1})}$$

for $f, g \in \mathbb{C}[G]$ or $l^1(G)$ and $t \in G$, and with the l^1-norm as $\|f\|_1 = \sum_{s\in G} |f(s)|$.

The universal representation Φ_G of G is defined to be the direct sum representation $\oplus_\pi \pi$ on the Hilbert space direct sum $\oplus_\pi H_\pi$, of all unitary representations π of G on representation Hilbert spaces H_π, and is extended canonically to be the universal representation of

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$\mathbb{C}[G]$ or $l^1(G)$, also denoted by Φ_G (as the same symbol), and defined by

$$\Phi_G(f) = \sum_{s\in G} f(s)\Phi_G(s) = \oplus_\pi[\sum_{s\in G} f(s)\pi(s)] = \oplus_\pi \pi(f)$$

and the representation space $\oplus_\pi H_\pi$ is said to be the universal Hilbert space. The full group C^*-algebra $C^*(G)$ of G is defined to be the C^*-algebra completion of $\mathbb{C}[G]$ or $l^1(G)$ by the universal representation. The C^*-norm on $\mathbb{C}[G]$ or $l^1(G)$ is defined to be

$$\|f\| = \|\Phi_G(f)\| = \| \oplus_\pi \pi(f)\| = \sup_\pi \|\pi(f)\|,$$

where π runs over the set of all $*$-representations of $\mathbb{C}[G]$ or $l^1(G)$.

Refer to [5] or [10] for more some details.

In particular, there is a left regular representation λ_G of G on $l^2(G)$ the Hilbert space of all square summable, complex valued functions on G, which extends to $l^1(G)$ as above. The reduced group C^*-algebra $C_r^*(G)$ of G is defined by replacing Φ_G with λ_G.

It follows that there is a C^*-algebra quotient map q_G from $C^*(G)$ to $C_r^*(G)$, and that q_G is an isomorphism if and only if G is amenable (see [10]).

We use the symbol $\equiv$ for definition and the symbol $\cong$ for isomorphism in several senses such as groups and C^*-algebras.

For a C^*-algebra $\mathfrak{A}$, we denote by $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ the K-theory groups of $\mathfrak{A}$. For these, refer to [2] and [16]. For (separable) C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, we denote by $KK(\mathfrak{A},\mathfrak{B})$ the Kasparov KK-theory group for $\mathfrak{A}$ and $\mathfrak{B}$ (see [2]).

We denote by $\mathbb{Z}$ the ring of integers and by $SL_n(\mathbb{Z})$ the special linear group of all $n\times n$ invertible matrices over $\mathbb{Z}$ with determinant one. Denote by $GL_n(\mathbb{Z})$ the general linear group of all $n \times n$ invertible matrices over $\mathbb{Z}$.

The group $SL_n(\mathbb{Z})$ is a normal subgroup of $GL_n(\mathbb{Z})$. Indeed, if $x \in SL_n(\mathbb{Z})$ and $g \in GL_n(\mathbb{Z})$, then we have the determinant

$$\det(gxg^{-1}) = \det g \det x \det g^{-1} = \det(gg^{-1}) = \det 1_n = 1,$$

where 1_n is the $n \times n$ identity matrix.

There is a group homomorphism:

$$\det : GL_n(\mathbb{Z}) \to \{1,-1\} \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$$

since $\det(gh) = \det g \det h$ for $g,h \in GL_n(\mathbb{Z})$, and $1 = \det(gg^{-1}) = \det g \det g^{-1}$, so that $\det g \in \{\pm 1\}$ because $\det g \in \mathbb{Z}$ for any $g \in GL_n(\mathbb{Z})$ by definition of det.

Then there is a short exact sequence of groups:

$$1 \to SL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}) \to \mathbb{Z}_2 \to 1.$$

which splits, so that $GL_n(\mathbb{Z}) \cong SL_n(\mathbb{Z}) \rtimes \mathbb{Z}_2$ a semi-direct product of groups. because the section for the splitting is given by sending $1, -1 \in \mathbb{Z}_2$ to 1_n and $-1 \oplus 1_{n-1}$ the diagonal sum, respectively.

It is known that $SL_n(\mathbb{Z})$ and $GL_n(\mathbb{Z})$ are non-amenable. For instance, refer to [6] and [9]. In particular, there are K-theory group homomorphisms for $j = 0, 1$:

$$K_j(C^*(SL_n(\mathbb{Z}))) \to K_j(C_r^*(SL_n(\mathbb{Z})))$$

induced by the quotient map from $C^*(SL_n(\mathbb{Z}))$ to $C_r^*(SL_n(\mathbb{Z}))$. The same holds for $GL_n(\mathbb{Z})$. However, we do not know whether such maps in general are injective or not and are surjective or not. This would be considered elsewhere if possible. It is proved by [15] that the map on K_0 for $SL_n(\mathbb{Z})$ is not injective but surjective and the map on K_1 is surjective.

2 The group and subgroup C^*-algebras It follows from a review in the previous section and a basic fact of [2] for group crossed product C^*-algebras viewed as C^*-algebra crossed products that

Lemma 2.1. *We have the following C^*-algebra isomorphism:*

$$C^*(GL_n(\mathbb{Z})) \cong C^*(SL_n(\mathbb{Z})) \rtimes \mathbb{Z}_2,$$

where the right hand side means a C^*-algebra crossed product, so that

$$K_j(C^*(GL_n(\mathbb{Z}))) \cong K_j(C^*(SL_n(\mathbb{Z})) \rtimes \mathbb{Z}_2)$$

for $j = 0, 1$.

Remark. By Takai duality theorem for C^*-algebra crossed products (see [10] or [2]), we have

$$C^*(SL_n(\mathbb{Z})) \otimes M_2(\mathbb{C}) \cong C^*(GL_n(\mathbb{Z})) \rtimes \mathbb{Z}_2^\wedge,$$

where the right hand side means the C^*-algebra dual crossed product by the dual action of the dual group $\mathbb{Z}_2^\wedge \cong \mathbb{Z}_2$ and $M_2(\mathbb{C})$ is the 2×2 matrix C^*-algebra over $\mathbb{C}$, which is viewed as the C^*-algebra of all compact operators on the Hilbert space $\mathbb{C}^2$ (in this case), with $M_2(\mathbb{C}) \cong C^*(\mathbb{Z}^2) \rtimes \mathbb{Z}_2^\wedge$, so that

$$K_j(C^*(SL_n(\mathbb{Z}))) \cong K_j(C^*(GL_n(\mathbb{Z})) \rtimes \mathbb{Z}_2^\wedge)$$

for $j = 0, 1$, by stability of K-theory groups.

Proposition 2.2. *Let G be a discrete group and H a subgroup of G. Then there is an injective C^*-algebra homomorphism:*

$$0 \to C^*(H) \xrightarrow{\quad i \quad} C^*(G)$$

induced by the inclusion map $i : H \to G$. Then there are K-theory group homomorphisms:

$$K_j(C^*(H)) \xrightarrow{\ i_{*,j}\ } K_j(C^*(G))$$

induced by the inclusion map i, for $j = 0, 1$.

Proof. Let Φ_G be the universal representation of G or $\mathbb{C}[G]$ or $l^1(G)$. The composite $\Phi_G \circ i$ on H extends to a $*$-algebra homomorphism from $\mathbb{C}[H]$ to $\Phi_G(\mathbb{C}[G])$ in $C^*(G)$. It extends to a C^*-algebra homomorphism from $C^*(H)$ to $C^*(G)$ by density of $\mathbb{C}[H]$ in $C^*(H)$ and continuity of the map $\Phi_G \circ i$.

Note that any representation of H extends trivially to G because G is discrete. Therefore, the universal representaion Φ_H of H is viewed as a subrepresentation of Φ_G by restriction. It follows that the extended map $\Phi_G \circ i$ from $C^*(H)$ to $C^*(G)$, denoted by the same symbol, is injective.

As a fact, a homomophism between C^*-algebras induces a group homomorphism of their K-theory groups as a functoriality of K-theory (see [16]). □

Remark. We do not know whether the maps $i_{*,j}$ are injective or not in general. But possibly, they are injective in that case.

Example 2.3. Let $\mathbb{Z}$ be the group of integers and $n\mathbb{Z}$ a subgroup of $\mathbb{Z}$ of multiplies by a natural number n. By the Fourier transform F, the group C^*-algebra $C^*(\mathbb{Z})$ is isomorphic to $C(\mathbb{T})$ the C^*-algebra of all continuous, complex-valued functions on the one-torus $\mathbb{T}$. Indeed, let χ_1 be the characteristic function at the generator 1 of $\mathbb{Z}$. Then $\chi_1^*(t) = \overline{\chi_1(-t)} = \chi_{-1}(t)$ and

$$\chi_1 * \chi_{-1}(t) = \sum_{s\in G} \chi_1(s)\chi_{-1}(-s+t) = \chi_{-1}(-1+t) = \chi_0(t)$$

with

$$\chi_0 * f(t) = \sum_{s\in G} \chi_0(s) f(-s+t) = f(t)$$

for any $f \in \mathbb{C}[G]$ or $l^1(G)$, so that χ_1 is a unitary and χ_0 is the unit of $C^*(\mathbb{Z})$. By the Fourier (inverse) transform,

$$F(\chi_1)(z) = \sum_{s\in G} \chi_1(s) z^s = z$$

for $z \in \mathbb{T}$. Similarly, each element $nk \in n\mathbb{Z}$ is identified with $\chi_{nk} \in C^*(n\mathbb{Z})$, and $F(\chi_{nk}) = z^{nk}$. Hence, $C^*(n\mathbb{Z})$ is isomorphic to the unital sub-C^*-algebra of $C(\mathbb{T})$ generated by z^n.

It is known ([16]) that $K_0(C(\mathbb{T})) \cong \mathbb{Z}[1] = \mathbb{Z}[F(\chi_0)]$ and $K_1(C(\mathbb{T})) \cong \mathbb{Z}[z] = \mathbb{Z}[F(\chi_1)]$. Therefore, the induced map $i_{*,0} : K_0(C^*(n\mathbb{Z})) \to K_0(C^*(\mathbb{Z}))$ is an isomorphism but $i_{*,1}$ is an injection by multiplication by n, so that

$$K_1(C^*(\mathbb{Z}))/K_1(C^*(n\mathbb{Z})) \cong \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n.$$

□

Example 2.4. By Lemma 2.1 or Proposition 2.2, there is an injective C^*-algebra homomorphism

$$0 \to C^*(SL_n(Z)) \xrightarrow{\ i\ } C^*(GL_n(\mathbb{Z}))$$

induced by the inclusion map $i : SL_n(\mathbb{Z}) \to GL_n(\mathbb{Z})$. Then there are K-theory group homomorphisms

$$K_j(C^*(SL_n(\mathbb{Z}))) \xrightarrow{\ i_{*,j}\ } K_j(C^*(GL_n(\mathbb{Z})))$$

induced by the inclusion map i for $j = 0, 1$.

Example 2.5. It is well known that $SL_2(\mathbb{Z})$ is isomorphic to the amalgam $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ of cyclic groups, with $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$, where the generators of $\mathbb{Z}_2$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ are identified with the following matrices respectively,

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$$

([6] and [9]). Note that $C^*(SL_2(\mathbb{Z})) \cong C^*(\mathbb{Z}_4) *_{C^*(\mathbb{Z}_2)} C^*(\mathbb{Z}_6)$ the full amalgam of C^*-algebras ([2]), so that there are injective C^*-algebra homomorphisms from $C^*(\mathbb{Z}_4)$ and $C^*(\mathbb{Z}_6)$ to $C^*(SL_2(\mathbb{Z}))$. It then follows ([3] and [2]) that

$$\begin{aligned} K_j(C^*(SL_2(\mathbb{Z}))) &\cong K_j(C^*(\mathbb{Z}_4) *_{C^*(\mathbb{Z}_2)} C^*(\mathbb{Z}_6)) \\ &\cong [K_j(\mathbb{C}^4) \oplus K_j(\mathbb{C}^6)]/K_j(\mathbb{C}^2) \\ &\cong \begin{cases} \mathbb{Z}^{10}/\mathbb{Z}^2 \cong \mathbb{Z}^8 & j = 0, \\ [0 \oplus 0]/0 \cong 0 & j = 1, \end{cases} \end{aligned}$$

where $C^*(\mathbb{Z}_k) \cong C(\mathbb{Z}_k^\wedge) \cong \mathbb{C}^k$, so that the induced K-theory group homomorphisms are injective at both K_0 and K_1, where each component $\mathbb{Z}^4$ and $\mathbb{Z}^6$ at K_0 are mapped injectively to $(\mathbb{Z}^4 \oplus \mathbb{Z}^2)/\mathbb{Z}^2 \cong \mathbb{Z}^4$ and $(\mathbb{Z}^2 \oplus \mathbb{Z}^6)/\mathbb{Z}^2 \cong \mathbb{Z}^6$ respectively. □

For a positive integer $n \geq 1$, we consider the following subgroup of $SL_{n+1}(\mathbb{Z})$ with the decomposition as block matrices:

$$\mathbb{Z}^n \cong Z_n \equiv \begin{pmatrix} 1_n & \mathbb{Z}^n \\ 0_n^t & 1 \end{pmatrix} \subset SL_{n+1}(\mathbb{Z}) \subset GL_{n+1}(\mathbb{Z}),$$

where $\mathbb{Z}^n$ is the group of all the $n \times 1$ matrices (or column vectors) over $\mathbb{Z}$ and $0_n = (0, \cdots, 0)^t$ is the column zero vector of $\mathbb{Z}^n$ and its transpose $0_n^t = (0, \cdots, 0)$ is the row zero vector, and this subgroup is isomorphism to $\mathbb{Z}^n$ and is denoted by Z_n.

Moreover, as the transpose of Z_n we consider the following subgroup of $SL_{n+1}(\mathbb{Z})$:

$$(\mathbb{Z}^n)^t \cong (Z_n)^t \equiv \begin{pmatrix} 1_n & 0_n \\ (\mathbb{Z}^n)^t & 1 \end{pmatrix} \subset SL_{n+1}(\mathbb{Z}) \subset GL_{n+1}(\mathbb{Z}).$$

Note that $\mathbb{Z}^n \cong (\mathbb{Z}^n)^t \cong Z_n \cong (Z_n)^t$ as a group. We identity $\mathbb{Z}^n$ with $(\mathbb{Z}^n)^t$.

Example 2.6. By Proposition 2.2, there is an injective C^*-algebra homomorphism

$$0 \to C^*(\mathbb{Z}^n) \xrightarrow{\ i\ } C^*(SL_{n+1}(\mathbb{Z}))$$

induced by the inclusion map $i : \mathbb{Z}^n \to Z_n \subset SL_{n+1}(\mathbb{Z})$ or $i : \mathbb{Z}^n \to (Z_n)^t$. Then there are K-theory group homomorphisms

$$K_j(C^*(\mathbb{Z}^n)) \xrightarrow{\ i_{*,j}\ } K_j(C^*(SL_{n+1}(\mathbb{Z})))$$

induced by the inclusion map i. for $j = 0, 1$.

Furthermore, all the same holds by replacing $SL_{n+1}(\mathbb{Z})$ with $GL_{n+1}(\mathbb{Z})$. □

For a positive integer $n \geq 1$, we next consider the following subgroup of $SL_{n+1}(\mathbb{Z})$ with the decomposition as block matrices:

$$\begin{pmatrix} SL_n(\mathbb{Z}) & \mathbb{Z}^n \\ 0_n^t & 1 \end{pmatrix} \subset SL_{n+1}(\mathbb{Z}),$$

where $SL_1(\mathbb{Z}) = \{1\}$ the trivial group. This subgroup is isomorphic to the semi-direct product $\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})$ and is denoted by H_n. In particular, we have $H_1 \cong \mathbb{Z}$.

Moreover, as the transpose of H_n we consider the following subgroup of $SL_{n+1}(\mathbb{Z})$:

$$\begin{pmatrix} SL_n(\mathbb{Z}) & 0_n \\ (\mathbb{Z}^n)^t & 1 \end{pmatrix} \cong (\mathbb{Z}^n)^t \rtimes SL_n(\mathbb{Z}) \cong \begin{pmatrix} SL_n(\mathbb{Z}) & \mathbb{Z}^n \\ 0_n^t & 1 \end{pmatrix}^t \equiv H_n^t.$$

We identify $\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})$ with $(\mathbb{Z}^n)^t \rtimes SL_n(\mathbb{Z})$.

Example 2.7. By Proposition 2.2, there is an injective C^*-algebra homomorphism

$$0 \to C^*(\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})) \xrightarrow{\ i\ } C^*(SL_{n+1}(\mathbb{Z}))$$

induced by the inclusion map $i : \mathbb{Z}^n \rtimes SL_n(\mathbb{Z}) \to H_n \subset SL_{n+1}(\mathbb{Z})$ or $i : (\mathbb{Z}^n)^t \rtimes SL_n(\mathbb{Z}) \to H_n^t$. Then there are K-theory group homomorphisms

$$K_j(C^*(\mathbb{Z}^n \rtimes SL_n(\mathbb{Z}))) \xrightarrow{\ i_{*,j}\ } K_j(C^*(SL_{n+1}(\mathbb{Z})))$$

induced by the inclusion map i for $j = 0, 1$.

Furthermore, all the same holds by replacing $SL_{n+1}(\mathbb{Z})$ with $GL_{n+1}(\mathbb{Z})$. □

Next consider the following amalgam of groups:

$$(\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})) *_{SL_n(\mathbb{Z})} ((\mathbb{Z}^n)^t \rtimes SL_n(\mathbb{Z})).$$

The group isomorphisms as

$$\mathbb{Z}^n \rtimes SL_n(\mathbb{Z}) \to \begin{pmatrix} SL_n(\mathbb{Z}) & \mathbb{Z}^n \\ (0_n)^t & 1 \end{pmatrix} \equiv H_n$$

and

$$(\mathbb{Z}^n)^t \rtimes SL_n(\mathbb{Z}) \to \begin{pmatrix} SL_n(\mathbb{Z}) & 0_n \\ (\mathbb{Z}^n)^t & 1 \end{pmatrix} \equiv H_n^t$$

induce a group homomorphism φ:

$$\begin{array}{c} (\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})) *_{SL_n(\mathbb{Z})} ((\mathbb{Z}^n)^t \rtimes SL_n(\mathbb{Z})) \\ \varphi \big\downarrow \\ \left\langle \begin{pmatrix} SL_n(\mathbb{Z}) & \mathbb{Z}^n \\ 0_n^t & 1 \end{pmatrix}, \begin{pmatrix} SL_n(\mathbb{Z}) & 0_n \\ (\mathbb{Z}^n)^t & 1 \end{pmatrix} \right\rangle \equiv \langle H_n, H_n^t \rangle, \end{array}$$

where $\langle H_n, H_n^t \rangle$ means the group generated by the subgroups H_n and H_n^t of $SL_{n+1}(\mathbb{Z})$.

Remark. It is known (see [9, Theorem VII.3] and also [6, Chapter III]) that $SL_n(\mathbb{Z})$ is generated by the following two matrices:

$$(1) \quad \sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus 1_{n-2},$$

the diagonal sum with 1_k the $k \times k$ identity matrix and (2) the other as a sort of the matrices of permutation $p_n = (p_{ij})_{i,j=1}^n$ with $p_{n1} = (-1)^{n-1}$, $p_{k,k+1} = 1$ for $1 \le k \le n-1$ and $p_{ij} = 0$ otherwise, as

$$p_n = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ (-1)^{n-1} & & & 0 \end{pmatrix}$$

Note that $(p_n)^n = (-1)^{n-1} 1_n$, and

$$\sigma^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \oplus 1_{n-2}.$$

It is shown below that the matrix p_{n+1} does belong to $\langle H_n, H_n^t \rangle$.

In fact, we obtain

Proposition 2.8. *We have* $\langle H_1, H_1^t \rangle = SL_2(\mathbb{Z})$.

Proof. For $n, m \in \mathbb{Z}$, we compute

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} = \begin{pmatrix} 1 + nm & n \\ m & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ m & mn + 1 \end{pmatrix}$$

so that in particular we obtain

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in \langle H_1, H_1^t \rangle.$$

Then

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = p_2 \in \langle H_1, H_1^t \rangle.$$

It then follows that $\langle H_1, H_1^t \rangle = SL_2(\mathbb{Z})$. □

Moreover, we obtain

Proposition 2.9. *We have* $\langle H_n, H_n^t \rangle = SL_{n+1}(\mathbb{Z})$ *for any* $n \geq 1$.

Proof. We compute

$$\begin{pmatrix} 1_n & 0_n \\ (-1, 0_{n-1}^t) & 1 \end{pmatrix} \begin{pmatrix} 1_n & (1, 0_{n-1}^t)^t \\ 0_n & 1 \end{pmatrix} = \begin{pmatrix} 1_n & (1, 0_{n-1}^t)^t \\ (-1, 0_{n-1}^t) & 0 \end{pmatrix}.$$

We next compute

$$\begin{pmatrix} 1_n & (1, 0_{n-1}^t)^t \\ (-1, 0_{n-1}^t) & 0 \end{pmatrix} \begin{pmatrix} 1_n & 0_n \\ (-1, 0_{n-1}^t) & 1 \end{pmatrix} = \begin{pmatrix} 0 \oplus 1_{n-1} & (1, 0_{n-1}^t)^t \\ (-1, 0_{n-1}^t) & 0 \end{pmatrix}.$$

Assume now that n is odd. Then we compute

$$\begin{pmatrix} p_n & 0_n \\ 0_n^t & 1 \end{pmatrix} \begin{pmatrix} 0 \oplus 1_{n-1} & (1, 0_{n-1}^t)^t \\ (-1, 0_{n-1}^t) & 0 \end{pmatrix} = p_{n+1}.$$

Hence $p_{n+1} \in \langle H_n, H_n^t \rangle$.
Similarly, we compute

$$\begin{pmatrix} 1_n & 0_n \\ (1, 0_{n-1}^t) & 1 \end{pmatrix} \begin{pmatrix} 1_n & (-1, 0_{n-1}^t)^t \\ 0_n & 1 \end{pmatrix} = \begin{pmatrix} 1_n & (-1, 0_{n-1}^t)^t \\ (1, 0_{n-1}^t) & 0 \end{pmatrix}.$$

We next compute

$$\begin{pmatrix} 1_n & (-1, 0_{n-1}^t)^t \\ (1, 0_{n-1}^t) & 0 \end{pmatrix} \begin{pmatrix} 1_n & 0_n \\ (1, 0_{n-1}^t) & 1 \end{pmatrix} = \begin{pmatrix} 0 \oplus 1_{n-1} & (-1, 0_{n-1}^t)^t \\ (1, 0_{n-1}^t) & 0 \end{pmatrix}.$$

Assume now that n is even. Then we compute

$$\begin{pmatrix} p_n & 0_n \\ 0_n^t & 1 \end{pmatrix} \begin{pmatrix} 0 \oplus 1_{n-1} & (-1, 0_{n-1}^t)^t \\ (1, 0_{n-1}^t) & 0 \end{pmatrix} = p_{n+1}.$$

Hence $p_{n+1} \in \langle H_n, H_n^t \rangle$.

□

Therefore, we obtain

Proposition 2.10. *There is a surjective* C^**-algebra homomorphism as*

$$C^*(H_n *_{SL_n(\mathbb{Z})} H_n^t) \xrightarrow{\varphi} C^*(H_n, H_n^t) = C^*(SL_{n+1}(\mathbb{Z})) \to 0,$$

which is induced by φ *on groups above and is denoted by the same symbol, where* $C^*(H_n, H_n^t)$ *is the full group* C^**-algebra of* $\langle H_n, H_n^t \rangle$.

Moreover, we have K-theory group homomorphisms as

$$K_j(C^*(H_n *_{SL_n(\mathbb{Z})} H_n^t)) \xrightarrow{\varphi_*} K_j(C^*(H_n, H_n^t))$$

induced by φ *for* $j = 0, 1$. *Furthermore, we have a KK-theory class*

$$[\varphi] \in KK(C^*(H_n *_{SL_n(\mathbb{Z})} H_n^t), C^*(H_n, H_n^t))$$

induced by φ.

Proof. The group homomorphism φ defined above extends to the C^*-algebra homomorphism denoted also by φ between those full group C^*-algebras by universality of the full group C^*-algebras. It induces the K-theory group homomorphism φ_* and as well the KK-theory class $[\varphi]$ as in the statement. □

Lemma 2.11. *There is a* C^**-algebra homomorphism from* $C^*(\mathbb{Z}^n \rtimes SL_n(\mathbb{Z}))$ *onto* $C^*(SL_n(\mathbb{Z}))$ *induced by the quotient map from* $\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})$ *to* $SL_n(\mathbb{Z})$*, which splits. Thus, there are surjective K-theory group homomorphisms*:

$$K_j(C^*(\mathbb{Z}^n \rtimes SL_n(\mathbb{Z}))) \to K_j(C^*(SL_n(\mathbb{Z}))) \to 0 \quad (j = 0, 1)$$

which split.

Proof. The trivial homomorphism from $\mathbb{Z}^n$ to the trivial group induces a quotient homomorphism from $\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})$ to $SL_n(\mathbb{Z})$. In other words, there is a short exact sequence of groups:

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^n \rtimes SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}) \to 0,$$

which splits. It then follows that there is a quotient homomorphism from $C^*(\mathbb{Z}^n \rtimes SL_n(\mathbb{Z}))$ to $C^*(SL_n(\mathbb{Z}))$ by the group quotient homomorphism and universality, which splits. It then follows that the induced K-theory group homomorphisms also split (see [16]). □

More generally, the same proof as above implies

Proposition 2.12. *Let* $H \rtimes G$ *be a semi-direct product of discrete groups. Then there is a* C^**-algebra homomorphisms from* $C^*(H \rtimes G)$ *onto* $C^*(G)$*, which splits. Thus, there are surjective K-theory group homomorphisms*:

$$K_j(C^*(H \rtimes G)) \to K_j(C^*(G)) \to 0 \quad (j = 0, 1)$$

which split.

Example 2.13. Let $H \times G$ be a direct product of discrete groups. This is the trivial case of semi-direct products. The quotient map from $H \times G$ to G implies the quotient map from $C^*(H \times G)$ to $C^*(G)$. Also $C^*(H \times G) \cong C^*(H) \otimes_{\max} C^*(G)$ when $\otimes_{\max}$ means the maximal C^*-tensor product, so that there is an injective C^*-algebra homomorphism from $C^*(G)$ to $C^*(H \times G)$ as a splitting section for the quotient map.

Example 2.14. Since $GL_n(\mathbb{Z}) \cong SL_n(\mathbb{Z}) \rtimes \mathbb{Z}_2$, there is a C^*-homomorphism from $C^*(GL_n(\mathbb{Z}))$ to $C^*(\mathbb{Z}_2) \cong \mathbb{C}^2$, which splits. Thus, there is a K-theory group homomorphism:

$$K_j(C^*(GL_n(\mathbb{Z}))) \to K_j(C^*(\mathbb{Z}_2)) \to 0,$$

which splits, so that we obtain

$$K_j(C^*(SL_n(\mathbb{Z}))) \xrightarrow{i_{*,j}} K_j(C^*(GL_n(\mathbb{Z}))) \rightleftarrows K_j(C^*(\mathbb{Z}_2)) \rightleftarrows 0$$

As a note, if we assume that $i_{*,j}$ are injective as a possible case (but we do not know the very reason for injectivity and short exactness), then

$$K_j(C^*(GL_n(\mathbb{Z}))) \cong \begin{cases} K_0(C^*(SL_n(\mathbb{Z}))) \rtimes \mathbb{Z}^2 & j = 0, \\ K_1(C^*(SL_n(\mathbb{Z}))) & j = 1. \end{cases}$$

Note that $C^*(SL_n(\mathbb{Z}))$ is not a closed ideal of $C^*(GL_n(\mathbb{Z}))$ but a C^*-subalgebra.

Furthermore, as a contrast,

Example 2.15. There is the full crossed product C^*-algebra $\mathfrak{A} \rtimes G$ of a unital C^*-algebra $\mathfrak{A}$ by an action of a discrete group G such that $\mathfrak{A} \rtimes G$ has no homomorphisms to $C^*(G)$. For instance, let $\mathfrak{A} = C(\mathbb{T})$ and $G = \mathbb{Z}$ and $C(\mathbb{T}) \rtimes \mathbb{Z}$ by the rotation action with an irrational, which is the irrational rotation C^*-algebra. Then $C(\mathbb{T}) \rtimes \mathbb{Z}$ is known to be simple (see [5]).

Also, for $n \geq 2$, if $\mathfrak{A} = C^*(\mathbb{Z}_n) \cong C(\mathbb{Z}_n^\wedge) \cong \mathbb{C}^n$ with $K_0(\mathfrak{A}) \cong \mathbb{Z}^n$ and $K_1(\mathfrak{A}) \cong 0$ and let $\mathbb{Z}_n$ act on $C(\mathbb{Z}_n)$ by translation. Then $\mathfrak{A} \rtimes \mathbb{Z}_n \cong M_n(\mathbb{C})$ with $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$ and $K_1(M_n(\mathbb{C})) \cong 0$. Hence, there are no injections from $K_0(\mathfrak{A})$ to $K_0(\mathfrak{A} \rtimes \mathbb{Z}_n)$ and no surjections from $K_0(\mathfrak{A} \rtimes \mathbb{Z}_n)$ to $K_0(C^*(\mathbb{Z}_n))$ in general. □

We denote by $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ the full amalgam of C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ over a common C^*-subalgebra $\mathfrak{C}$.

Lemma 2.16. *There is a short exact sequence of K-theory groups*:

$$\begin{aligned} 0 &\to K_j(C^*(SL_n(\mathbb{Z}))) \to K_j(C^*(H_n)) \oplus K_j(C^*(H_n^t)) \\ &\to K_j(C^*(H_n) *_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t)) \cong K_j(C^*(H_n *_{SL_n(\mathbb{Z})} H_n^t)) \to 0 \end{aligned}$$

for $j = 0, 1$.

Proof. We use a formula of Cuntz ([3] or [2]) on computing the K-theory groups for an amalgam $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ of C^*-algebras with retractions (that are homomorphisms onto an intersection C^*-algebra $\mathfrak{C}$. Note as well that the full group C^*-algebra of an amalgam $G *_K H$ of discrete groups G and H over a common subgroup K is isomorphic to the amalgam of the full group C^*-algebras as (see [2])

$$C^*(G *_K H) \cong C^*(G) *_{C^*(K)} C^*(H).$$

□

Proposition 2.17. *The same statement holds for an amalgam* $(H \rtimes G) *_G (K \rtimes G)$ *of semi-direct products* $H \rtimes G$ *and* $K \rtimes G$ *of discrete groups.*

Recall that the pull back C^*-algebra $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ of C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ over a C^*-algebra $\mathfrak{C}$ is defined to be the C^*-subalgebra of the direct sum $\mathfrak{A} \oplus \mathfrak{B}$, which satisfies the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B} & \xrightarrow{p_1} & \mathfrak{A} \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q_1} \\ \mathfrak{B} & \xrightarrow{q_2} & \mathfrak{C} \end{array}$$

where each p_j is the canonical projection to the j-th component and each q_j is a $*$-homomorphism. But in applications such as to Mayer-Vietoric K-theory diagram and below, we need to assume that each q_j is a surjective homomorphism. In this case we may call $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ a surjective pull back C^*-algebra.

Proposition 2.18. *There is a surjective C^*-algebra homomorphism as*

$$C^*(SL_{n+1}(\mathbb{Z})) = C^*(H_n, H_n^t) \xrightarrow{\psi} C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t) \to 0$$

where the quotient is the surjective pull back C^*-algebra obtained from the following commutative diagram:

$$\begin{array}{ccccc} C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t) & \xrightarrow{p_1} & C^*(\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})) & & \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q_1} & & \\ C^*((\mathbb{Z}^n)^t \rtimes SL_n(\mathbb{Z})) & \xrightarrow{q_2} & C^*(SL_n(\mathbb{Z})) & \longrightarrow & 0 \end{array}$$

where each p_j is the canonical projection map and each q_j is the canonical quotient map.

Moreover, we have K-theory group homomorphisms as

$$K_j(C^*(H_n, H_n^t)) \xrightarrow{\psi_*} K_j(C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t))$$

induced by ψ for $j = 0, 1$. Furthermore, we have a KK-theory class

$$[\psi] \in KK(C^*(H_n, H_n^t), C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t))$$

induced by ψ.

Proof. There are canonical C^*-algebra homomorphisms

$$\psi_1 : C^*(H_n) \to C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t)$$

and

$$\psi_2 : C^*(H_n^t) \to C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t),$$

defined as

$$\psi_1(u_{(z,g)}) = (u_{(z,g)}, u_g) \quad \text{and} \quad \psi_2(u_{(z,h)}) = (u_h, u_{(z,h)})$$

for $u_{(z,g)} \in C^*(H_n)$ and $u_{(z,h)} \in C^*(H_n^t)$, with $(z,g) \in \mathbb{Z}^n \rtimes SL_n(\mathbb{Z})$ and $(z,h) \in (\mathbb{Z}^n)^t \rtimes SL_n(\mathbb{Z})$. Indeed, for $u_{(z,g)}$, $u_{(z',g')} \in C^*(H_n)$,

$$\begin{aligned} \psi_1(u_{(z,g)}u_{(z',g')}) &= \psi_1(u_{(z+g\cdot z', gg')}) \\ &= (u_{(z+g\cdot z', gg')}, u_{gg'}) \\ &= (u_{(z,g)}u_{(z',g')}, u_g u_{g'}) \\ &= (u_{(z,g)}, u_g)(u_{(z',g')}, u_{g'}) = \psi_1(u_{(z,g)})\psi_1(u_{(z',g')}), \end{aligned}$$

where $g\cdot z'$ means the action involved in the semi-direct product H_n. Those homomorphisms ψ_1 and ψ_2 induce the homomorphism ψ as in the statement. It induces the K-theory group homomorphism ψ_* and as well the KK-theory class $[\psi]$ as in the statement. □

Combining Proposition 2.10 with Proposition 2.18, we obtain

Corollary 2.19. *There are successive, surjective C^*-algebra homomorphisms as*

$$\begin{array}{ccc} C^*(H_n *_{SL_n(\mathbb{Z})} H_n^t) & & \\ {\scriptstyle \varphi}\downarrow & & \\ C^*(H_n, H_n^t) = C^*(SL_{n+1}(\mathbb{Z})) & \longrightarrow & 0 \\ {\scriptstyle \psi}\downarrow & & \\ C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t) & \longrightarrow & 0. \end{array}$$

It is shown by [2] and [3] that the K-theory groups of the amalgam $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ of C^*-algebras with retractions to $\mathfrak{C}$ is isomorphic to those of the surjective pull back C^*-algebra $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$. Namely,

$$K_j(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \cong K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B})$$

for $j = 0, 1$ under the condition with retractions. Also may refer to [13] for a revised complete proof. In fact, the isomorphism k_* from $K_j(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B})$ to $K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B})$ is induced from the homomorphism k defined by

$$k(a) = (a, q_1(a)) \quad \text{and} \quad k(b) = (q_2(b), b)$$

for $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$, where $q_1 : \mathfrak{A} \to \mathfrak{C}$ and $q_2 : \mathfrak{B} \to \mathfrak{C}$ are retractions, that are onto $*$-homomorphisms. On the other hand, its inverse is given by $f_* - h_*$, where for $(a, b) \in \mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$, $f(a, b) = a \oplus b$ the diagonal sum in the 2×2 matrix algebra $M_2(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B})$ over $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ and $h(a, b) = i \circ q_1(a) = i \circ q_2(b)$, where $i : \mathfrak{C} \to \mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ is the inclusion map.

Therefore, we obtain

Theorem 2.20. *Suppose that $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ on K-theory groups, which looks automatic, but not in general. Then there is an injective K-theory group homomorphism φ_* and a surjective ψ_* :*

$$0 \to K_j(C^*(H_n *_{SL_n(\mathbb{Z})} H_n^t)) \xrightarrow{\varphi_*} K_j(C^*(SL_{n+1}(\mathbb{Z}))), \quad \textit{and}$$

$$K_j(C^*(SL_{n+1}(\mathbb{Z}))) \xrightarrow{\psi_*} K_j(C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t)) \to 0$$

where the maps φ_* and ψ_* are induced from the quotient maps φ and ψ of Propositions 2.10 *and* 2.18 *respectively, and it holds that the image of φ_* is isomorphic to the image of ψ_*.*

Moreover, without the assumption above, the Kasparov product $[\psi] \otimes [\varphi] = [\psi \circ \varphi]$ of $[\varphi]$ and $[\psi]$ is a KK-equivalence.

Proof. Note that the composite $\psi \circ \varphi$ is the canonical homomorphism from the amalgam C^*-algebra with retractions onto the surjective pull back C^*-algebra, and then the induced K-theory group homomorphism $(\psi \circ \varphi)_*$ is an isomorphism as cited above. If $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$, then it must be that φ_* is injective and ψ_* is surjective, and the image of φ_* is isomorphic to the image of ψ_*.

Note as well that the Kasparov product $[\psi] \otimes [\varphi]$ is equal to $[\psi \circ \varphi]$, where

$$KK(C^*(H_n *_{SL_n(\mathbb{Z})} H_n^t), \mathfrak{D}) \times KK(\mathfrak{D}, C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t))$$

$$\otimes \Big\downarrow$$

$$KK(C^*(H_n *_{SL_n(\mathbb{Z})} H_n^t), C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t)),$$

with $\mathfrak{D} = C^*(H_n, H_n^t)$, and that the KK-theory equivalence is equivalent to the K-theory equivalence, that is an equivalence of K-theory group isomorphisms, as cited above ([2]). $\square$

Similarly,

Proposition 2.21. *If a C^*-algebra $\mathfrak{D}$ is a quotient of an amalgam C^*-algebra $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ with retractions to $\mathfrak{C}$ by φ a homomorphism and has the associated surjective pull back C^*-algebra $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ as a quotient by ψ a homomorphism, and if the composite $\psi \circ \varphi$ is the canonical homomorphism from $\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}$ to $\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}$ and if $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ on their K-theory groups, then*

$$0 \to K_j(\mathfrak{A} *_{\mathfrak{C}} \mathfrak{B}) \xrightarrow{\varphi_*} K_j(\mathfrak{D}), \quad \textit{and}$$

$$K_j(\mathfrak{D}) \xrightarrow{\psi_*} K_j(\mathfrak{A} \oplus_{\mathfrak{C}} \mathfrak{B}) \to 0$$

for $j = 0, 1$, and it holds that the image of φ_* is isomorphic to the image of ψ_*.

Remark. In general, a KK-equivalence class in $KK(\mathfrak{A}, \mathfrak{B})$ for (separable) C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ implies the isomorhisms

$$KK(\mathfrak{B}, \mathfrak{D}) \to KK(\mathfrak{A}, \mathfrak{D}) \quad \text{and} \quad KK(\mathfrak{D}, \mathfrak{A}) \to KK(\mathfrak{D}, \mathfrak{B})$$

by the Kasparov product of the KK-equivalence class from the left and right respectively, for any (separable) C^*-algebra $\mathfrak{D}$. It follows that the KK-theory class $[\varphi]$ may correspond to the KK-theory class of a cross section with respect to ψ, and the similar holds for $[\psi]$.

Theorem 2.22. *There are K-theory group homomorphisms*:

$$[K_j(C^*(H_n)) \oplus K_j(C^*(H_n^t))]/K_j(C^*(SL_n(\mathbb{Z}))) \to K_j(C^*(SL_{n+1}(\mathbb{Z})))$$

for $j = 0, 1$. These are injective only when $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

Corollary 2.23. *Suppose that $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$. Then for $n \geq 1$, we have the following inclusions as groups*:

$$K_j(C^*(SL_{n+1}(\mathbb{Z}))) \supset [K_*(C^*(H_n)) \oplus K_*(C^*(H_n^t))]/K_*(SL_n(\mathbb{Z}))$$

for $* = 0, 1$.

In particular, we do have the following unexpected:

Example 2.24. Without the assumption above, it holds that

$$\begin{aligned} K_0(C^*(SL_2(\mathbb{Z}))) &\supsetneq [K_0(C^*(H_1)) \oplus K_0(C^*(H_1^t))]/K_0(C^*(SL_1(\mathbb{Z}))) \\ &\cong [\mathbb{Z} \oplus \mathbb{Z}]/\mathbb{Z} \cong \mathbb{Z} \end{aligned}$$

but

$$\begin{aligned} K_1(C^*(SL_2(\mathbb{Z}))) &\not\supset [K_1(C^*(H_1)) \oplus K_1(C^*(H_1^t))]/K_0(C^*(SL_1(\mathbb{Z}))) \\ &\cong [\mathbb{Z} \oplus \mathbb{Z}]/0 \cong \mathbb{Z}^2. \end{aligned}$$

It then follows that $(\psi \circ \varphi)_* \neq \psi_* \circ \varphi_*$ in the last case. Indeed, it is known ([2] or [3]) that $K_0(C^*(SL_2(\mathbb{Z}))) \cong \mathbb{Z}^8$ and $K_1(C^*(SL_2(\mathbb{Z}))) \cong 0$. Also, the assumption $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ does not hold in the K_1-case above because the left hand side of the equation is an isomorphism but the right hand side is zero since φ_* and ψ_* are zero. □

Remark. We do not know whether the assumption on K-theory group homomorphisms above which looks like to be natural is correct or not in general, and even if correct, then whether the inclusions are strict or not. It is desirable as a goal that the assumption is correct and the inclusions are equal for $SL_n(\mathbb{Z})$ with $n \geq 3$. As a reason which supports our conjecture, note that the group C^*-algebras $C^*(SL_n(\mathbb{Z}))$ as well as the groups $SL_n(\mathbb{Z})$ are highly noncommutative in a sense that it is known that $SL_n(\mathbb{Z})$ for $n \geq 3$ are not isomorphic to amalgams, but isomorphic to multi-amalgams ([12]).

As a contrast, but in the case of a commutative C^*-algebra involved,

Example 2.25. There is a surjective C^*-algebra homomorphism from the unital full free product C^*-algebra $C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(\mathbb{Z}^m)$ to the group C^*-algebra $C^*(\mathbb{Z}^{n+m}) \cong C^*(\mathbb{Z}^n) \otimes C^*(\mathbb{Z}^m)$ the (full or reduced) C^*-algebra tensor product. Also, there is a surjective C^*-algebra homomorphism from $C^*(\mathbb{Z}^{n+m})$ to the pull back C^*-algebra $C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} C^*(\mathbb{Z}^m)$. It follows that

$$\begin{aligned}
&K_0(C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(\mathbb{Z}^m)) \cong K_0(C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} C^*(\mathbb{Z}^m)) \\
&\cong [K_0(C^*(\mathbb{Z}^n)) \oplus K_0(C^*(\mathbb{Z}^m))]/K_0(\mathbb{C}) \cong [\mathbb{Z}^{2^{n-1}} \oplus \mathbb{Z}^{2^{m-1}}]/\mathbb{Z} \cong \mathbb{Z}^{2^{n-1}+2^{m-1}-1}, \\
&K_1(C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(\mathbb{Z}^m)) \cong K_1(C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} C^*(\mathbb{Z}^m)) \\
&\cong [K_1(C^*(\mathbb{Z}^n)) \oplus K_1(C^*(\mathbb{Z}^m))]/K_1(\mathbb{C}) \cong \mathbb{Z}^{2^{n-1}+2^{m-1}}
\end{aligned}$$

but

$$K_j(C^*(\mathbb{Z}^{n+m})) \cong \mathbb{Z}^{2^{n+m-1}}$$

(see [16]). The equality for the respective K_0 and K_1-groups only holds when $n = m = 1$ and $j = 1$. Note that $K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(\mathbb{C}) \cong 0$ and that by the Fourier transform, $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$ the C^*-algebra of all continuous, complex-valued functions on the n-dimensional torus $\mathbb{T}^n$.

As another consequence, of some interest, we obtain

Corollary 2.26. *There is a continuous field of C^*-algebras over the closed interval $[0, 1]$ with fibers changing continuously from $C^*(H_n *_{SL_n(\mathbb{Z})} H_n^t)$ at 0 to $C^*(H_n, H_n^t) = C^*(SL_{n+1}(\mathbb{Z}))$ at some $0 < t < 1$ and to $C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t)$ at 1 along φ and ψ respectively, such that their canonical generators associated with H_n and H_n^t are defined to be continuous.*

Proof. The constructions is as follows. Deform generators of $H_n *_{SL_n(\mathbb{Z})} H_n^t$ to have the relations for those of $\langle H_n, H_n^t\rangle$, with a certain continuous parameter, and doing so may induce a C^*-algebra deformation. And similarly deform generators of $C^*(H_n, H_n^t)$ to have relations for those of $C^*(H_n) \oplus_{C^*(SL_n(\mathbb{Z}))} C^*(H_n^t)$, as desired. We omit the detailed construction about this. Also, equivalently, we may deform the identity map on $C^*(H_n *_{SL_n(\mathbb{Z})} H_n^t)$ to φ and to ψ as well. □

Remark. Such a consequence may be well known. For instance, it is also known as softening C^*-algebras or soft C^*-algebras (see [7], [8], and [14]). Unfortunately, note that the K-theory groups of fibers of a continuous field of C^*-algebras are not necessarily isomorphic.

Example 2.27. In the case of $n = 1$, there is such a continuous field of C^*-algebras on $[0, 1]$ with fibers given by $C^*(\mathbb{Z} *_{1_{\mathbb{Z}}} \mathbb{Z})$ at 0, $C^*(SL_2(\mathbb{Z}))$ at some $0 < t < 1$, and $C^*(\mathbb{Z}) \oplus_{\mathbb{C}} C^*(\mathbb{Z})$ at 1, where $1_{\mathbb{Z}}$ is the unit of $\mathbb{Z}$. Moreover,

$$\begin{aligned}
K_0(C^*(\mathbb{Z} *_{1_{\mathbb{Z}}} \mathbb{Z})) \cong K_0(C^*(\mathbb{Z}) \oplus_{\mathbb{C}} C^*(\mathbb{Z})) \cong [\mathbb{Z} \oplus \mathbb{Z}]/\mathbb{Z} \cong \mathbb{Z}, \\
K_1(C^*(\mathbb{Z} *_{1_{\mathbb{Z}}} \mathbb{Z})) \cong K_1(C^*(\mathbb{Z}) \oplus_{\mathbb{C}} C^*(\mathbb{Z})) \cong [\mathbb{Z} \oplus \mathbb{Z}]/0 \cong \mathbb{Z}^2,
\end{aligned}$$

but $K_0(C^*(SL_2(\mathbb{Z})) \cong \mathbb{Z}^8$ and $K_1(C^*(SL_2(\mathbb{Z})) \cong 0$ as confirmed.

Example 2.28. As a contrast, let G be the discrete Heisenberg solvable group of rank 3, of all upper triangular 3×3 matrices over $\mathbb{Z}$ with 1 on the diagonal. It is well known that $C^*(G)$ can be viewed as a continuous field of C^*-algebras over the one torus $\mathbb{T}$ as the dual group of $\mathbb{Z}$ as the center of G, with fibers given by rational or irrational rotation C^*-algebras, or 2-dimensional noncommutative tori. In this case, all the fibers have the same K-theory groups isomorphic to $K_j(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ for $j = 0, 1$ (see [1]).

3 The crossed product C^*-algebras The full group C^*-algebras $C^*(H_n)$ and $C^*(H_n^t)$ are viewed as the full crossed product C^*-algebras by actions of $SL_n(\mathbb{Z})$ as

$$C^*(H_n) = C^*(\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})) \cong C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z})$$

and

$$C^*(\mathbb{Z}^n) \rtimes \langle H_{n-1}, H_{n-1}^t \rangle = C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z}).$$

As obtained as Proposition 2.10 in the previous section,

Proposition 3.1. *There is a surjective C^*-algebra homomorphism from the full unital free product C^*-algebra to the full unital crossed product C^*-algebra as*

$$C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(H_{n-1} *_{SL_{n-1}(\mathbb{Z})} H_{n-1}^t) \xrightarrow{\mathrm{id} *_{\mathbb{C}} \varphi} C^*(\mathbb{Z}^n) \rtimes \langle H_{n-1}, H_{n-1}^t \rangle \to 0,$$

which is induced by φ and the identity map id *on $C^*(\mathbb{Z}^n)$ and is denoted by the symbol* $\mathrm{id} *_{\mathbb{C}} \varphi$.

Moreover, we have K-theory group homomorphisms as

$$\begin{array}{c} K_j(C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(H_{n-1} *_{SL_{n-1}(\mathbb{Z})} H_{n-1}^t) \\ {\scriptstyle (\mathrm{id} *_{\mathbb{C}} \varphi)_*}\downarrow \\ K_j(C^*(\mathbb{Z}^n) \rtimes \langle H_{n-1}, H_{n-1}^t \rangle) = K_j(C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z})) \end{array}$$

induced by $\mathrm{id} *_{\mathbb{C}} \varphi$ *for $j = 0, 1$. Furthermore, we have a KK-theory class*

$$[\mathrm{id} *_{\mathbb{C}} \varphi] \in KK(C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(H_{n-1} *_{SL_{n-1}(\mathbb{Z})} H_{n-1}^t), C^*(\mathbb{Z}^n) \rtimes \langle H_{n-1}, H_{n-1}^t \rangle)$$

induced by $\mathrm{id} *_{\mathbb{C}} \varphi$.

Proof. The surjective homomorphism φ between those full group C^*-algebras defined in the previous section extends trivially to the surjective C^*-algebra homomorphism by universality of the full untal free product C^*-algebras. It induces the K-theory group homomorphism $(\mathrm{id} *_{\mathbb{C}} \varphi)_*$ and the KK-theory class $[\mathrm{id} *_{\mathbb{C}} \varphi]$ as in the statement. □

As obtained as Proposition 2.18 in the previous section,

Proposition 3.2. *There is a surjective C^*-algebra homomorphism as*

$$\begin{array}{c} C^*(\mathbb{Z}^n) \rtimes \langle H_{n-1}, H_{n-1}^t \rangle = C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z}) \\ {\scriptstyle \mathrm{id} \oplus_{\mathbb{C}} \psi}\downarrow \\ C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} [C^*(H_{n-1}) \oplus_{C^*(SL_{n-1}(\mathbb{Z}))} C^*(H_{n-1}^t)] \longrightarrow 0 \end{array}$$

where the image is the successive, surjective pull back C^*-algebra obtained as before.

Moreover, we have K-theory group homomorphisms as

$$\begin{array}{c} K_j(C^*(\mathbb{Z}^n) \rtimes \langle H_{n-1}, H_{n-1}^t \rangle) = K_j(C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z})) \\ {\scriptstyle (\mathrm{id} \oplus_{\mathbb{C}} \psi)_*}\downarrow \\ K_j(C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} [C^*(H_{n-1}) \oplus_{C^*(SL_{n-1}(\mathbb{Z}))} C^*(H_{n-1}^t)]) \end{array}$$

induced by $\mathrm{id} \oplus_{\mathbb{C}} \psi$ *for $j = 0, 1$. Furthermore, we have a KK-theory class*

$$[\mathrm{id} \oplus_{\mathbb{C}} \psi] \in KK(C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z}), C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} [C^*(H_{n-1}) \oplus_{C^*(SL_{n-1}(\mathbb{Z}))} C^*(H_{n-1}^t)])$$

induced by $\mathrm{id} \oplus_{\mathbb{C}} \psi$.

Proof. There are canonical C^*-algebra homomorphisms

$$\psi_1 = \mathrm{id} : C^*(\mathbb{Z}^n) \to C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} [C^*(H_{n-1}) \oplus_{C^*(SL_{n-1}(\mathbb{Z}))} C^*(H_{n-1}^t)]$$

and

$$\psi_2 = \psi : C^*(H_{n-1}, H_{n-1}^t) \to C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} [C^*(H_{n-1}) \oplus_{C^*(SL_{n-1}(\mathbb{Z}))} C^*(H_{n-1}^t)],$$

defined as

$$\psi_1(u_z)) = (u_z, 1) \quad \text{and} \quad \psi_2(u_g) = (1, \psi(u_g))$$

for $u_z \in C^*(\mathbb{Z}^n)$ and $u_g \in C^*(H_{n-1}, H_{n-1}^t)$, Those homomorphisms ψ_1 and ψ_2 induce the homomorphism $\mathrm{id} \oplus_{\mathbb{C}} \psi$ as in the statement. It induces the K-theory group homomorphism $(\mathrm{id} \oplus_{\mathbb{C}} \psi)_*$ and the KK-theory class $[\mathrm{id} \oplus_{\mathbb{C}} \psi]$ as in the statement. □

Corollary 3.3. *There are successive, surjective C^*-algebra homomorphisms as*

$$\begin{array}{ccc} C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(H_{n-1} *_{SL_{n-1}(\mathbb{Z})} H_{n-1}^t) & & \\ \Big\downarrow {\scriptstyle \mathrm{id} *_{\mathbb{C}} \varphi} & & \\ C^*(\mathbb{Z}^n) \rtimes \langle H_{n-1}, H_{n-1}^t \rangle = C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z}) & \longrightarrow & 0 \\ \Big\downarrow {\scriptstyle \mathrm{id} \oplus_{\mathbb{C}} \psi} & & \\ C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} [C^*(H_{n-1}) \oplus_{C^*(SL_{n-1}(\mathbb{Z}))} C^*(H_{n-1}^t)] & \longrightarrow & 0. \end{array}$$

We then obtain

Theorem 3.4. *Suppose that $((\mathrm{id} \oplus_{\mathbb{C}} \psi) \circ (\mathrm{id} *_{\mathbb{C}} \varphi))_* = (\mathrm{id} \oplus_{\mathbb{C}} \psi)_* \circ (\mathrm{id} *_{\mathbb{C}} \varphi)_*$. Then there is an injective K-theory group homomorphism $(\mathrm{id} *_{\mathbb{C}} \varphi)_*$ and a surjective $(\mathrm{id} \oplus_{\mathbb{C}} \psi)_*$:*

$$\begin{array}{cc} 0 \longrightarrow & K_j(C^*(\mathbb{Z}^n) *_{\mathbb{C}} [C^*(H_{n-1}) *_{C^*(SL_{n-1}(\mathbb{Z}))} C^*(H_{n-1}^t)]) \\ & \Big\downarrow {\scriptstyle (\mathrm{id} *_{\mathbb{C}} \varphi)_*} \\ & K_j(C^*(\mathbb{Z}^n) \rtimes \langle H_{n-1}, H_{n-1}^t \rangle) = K_j(C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z}))), \\ & K_j(C^*(\mathbb{Z}^n) \rtimes \langle H_{n-1}, H_{n-1}^t \rangle) = K_j(C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z}))) \\ & \Big\downarrow {\scriptstyle (\mathrm{id} \oplus_{\mathbb{C}} \psi)_*} \\ 0 \longleftarrow & K_j(C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} [C^*(H_{n-1}) \oplus_{C^*(SL_{n-1}(\mathbb{Z}))} C^*(H_{n-1}^t)]) \end{array}$$

where the maps $(\mathrm{id} *_{\mathbb{C}} \varphi)_*$ and $(\mathrm{id} \oplus_{\mathbb{C}} \psi)_*$ are induced from the quotient maps $\mathrm{id} *_{\mathbb{C}} \varphi$ and $\mathrm{id} \oplus_{\mathbb{C}} \psi$ of Propositions 3.1 and 3.2 respectively, and it holds that the image of $(\mathrm{id} *_{\mathbb{C}} \varphi)_*$ is isomorphic to the image of $(\mathrm{id} \oplus_{\mathbb{C}} \psi)_*$.

Moreover, without the assumption above, the Kasparov product of $[\mathrm{id} *_{\mathbb{C}} \varphi]$ and $[\mathrm{id} \oplus_{\mathbb{C}} \psi]$ is a KK-equivalence.

Proof. Note that the composite $(\mathrm{id} \oplus_{\mathbb{C}} \psi) \circ (\mathrm{id} *_{\mathbb{C}} \varphi)$ is the canonical homomorphism from the successive amalgam C^*-algebra with retractions onto the successive, surjective pull back C^*-algebra, and then the induced K-theory group homomorphism $(\mathrm{id} \oplus_{\mathbb{C}} \psi)_* \circ (\mathrm{id} *_{\mathbb{C}} \varphi)_*$ is an isomorphism as cited in the previous section. Therefore, it must be that $(\mathrm{id} *_{\mathbb{C}} \varphi)_*$ is injective and $(\mathrm{id} \oplus_{\mathbb{C}} \psi)_*$ is surjective, and the image of $(\mathrm{id} *_{\mathbb{C}} \varphi)_*$ is isomorphic to the image of $(\mathrm{id} \oplus_{\mathbb{C}} \psi)_*$.

Note as well that the Kasparov product $[\mathrm{id} \oplus_{\mathbb{C}} \psi] \otimes [\mathrm{id} *_{\mathbb{C}} \varphi]$ is equal to $[(\mathrm{id} \oplus_{\mathbb{C}} \psi) \circ (\mathrm{id} *_{\mathbb{C}} \varphi)]$, and that KK-theory equivalence is equivalent to the K-theory equivalence [2]. □

Theorem 3.5. *There are K-theory group homomorphisms:*

$$\{K_j(C^*(\mathbb{Z}^n)) \oplus [(K_j(C^*(H_{n-1})) \oplus K_j(C^*(H_{n-1}^t)))/K_j(C^*(SL_{n-1}(\mathbb{Z}))]\}/K_j(\mathbb{C})$$
$$\downarrow$$
$$K_j(C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z}))$$

for $j = 0, 1$. *These are injective only when* $((\mathrm{id} \oplus_{\mathbb{C}} \psi) \circ (\mathrm{id} *_{\mathbb{C}} \varphi))_* = (\mathrm{id} \oplus_{\mathbb{C}} \psi)_* \circ (\mathrm{id} *_{\mathbb{C}} \varphi)_*$.

Corollary 3.6. *Suppose that* $((\mathrm{id} \oplus_{\mathbb{C}} \psi) \circ (\mathrm{id} *_{\mathbb{C}} \varphi))_* = (\mathrm{id} \oplus_{\mathbb{C}} \psi)_* \circ (\mathrm{id} *_{\mathbb{C}} \varphi)_*$. *Then for* $n \geq 2$, *we have the following inclusions as groups:*

$$\begin{aligned}
&K_0(C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z})) \supset \\
&\{\mathbb{Z}^{2^{n-1}} \oplus [(K_0(C^*(H_{n-1})) \oplus K_0(C^*(H_{n-1}^t)))/K_0(SL_{n-1}(\mathbb{Z}))]\}/\mathbb{Z}, \\
&K_1(C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z})) \supset \\
&\mathbb{Z}^{2^{n-1}} \oplus [(K_1(C^*(H_{n-1})) \oplus K_1(C^*(H_{n-1}^t)))/K_1(SL_{n-1}(\mathbb{Z}))].
\end{aligned}$$

In particular,

$$K_0(C^*(\mathbb{Z}^3) \rtimes SL_3(\mathbb{Z})) \supset \{\mathbb{Z}^4 \oplus [(K_0(C^*(H_2)) \oplus K_0(C^*(H_2^t)))/\mathbb{Z}^8]\}/\mathbb{Z},$$

and

$$K_1(C^*(\mathbb{Z}^3) \rtimes SL_3(\mathbb{Z})) \supset \mathbb{Z}^4 \oplus K_1(C^*(H_2)) \oplus K_1(C^*(H_2^t)).$$

Remark. We do not know whether the assumption is correct or not in general, and even if correct, then whether those inclusions are strict or not. We have the similar conjecture as mentioned in the remark after Corollary 2.23 and Example 2.24.

As a contrast, but in the case of a trivial action involved,

Example 3.7. There is a surjective C^*-algebra homomorphism from the unital full free product C^*-algebra $C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(SL_n(\mathbb{Z}))$ to the crossed product C^*-algebra $C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z})$, but with the trivial action, which is isomorphic to the (full or reduced) C^*-algebra tensor product $C^*(\mathbb{Z}^n) \otimes C^*(SL_n(\mathbb{Z}))$ (since $C^*(\mathbb{Z}^n)$ is nuclear). Also, there is a surjective C^*-algebra homomorphism from this tensor product to the pull back C^*-algebra $C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} C^*(SL_n(\mathbb{Z}))$. It follows that

$$\begin{aligned}
&K_0(C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(SL_n(\mathbb{Z}))) \cong K_0(C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} C^*(SL_n(\mathbb{Z}))) \\
&\cong [K_0(C^*(\mathbb{Z}^n)) \oplus K_0(C^*(SL_n(\mathbb{Z})))]/K_0(\mathbb{C}) \cong [\mathbb{Z}^{2^{n-1}} \oplus K_0(C^*(SL_n(\mathbb{Z})))]/\mathbb{Z}, \\
&K_1(C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(SL_n(\mathbb{Z}))) \cong K_1(C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} C^*(SL_n(\mathbb{Z}))) \\
&\cong [K_1(C^*(\mathbb{Z}^n)) \oplus K_1(C^*(SL_n(\mathbb{Z})))]/K_1(\mathbb{C}) \cong \mathbb{Z}^{2^{n-1}} \oplus K_1(C^*(SL_n(\mathbb{Z})))
\end{aligned}$$

but the Künneth theorem in K-theory for C^*-algebras implies that

$$\begin{aligned}
&K_0(C^*(\mathbb{Z}^n) \otimes C^*(SL_n(\mathbb{Z}))) \cong \\
&[\mathbb{Z}^{2^{n-1}} \otimes K_0(C^*(SL_n(\mathbb{Z})))] \oplus [\mathbb{Z}^{2^{n-1}} \otimes K_1(C^*(SL_n(\mathbb{Z})))] \\
&\cong K_1(C^*(\mathbb{Z}^n) \otimes C^*(SL_n(\mathbb{Z}))),
\end{aligned}$$

both K_0 and K_1-groups of which contain strictly the above K_0 and K_1-groups respectively, for $n \geq 2$.

As another consequence, we obtain

Corollary 3.8. *There is a continuous field of C^*-algebras over the closed interval $[0,1]$ with fibers changing from $C^*(\mathbb{Z}^n) *_{\mathbb{C}} C^*(H_{n-1} *_{SL_{n-1}(\mathbb{Z})} H_{n-1}^t)$ at 0 to $C^*(\mathbb{Z}^n) \rtimes \langle H_{n-1}, H_{n-1}^t \rangle = C^*(\mathbb{Z}^n) \rtimes SL_n(\mathbb{Z})$ at some $0 < t < 1$ and to $C^*(\mathbb{Z}^n) \oplus_{\mathbb{C}} [C^*(H_{n-1}) \oplus_{C^*(SL_{n-1}(\mathbb{Z}))} C^*(H_{n-1}^t)]$ at 1 along $\mathrm{id} *_{\mathbb{C}} \varphi$ and $\mathrm{id} \oplus_{\mathbb{C}} \psi$ respectively, such that their canonical generators associated with $\mathbb{Z}^n$, H_{n-1}, and H_{n-1}^t are defined to be continuous.*

Proof. The constructions is similar as given in the proof of Corollary 2.26. □

References

[1] J. Anderson and W. Paschke, *The rotation algebra*, Houston J. Math. **15** (1989), 1-26.

[2] Bruce Blackadar, *K-theory for Operator Algebras*, Second Edition, Cambridge, (1998).

[3] Joachim Cuntz, *K-groups for free products of C^*-algebras*, Proc. Sympos. Pure Math., **38** Part I, AMS (1982), 81-84.

[4] J. Cuntz, *K-theoretic amenability for discrete groups*, J. reine angew. Math., **344** (1983), 180-195.

[5] Kenneth R. Davidson, *C^*-algebras by Example*, Fields Institute Monographs 6, AMS (1996).

[6] Pierre de la Harpe, *Topics in Geometric Group Theory*, The University of Chicago Press, (2000).

[7] R. Exel, *The soft torus and applications to almost commuting matrices*, Pacific J. Math. **160**, No. 2 (1993), 207-217.

[8] R. Exel, *The soft torus: A variational analysis of commutator norms*, J. Funct. Anal. **126** (1994), 259-273.

[9] Morris Newman, *Integral Matrices*, Academic Press (1972).

[10] Gert K. Pedersen, *C^*-Algebras and their Automorphism Groups*, Academic Press (1979).

[11] Jean-Paul Pier, *Amenable locally compact groups*, A Wiley-Interscience Publication (1984).

[12] C. Soulé, *The cohomology of $SL_3(\mathbb{Z})$*, Topology, **17** (1978), 1-22.

[13] T. Sudo, *K-theory of the pullback and pushout C^*-algebras*, Scientiae Mathematicae Japonicae, **65**, No. 1 (2007), 53-60, :e2006, 1061-1068.

[14] T. Sudo, *Noncommutative continuous deformation theory by soft C^*-algebras — a review*, Ryukyu Math. J. **22** (2009), 31-114.

[15] T. Sudo, *K-theory for the group C^*-algebras of a residually finite discrete group with Kazhdan property T*, Cubo A Mathematical Journal, Vol. **16**, No. 2 (2014), 49-52.

[16] N. E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford Univ. Press (1993).

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