SIG-DIMENSION OF $K_{2,2}$−FREE GRAPHS

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Abstract. This paper introduces an algorithmic approach to investigate into the SIG-dimension of graphs, under the sup-norm. We provide an upper bound for the SIG-dimension of graphs, without isolated vertices, which do not contain an induced subgraph isomorphic to $K_{2,2}$.

1 Introduction The sphere-of-influence graph (SIG) on a set of points, each with an open ball centered about it of radius equal to the distance between that point and its nearest neighbor, is defined to be the intersection graph of these balls.

The notion of the sphere of influence graphs was introduced by Toussaint to model situations in pattern recognition and computer vision. These are used to help separate objects or otherwise capture perceptual relevance, see [6, 7, 8].

Toussaint has used the SIGs under L_2-norm to capture low-level perceptual information in certain dot patterns. The SIGs in general metric spaces are considered in [3]. It is known that the SIGs under the L_∞-norm perform better for this purpose, see [4]. Below we provide the construction of SIGs in this case.

Let d be a natural number and $\mathbb{R}^d$ denotes the d-dimensional Euclidean space. For any $z \in \mathbb{R}^d$, let $z[j]$ denotes the j^{th} component of z. The distance between any $x, y \in \mathbb{R}^d$ under the L_∞-metric, denoted by $\rho(x,y)$, is defined as,

$$\rho(x,y) := \max\{|x[j] - y[j]| : j = 1, 2, \dots, d\}.$$

Let $P \subset \mathbb{R}^d$ be a finite set having atleast two points. For a point $v \in P$, let r_v denotes the distance of v to its *nearest neighbor,* that is

$$r_v = \min\{\rho(u,v) : u \in P \setminus \{v\}\}.$$

The open ball $B_v := \{u \in \mathbb{R}^d : \rho(u,v) < r_v\}$ is known as the sphere of influence at v. The sphere of influence graph of P, denoted by $SIG^d_\infty(P)$, is the graph with vertex set P and edges corresponding to the pairs of intersecting spheres of influence. That is, the edge set of $SIG^d_\infty(P)$ is

$$\{uv : B_u \cap B_v \neq \emptyset; u, v \in P\}.$$

Throughout this paper, $E(G)$ and $V(G)$ will denote the vertex set and the edge set of a graph G. Note that for $G = SIG^d_\infty(P)$ and $u, v \in P$,

$$uv \in E(G) \iff \rho(u,v) < r_u + r_v.$$

A graph G is said to be *realizable* in $\mathbb{R}^d$ if there exists a finite set $P \subset \mathbb{R}^d$ such that G is isomorphic to $SIG^d_\infty(P)$. Note that if G is realized in $\mathbb{R}^d$, then it is realizable in $\mathbb{R}^{d+e}$ for every $e \in \mathbb{N}$. This can be observed by appending e zero coordinates to each point in the vertex set. The smallest such d is called the SIG-dimension of a graph G, denoted by $SIG(G)$. That is,

$$SIG(G) = \min\{d : G \text{ is realizable in } \mathbb{R}^d\}.$$

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It is trivial to see that if a graph with at least two vertices is realizable in some $\mathbb{R}^d$, then it can not have isolated vertices. Also, each graph G with atleast two vertices and no isolated vertices can be realized in $\mathbb{R}^d$, for some $d \in \mathbb{N}$. This can be seen as the rows of the matrix $2I + A$ realize G, where A is the adjacency matrix for G and I is the identity matrix, for more details see [4, Theorem 1].

Recently in [9], Taussaint has surveyed the theory and applications of sphere of influence graphs. In [4], several open problems on SIG-dimension have been discussed, the one regarding SIG-dimension of trees has already been solved, for details see [2]. In [5], we have proved the SIG-dimension conjecture for graphs having a perfect matching. A few partial results regarding the SIG-dimension for some particular graphs are proved in [1, 4].

It is easy to see that if G is path of size n, then $SIG(G) = 1$. Also it is known that if G is a graph of size n with no isolated vertex, then $SIG(G) \leq n - 1$, for details see [4].

In this paper, we consider the graphs which do not contain an induced subgraph isomorphic to $K_{2,2}$. We call them $K_{2,2}$-*free* graphs. We prove that if G is a $K_{2,2}$-free graph of order n which has no isolated vertex, then

$$SIG(G) \leq \left\lfloor \frac{3n}{4} \right\rfloor + \lceil \log_2 n \rceil + 1.$$

2 Definitions and Notations To establish our main result for $K_{2,2}$-free graphs, we will map our graph to a suitably required finite dimensional Euclidean space. But before that, we simply categorize the vertices in terms of triplets and pairs as per the following algorithm.

We start with a $K_{2,2}$-free graph, of size n, without an isolated vertex. The fact that G is $K_{2,2}$-free will be used later in our constructions, not for the following algorithm.

Algorithm 1 *Step I. Let G be a $K_{2,2}$-free graph, of size n, without an isolated vertex.*

Step II. Take an edge $pq \in E(G)$. There are two possible cases:

Case 1. There is a vertex $s \in V(G)$ such that exactly one of ps or qs is an edge. That is,

$$\text{(1)} \qquad \text{either } 'ps \in E(G) \ \ \& \ \ qs \notin E(G)' \text{ or } 'ps \notin E(G) \ \ \& \ \ qs \in E(G)'.$$

Define $n(p) = n(q) = n(s) = 0$. The set $\{p, q, s\}$ will be called a root of G.

Case 2. There is no vertex $s \in V(G)$ satisfying (1). That is, for all $s \in V(G)$,

$$ps \in E(G) \iff qs \in E(G).$$

Define $n'(p) = n'(q) = 0$. The set $\{p, q\}$ will be called a root of G.

Step III. Let $G_1 = G \setminus R$, where $R \neq \emptyset$ is a root of G and $r \in R$. Let

$$k = \begin{cases} n'(r) + 1, & \text{if } |R| = 2 \\ n(r) + 1, & \text{if } |R| = 3 \end{cases}$$

Case 1. $E(G_1) \neq \emptyset$. As above, let R_1 be a root of G_1.

$$\text{If } |R_1| = 2, \text{ define } n'(u) = k \text{ and if } |R_1| = 3, \text{ define } n(u) = k, \text{ for all } u \in R_1.$$

Set $G = G_1$ and repeat Step 2.

Case 2. $E(G_1) = \emptyset$. For all $v \in V(G_1)$, define $n''(v) = k$.

Note that, the vertices v for which $n''(v)$ is defined, form an independent set. Therefore, the vertices of our graph are divided into triplets, pairs and the remaining independent set.

In order to facilitate our argument, we now fix up few notations. Note that for any $v \in V(G)$, exactly one of $n(v)$, $n'(v)$ and $n''(v)$ is defined.

Notations 2. 1. For any $v \in V(G)$, the *index* of v, denoted by $m(v)$, is defined as follows:

$$m(v) := \begin{cases} n(v) & \text{if } n(v) \text{ is defined} \\ n'(v) & \text{if } n'(v) \text{ is defined} \\ n''(v) & \text{if } n''(v) \text{ is defined.} \end{cases}$$

2. Let α denotes the maximum value of $m(v); v \in V(G)$.

3. If v is a vertex such that $n''(v)$ is defined, choose a vertex u such that $uv \in E(G)$ and call it $N(v)$. That is, $N(v) = u$.

   Comment: There can be more than one such vertices u, which have an edge with v. In that case we fix up any one of these and call it $N(v)$.

4. Let $r > 0$ be any real and $\delta := \frac{r}{n+2}$.

5. For $0 \leq k \leq \alpha$ and for $v \in V(G)$, let $r(v) := r + \delta m(v)$.

As a common practice in most analytic proofs, the purpose of the above particular choice of $\delta > 0$ will be cleared later, in our proofs.

Remark 3. For any triplet $\{p, q, s\}$, $r(p) = r(q) = r(s) = r + \delta m(p)$. Similarly, it is same on every pair and on the residual independent set.

3 Mapping the graph to a Euclidean space In this section, we map the vertices of our given graph to a Euclidean space. This mapping will be done in a way that the corresponding SIG becomes isomorphic to the given graph. The bijection will be proved in the next section.

Each triplet, as per the previus section, will determine two dimensions of the Euclidean space, while the pairs will determine a single dimension. The final independent set will be considered in a separate manner later, while assigning new dimensions to the vertices.

Below we present the detailed algorithm to ensure the same.

Algorithm 4. *Step 1. Let G be a $K_{2,2}$-free graph, of size $n(\geq 2)$, without an isolated vertex.*

Step 2. Apply Algorithm 1 on G to categorize its vertices into triplets, pairs and an independent set.

Step 3. Repeat this Step, for each $k = 0, 1, \ldots \alpha$. Find $v \in V(G)$ with $m(v) = k$.

Case 1. There is a triplet $\{p, q, s\}$ such that $m(p) = m(q) = m(s) = k$ and $n(p)$ is defined. Without loss of generality, we assume that $qs \notin E(G)$. We define $c_{1(k)}$ and $c_{2(k)}$ on vertices of G as follows. Let $v \in V(G)$.

Case 1.1. If $m(v) < k$, then we define

$$c_{1(k)}(v) := c_{2(k)}(v) := \frac{3}{2}r(p).$$

Case 1.2. If $m(v) = k$, then $v \in \{p, q, s\}$. Define

$$c_{1(k)}(v) := \begin{cases} 0 & \text{if } v = q \\ r(p) & \text{if } v = p \\ 2r(p) & \text{if } v = s \end{cases} \quad \text{and } c_{2(k)}(v) := \begin{cases} 0 & \text{if } v = s \\ r(p) & \text{if } v = p \\ 2r(p) & \text{if } v = q. \end{cases}$$

Case 1.3. If $k < m(v) < \alpha$, then we define

$$c_{1(k)}(v) := \begin{cases} 2r(p) + r(v) & \text{if } vp \notin E(G), vq \notin E(G) \text{ and } vs \notin E(G) \\ 2r(p) + r(v) & \text{if } vp \notin E(G), vq \notin E(G) \text{ and } vs \in E(G) \\ r(p) + r(v) & \text{if } vp \in E(G), vq \notin E(G) \text{ and } vs \notin E(G) \\ r(p) + r(v) & \text{if } vp \in E(G), vq \notin E(G) \text{ and } vs \in E(G) \\ r(p) + r(v) - \delta & \text{if } vp \in E(G), vq \in E(G) \text{ and } vs \in E(G) \\ r(p) + r(v) - \delta & \text{if } vp \notin E(G), vq \in E(G) \text{ and } vs \notin E(G) \\ r(p) + r(v) - \delta & \text{if } vp \in E(G), vq \in E(G) \text{ and } vs \notin E(G) \end{cases}$$

and

$$c_{2(k)}(v) := \begin{cases} 2r(p) + r(v) & \text{if } vp \notin E(G), vq \notin E(G) \text{ and } vs \notin E(G) \\ r(p) + r(v) - \delta & \text{if } vp \notin E(G), vq \notin E(G) \text{ and } vs \in E(G) \\ r(p) + r(v) & \text{if } vp \in E(G), vq \notin E(G) \text{ and } vs \notin E(G) \\ r(p) + r(v) - \delta & \text{if } vp \in E(G), vq \notin E(G) \text{ and } vs \in E(G). \\ r(p) + r(v) - \delta & \text{if } vp \in E(G), vq \in E(G) \text{ and } vs \in E(G) \\ 2r(p) + r(v) & \text{if } vp \notin E(G), vq \in E(G) \text{ and } vs \notin E(G) \\ r(p) + r(v) & \text{if } vp \in E(G), vq \in E(G) \text{ and } vs \notin E(G) \end{cases}$$

Note that if $vp \notin E(G), vq \in E(G)$ and $vs \in E(G)$, then the induced subgraph of G on the vertices p, q, s and v is isomorphic to $K_{\{2,2\}}$, which is not possible.

Case 1.4. If $m(v) = \alpha$, define

$$c_{1(k)}(v) := c_{2(k)}(v) := \begin{cases} r(p) & \text{if } N(v) \in \{p, q, s\} \\ 2r(p) & \text{if } N(v) \notin \{p, q, s\}. \end{cases}$$

Case 2. There is a pair $\{p, q\}$ such that $n'(p)$ is defined and $m(p) = m(q) = k$. We define $c_{1(k)}$ on vertices $v \in V(G)$ as follows:

Case 2.1. If $m(v) < k$, define $c_{1(k)}(v) := \frac{3}{2}r(p)$.

Case 2.2. If $m(v) = k$, then $v \in \{p, q\}$. Define

$$c_{1(k)}(v) := \begin{cases} 0 & \text{if } v = p \\ r(p) & \text{if } v = q. \end{cases}$$

Case 2.3. If $k < m(v) < \alpha$, then we define

$$c_{1(k)}(v) := \begin{cases} 2r(p) + r(v) & \text{if } vp \notin E(G) \text{ and } vq \notin E(G) \\ r(p) + r(v) - \delta & \text{if } vp \in E(G) \text{ and } vq \in E(G). \end{cases}$$

Case 2.4. If $m(v) = \alpha$, we define

$$c_{1(k)}(v) := \begin{cases} r(p) & \text{if } N(v) \in \{p, q\} \\ 2r(p) & \text{if } N(v) \notin \{p, q\}. \end{cases}$$

Case 3. $k = \alpha$. Assume that there are exactly n_0 vertices $v_1, \ldots, v_{n_0}$ such that $m(v_1) = \cdots = m(v_{n_0}) = \alpha$. For each $l = 1, \ldots, n_0$, we define

$$c_{v_l(k)}(v) := \begin{cases} 0 & \text{if } v = v_l \\ r(v_l) & \text{if } vv_l \in E(G) \\ r(v_l) + r(v) & \text{if } vv_l \notin E(G) \end{cases}$$

Step 4. Define $c_{1'}$ and $c_{2'}$ on vertices $u \in V(G)$ as follows:

Case 1. $n(u)$ is defined. Then there exist two other vertices v_1 and v_2 such that $m(v_1) = m(v_2) = m(u)$. Define

$$\begin{aligned} & (c_{1'}(u), c_{2'}(u)) && := (0, r(u)), \\ & (c_{1'}(v_1), c_{2'}(v_1)) && := (r(u), 0) \\ \text{and} \quad & (c_{1'}(v_2), c_{2'}(v_2)) && := (r(u), r(u)). \end{aligned}$$

Case 2. $n'(u)$ is defined. Then there exists only one other vertex v such that $m(v) = m(u)$. Define

$$(c_{1'}(u), c_{2'}(u)) := (0, r(u)) \text{ and } (c_{1'}(v), c_{2'}(v)) := (r(u), 0).$$

Case 3. $n''(u)$ is defined. Define $(c_{1'}(u), c_{2'}(u)) := (0, 0)$.

Step 5. Let $d_0 := \lceil \log_2 \alpha \rceil$ and $P := \{p : p[j] = 1 \text{ or } -1\} \subset \mathbb{R}^{d_0}$. In this step we choose a point in $\mathbb{R}^{d_0}$, corresponding to every triplet and pair.

For each $k = 0, 1, \ldots, \alpha - 1$, pick a different point from P, say $p'_k \in P$ and let $p_k = (r - \delta)p'_k$. Also let

$$S_k = \{v : m(v) < \alpha \ \ \& \ \ m(v) = k\} \cup \{v : m(v) = \alpha \ \ \& \ \ m(N(v)) = k \ \ \}.$$

Now append p_k to each $s \in S_k$.

Remark 5. Note that $S_k \cap S_{k'} = \emptyset$, for all $k \neq k'$. Therefore the last step above, won't add more than $\lceil \log_2 \alpha \rceil$ dimensions to our mapping.

Remark 6. Further, every vertex is appended with $\lceil \log_2 \alpha \rceil$ coordinates in the mapping, as for every vertex v such that $n''(v)$ is defined, there exists at least one u such that $u = N(v)$ and $n''(u)$ is not defined. Otherwise, v has to be an isolated vertex in our graph, which is not the case.

4 The mapping is an isomorphism In the previous section, we mapped the vertex set on a Euclidean space by assigning the coordinates with respect to each triplet, pair and the independent set. For convenience, we will use the same symbol v for the image of v, under this mapping.

In this sense, the vertex set $V(G)$ is now projected in a Euclidean space endowed with sup metric. We now prove that the SIG of this mapped vertex set is isomorphic to our given graph. We prove our main result through a series of claims.

In the sequel, for $u, v \in G$ we will use the notation $|c_k(u) - c_k(v)|$, even when c_k represents a pair of Euclidean dimensions. In that case, as an abuse of notation, it will represent the sup-norm in those two dimensions.

Lemma 7. *For all $v \in V(G)$, we have $r_v \leq r(v)$.*

Proof. Let $u \in V(G)$. We have the following cases:

Case 1. $n(u)$ is defined. Then there exist v_1 and v_2 such that $n(v_1) = n(v_2) = n(u)$.

Case 1.1. $v_1 v_2 \notin E(G)$. Then $uv_1 \in E(G)$ & $uv_2 \in E(G)$. Note that it is enough to prove that $\rho(u, v_1) \leq r(u)$. Therefore it is enough to prove that

$$|u[j] - v_1[j]| \leq r(u), \text{ for each } j = 1, 2, \ldots \tag{2}$$

We verify (2), for each co-ordinate separately. First let $k = \{0, 1, \ldots, \alpha\}$.

Case 1.1.1. $k = m(u)$. Then the only possibilities for $c_{1(k)}$ and $c_{2(k)}$ are

$$c_{1(k)}(u) = r(u) = c_{2(k)}(u), c_{1(k)}(v_1) \in \{0, 2r(u)\} \text{ and } c_{2(k)}(v_1) \in \{0, 2r(u)\}.$$

Thus (2) is verified for $c_{1(k)}$ and $c_{2(k)}$, as we have

$$|c_{1(k)}(u) - c_{1(k)}(v_1)| = r(u) = |c_{2(k)}(u) - c_{2(k)}(v_1)|.$$

Case 1.1.2. $k < m(u)$. Let $v_3 \in V(G)$ be such that $m(v_3) = k$.

Case 1.1.2.1. $n(v_3)$ is defined. In this case, we have
'$c_{1(k)}(u) = 2r(v_3) + r(u)$ or $r(v_3) + r(u)$ or $r(v_3) + r(u) - \delta$'
'$c_{1(k)}(v_1) = 2r(v_3) + r(v_1)$ or $r(v_3) + r(v_1)$ or $r(v_3) + r(v_1) - \delta$'
Hence we see that

$$|c_{1(k)}(u) - c_{1(k)}(v_1)| \leq r(v_3) + \delta = r + \delta k + \delta \leq r + \delta m(u) = r(u).$$

The second inequality above holds, as we have $m(u) \geq k + 1$. Similarly, we obtain ,

$$|c_{2(k)}(u) - c_{2(k)}(v_1)| \leq r(u).$$

Case 1.1.2.2. $n'(v_3)$ is defined. In this case, we have
'$c_{1(k)}(u) = 2r(v_3) + r(u)$ or $r(v_3) + r(u) - \delta$'
'$c_{1(k)}(v_1) = 2r(v_3) + r(v_1)$ or $r(v_3) + r(v_1) - \delta$'
Hence. as earlier, we see that

$$|c_{1(k)}(u) - c_{1(k)}(v_1)| \leq r(v_3) + \delta \leq r(u).$$

Case 1.1.3. $k > m(u)$. Let $v_3 \in V(G)$ be such that $m(v_3) = k$.

Case 1.1.3.1. $n(v_3)$ is defined. Then
$c_{1(k)}(u) = \frac{3}{2}r(v_3)$ and $c_{1(k)}(v_1) = \frac{3}{2}r(v_3)$.
Therefore $|c_{1(k)}(u) - c_{1(k)}(v_1)| = 0$.
Similarly, $|c_{2(k)}(u) - c_{2(k)}(v_1)| = 0$. Similarly we deal with the case when $n'(v_3)$ is defined.

Case 1.1.3.2. $n''(v_3)$ is defined. For each $l = 1, 2, \dots, i$, we have
$c_{v_l(k)}(u) = r(v_3)$ or $r(v_3) + r(u)$
and $c_{v_l(k)}(v_1) = r(v_3)$ or $r(v_3) + r(v_1)$
Therefore $|c_{v_l(k)}(u) - c_{v_l(k)}(v_1)| \leq r(u)$.

Also note that $\max\{|c_{1'}(u) - c_{1'}(v_1)|, |c_{2'}(u) - c_{2'}(v_1)|\} = r(u)$ and $\max\{|p_{m(u)}[j] - p_{m(v_1)}[j]| : j = 1, 2, \dots\} = 0$.

This verifies (2) and hence, in this case $r_u \leq r(u)$.

Case 1.2. Either $uv_1 \notin E(G)$ or $uv_2 \notin E(G)$. Let $uv_1 \notin E(G)$. Then we have $uv_2 \in E(G)$ and $v_1v_2 \in E(G)$. This case is similar to Case 1.1.

Case 2. $n'(u)$ is defined. This case is analogous to Case 1.

Case 3. $n''(u)$ is defined. Then there is v such that $N(u) = v$. Therefore $uv \in E(G)$. Let $k \in \{0, 1, \dots, \alpha\}$.

Case 3.1. $k = m(u)$. For $l = 1, 2, \dots, n_0$, $c_{v_l(k)}(u) = 0$ or $2r(u)$.
If $c_{v_l(k)}(u) = 0$, we have $c_{v_l(k)}(v) = r(u)$.
If $c_{v_l(k)}(u) = 2r(u)$, we have $c_{v_l(k)}(v) = r(u)$ or $r(u) + r(v)$.
In both cases, we have $|c_{v_l(k)}(u) - c_{v_l(k)}(v)| \leq r(u)$.

Case 3.2. $k < m(u)$. Let $w \in V(G)$ be such that $m(w) = k$.

Case 3.2.1. $n(w)$ is defined.

Case 3.2.1.1. $m(v) = k$. Then we have $c_{1(k)}(u) = r(v)$ and $c_{1(k)}(v) = 0, r(v)$ or $2r(v)$. Then

$$|c_{1(k)}(u) - c_{1(k)}(v)| \leq r(v) = r + \delta k < r + \delta m(u) = r(u).$$

Similarly, we have $|c_{2(k)}(u) - c_{2(k)}(v)| < r(u)$.

Case 3.2.1.2. $m(v) \neq k$. Then we have $c_{1(k)}(u) = 2r(v)$ and $c_{1(k)}(v) = 2r(w) + r(v), r(w) + r(v), r(w) + r(v) - \delta$ or $\frac{3}{2}r(w)$. Again, we have

$$|c_{1(k)}(u) - c_{1(k)}(v)| \leq r(v) < r(u).$$

Case 3.2.2. $n'(w)$ is defined.

Case 3.2.2.1. $m(v) = k$. Then we have $c_{1(k)}(u) = r(v)$ and $c_{1(k)}(v) = 0$ or $r(v)$. Then we see that

$$|c_{1(k)}(u) - c_{1(k)}(v)| \leq r(v) < r(u).$$

Case 3.2.2.2. $m(v) \neq k$. Then we have $c_{1(k)}(u) = 2r(v)$ and $c_{1(k)}(v) = 2r(w) + r(v), r(w) + r(v) - \delta$ or $\frac{3}{2}r(w)$. Hence

$$|c_{1(k)}(u) - c_{1(k)}(v)| \leq r(v) < r(u).$$

Also, as earlier, we have

$$\max\{|c_{1'}(u) - c_{1'}(v)|, |c_{2'}(u) - c_{2'}(v)|\} = r(v) < r(u) \text{ and } \max\{|p_{m(u)}[j] - p_{m(v)}[j]| : j = 1, 2, \ldots\} = 0.$$

This implies that $\rho(u, v) = r(u)$. Therefore $r_u \leq r(u)$.

Hence the result. $\square$

Lemma 8. For all $v \in V(G)$, we have $r_v \geq r(v)$.

Proof. Let $v_1, v_2 \in V(G)$.

Case 1. There is some $k < \alpha$ such that $v_1, v_2 \in S_k$.

Case 1.1. Either $m(v_1) < \alpha$ or $m(v_2) < \alpha$. Then we have

$$\max\{|c_{i'}(v_1) - c_{i'}(v_2)| : i = 1, 2\} = r(v_1).$$

Case 1.2. $m(v_1) = m(v_2) = \alpha$. Then $v_1 v_2 \notin E(G)$ and we have $c_{v_1(n''(v_1))}(v_1) = 0$ and $c_{v_1(n''(v_1))}(v_2) = r(v_1) + r(v_2)$. Therefore

$$|c_{v_1(n''(v_1))}(v_1) - c_{v_1(n''(v_1))}(v_2)| = r(v_1) + r(v_2) > r(v_1).$$

Case 2. $v_1 \in S_{k_1}$ and $v_2 \in S_{k_2}$, where $k_1 \neq k_2$. Then, by our construction

$$\begin{aligned}
&\max\{|p_{m(v_1)}[i] - p_{m(v_2)}[i]| : i = 1, 2, \ldots\} \\
&= 2(r - \delta) \qquad = 2\left(r - \tfrac{r}{n+2}\right) \\
&= 2r\left(\tfrac{n+1}{n+2}\right) \qquad = r\left(\tfrac{2n+2}{n+2}\right) \\
&\geq r\left(\tfrac{n+k_1+2}{n+2}\right) \qquad = r\left(1 + \tfrac{k_1}{n+2}\right) \\
&= r + k_1\left(\tfrac{r}{n+2}\right) \qquad = r + k_1\delta \\
&= r(v_1).
\end{aligned}$$

This implies $\rho(v_1, v_2) \geq r(v_1) \Rightarrow r_{v_1} \geq r(v_1)$, which establishes our lemma. $\square$

The following is immediate from Lemma 7 and Lemma 8.

Proposition 9. *For all $v \in V(G)$, we have $r_v = r(v)$.*

Lemma 10. *If $v_1, v_2 \in V(G)$ are such that $v_1v_2 \notin E(G)$, then $\rho(v_1, v_2) \geq r_{v_1} + r_{v_2}$.*

Proof. Case 1. Either $n''(v_1)$ or $n''(v_2)$ or both $n''(v_1)$ and $n''(v_2)$ are defined. Without loss of generality, let $n''(v_1)$ is defined. Then $c_{v_1(n''(v_1))}(v_1) = 0$ and $c_{v_1(n''(v_1))}(v_2) = r(v_1) + r(v_2)$. Hence

$$\begin{aligned}\rho(v_1, v_2) &= \max\{|v_1[j] - v_2[j]| : j = 1, 2, \dots\} \\ &\geq |c_{v_1(n''(v_1))}(v_1) - c_{v_1(n''(v_1))}(v_2)| \\ &= r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.\end{aligned}$$

Case 2. Both $n''(v_1)$ are $n''(v_2)$ not defined.

Case 2.1. $m(v_1) = m(v_2)$. Clearly by our construction, the case that both $n'(v_1)$ and $n'(v_2)$ are defined fails, as in that case $v_1v_2 \in E(G)$. Therefore both $n(v_1)$ and $n(v_2)$ must be defined and $n(v_1) = n(v_2)$. Then, we have $c_{1(n(v_1))}(v_1) = 0$ or $2r(v_1)$.
Also $c_{1(n(v_1))}(v_1) = 0$ implies $c_{1(n(v_1))}(v_2) = 2r(v_1)$
and $c_{1(n(v_1))}(v_1) = 2r(v_1)$ implies $c_{1(n(v_1))}(v_2) = 0$.
Therefore, $|c_{1(n(v_1))}(v_1) - c_{1(n(v_1))}(v_2)| = 2r(v_1) = r_{v_1} + r_{v_2}$ and hence

$$\begin{aligned}\rho(v_1, v_2) &= \max\{|v_1[j] - v_2[j]| : j = 1, 2, \dots\} \\ &\geq |c_{v_1(n(v_1))}(v_1) - c_{v_1(n(v_1))}(v_2)| \\ &= r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.\end{aligned}$$

Case 2.2. $m(v_1) \neq m(v_2)$. Let $m(v_1) = k_1$ and $m(v_2) = k_2$. Without loss of generality, assume that $k_1 < k_2$.

Case 2.2.1. $n(v_1)$ is defined. Then $c_{1(m(v_1))}(v_1) = 0$ or $r(v_1)$ or $2r(v_1)$. In each of the following arguments, we look at the possibilities from our construction.
If $c_{1(m(v_1))}(v_1) = 0$ then

$$c_{1(m(v_1))}(v_2) = 2r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2).$$

If $c_{1(m(v_1))}(v_1) = r(v_1)$ then

$$c_{1(m(v_1))}(v_2) = 2r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2) - \delta.$$

Incase $c_{1(m(v_1))}(v_2) = r(v_1) + r(v_2) - \delta$, we have

$$c_{2(m(v_1))}(v_2) = 2r(v_1) + r(v_2). \text{ Already } c_{2(m(v_1))}(v_1) = r(v_1).$$

If $c_{1(m(v_1))}(v_1) = 2r(v_1)$ then $c_{2(m(v_1))}(v_1) = 0$ and

$$c_{1(m(v_1))}(v_2) = r(v_1) + r(v_2), 2r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2) - \delta.$$

Therefore

$$c_{2(m(v_1))}(v_2) = 2r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2).$$

Hence we observe that

$$\begin{aligned}\rho(v_1, v_2) &= \max\{|v_1[j] - v_2[j]| : j = 1, 2, \dots\} \\ &\geq \max\{|c_{i(m(v_1))}(v_1) - c_{i(m(v_1))}(v_2)| : i = 1, 2\} \\ &\geq r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.\end{aligned}$$

Case 2.2.2. $n'(v_1)$ is defined. Then we have $c_{1(m(v_1))}(v_1) = 0$ or $r(v_1)$ and $c_{1(m(v_1))}(v_2) = 2r(v_1) + r(v_2)$. Hence

$$\begin{aligned}\rho(v_1, v_2) &= \max\{|v_1[j] - v_2[j]| : j = 1, 2, \dots\} \\ &\geq |c_{1(m(v_1))}(v_1) - c_{1(m(v_1))}(v_2)| \\ &\geq r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.\end{aligned}$$

This proves our lemma. □

Lemma 11. If $v_1, v_2 \in V(G)$ are such that $v_1 v_2 \in E(G)$, then $\rho(v_1, v_2) < r_{v_1} + r_{v_2}$.

Proof. Pick $v_1, v_2 \in V(G)$ with $v_1 v_2 \in E(G)$ and let $k_1 = m(v_1)$ and $k_2 = m(v_2)$.

Case 1. $k_1 = k_2$.

Case 1.1. $n''(v_1)$ is defined. Then $n''(v_2)$ is defined. This implies $v_1 v_2 \notin E(G)$, which is not the case.

Case 1.2. $n'(v_1)$ is defined. Then $n'(v_2)$ is defined. Repeat the following steps for $k = 0$ to α.

Case 1.2.1. $k = k_1$. Then $c_{1(k)}(v_1) = 0$ or $r(v_1)$.
If $c_{1(k)}(v_1) = 0$ then $c_{1(k)}(v_2) = r(v_1)$ and if $c_{1(k)}(v_1) = r(v_1)$ then $c_{1(k)}(v_2) = 0$. Hence

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| = r(v_1) = r_{v_1}.$$

Case 1.2.2. $k > k_1$.

Case 1.2.2.1. k denotes the index of vertices in the independent set (left at the end of our algorithm), if any.
Then $c_{v_l(k)}(v_1) = r(v_l)$ or $r(v_l) + r(v_1)$. Also $c_{v_l(k)}(v_2) = r(v_l)$ or $r(v_l) + r(v_1)$ and therefore

$$|c_{v_l(k)}(v_1) - c_{v_l(k)}(v_2)| \leq r(v_1) = r_{v_1}.$$

Case 1.2.2.2. Otherwise, $c_{1(k)}(v_1) = \frac{3}{2}r(v_0)$, with $v_0 \in V(G)$ is such that $m(v_0) = k$. Also $c_{1(k)}(v_2) = \frac{3}{2}r(v_0)$ and therefore

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| = 0.$$

If $n(v_0)$ is defined, then $c_{2(k)}$ is defined and we have $c_{2(k)}(v_1) = \frac{3}{2}r(v_0)$ and $c_{2(k)}(v_2) = \frac{3}{2}r(v_0)$. Therefore

$$|c_{2(k)}(v_1) - c_{2(k)}(v_2)| = 0.$$

Case 1.2.3. $k < k_1$.

Case 1.2.3.1. There exists a vertex $v_3 \in V(G)$ such that $n'(v_3)$ is defined with $k = n'(v_3)$. Then

$$c_{1(k)}(v_1) = 2r(v_3) + r(v_1) \text{ or } r(v_3) + r(v_1) - \delta.$$

Also $c_{1(k)}(v_2) = 2r(v_3) + r(v_1)$ or $r(v_3) + r(v_1) - \delta$. Hence

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| \leq r(v_3) + \delta \leq r(v_1) = r_{v_1}.$$

Case 1.2.3.2. There exists a vertex $v_3 \in V(G)$ such that $n(v_3)$ is defined with $k = n(v_3)$. Then both $c_{1(k)}(v_1)$ and $c_{1(k)}(v_2)$ are either

$$2r(v_3) + r(v_1), r(v_3) + r(v_1) \text{ or } r(v_3) + r(v_1) - \delta.$$

Therefore, we have

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| \leq r(v_3) + \delta \leq r(v_1) = r_{v_1}.$$

Similarly, $|c_{2(k)}(v_1) - c_{2(k)}(v_2)| \leq r_{v_1}$.

Case 1.2.4. $(c_{1'}(v_1), c_{2'}(v_1)) = (0, r(v_1))$ or $(r(v_1), 0)$.
If $(c_{1'}(v_1), c_{2'}(v_1)) = (0, r(v_1))$, then $(c_{1'}(v_2), c_{2'}(v_2)) = (r(v_1), 0)$.
If $(c_{1'}(v_1), c_{2'}(v_1)) = (r(v_1), 0)$, then $(c_{1'}(v_2), c_{2'}(v_2)) = (0, r(v_1))$. Hence $\max\{|c_{i'}(v_1) - c_{i'}(v_2)| : i = 1, 2\} = r(v_1) = r_{v_1}$.

Case 1.2.5. $\max\{|p_{m(v_1)}[i] - p_{m(v_2)}[i]| : i = 1, 2, \dots\} = 0$.

Therefore, if $n'(v_1)$ and $n'(v_2)$ are defined and $n'(v_1) = n'(v_2)$, then

$$\rho(v_1, v_2) \leq r_{v_1} < r_{v_1} + r_{v_2}.$$

Case 1.3. $n(v_1)$ is defined. Then $n(v_2)$ is also defined.

Case 1.3.1. $k = k_1$. Then $c_{1(k)}(v_1) = 0, r(v_1)$ or $2r(v_1)$.
If $c_{1(k)}(v_1) = 0$ then $c_{1(k)}(v_2) = r(v_1)$.
If $c_{1(k)}(v_1) = r(v_1)$ then $c_{1(k)}(v_2) = 0$ or $2r(v_1)$.
If $c_{1(k)}(v_1) = 2r(v_1)$ then $c_{1(k)}(v_2) = r(v_1)$.
Hence

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| = r(v_1) = r_{v_1}.$$

Case 1.3.2. $k > k_1$. This case is same as Case 1.2.2.

Case 1.3.3. $k < k_1$. This case is same as Case 1.2.3.

Case 1.3.4. $(c_{1'}(v_1), c_{2'}(v_1)) = (0, r(v_1)), (r(v_1), 0)$ or $(r(v_1), r(v_1))$.
$(c_{1'}(v_2), c_{2'}(v_2)) = (0, r(v_1)), (r(v_1), 0)$ or $(r(v_1), r(v_1))$. Hence

$$\max\{|c_{i'}(v_1) - c_{i'}(v_2)| : i = 1, 2\} \leq r(v_1) = r_{v_1}.$$

Case 1.3.5. $\max\{|p_{m(v_1)}[i] - p_{m(v_2)}[i]| : i = 1, 2, \dots\} = 0$.

Therefore, if $n(v_1)$ and $n(v_2)$ are defined such that $n(v_1) = n(v_2)$ and $v_1v_2 \in E(G)$, then we have

$$\rho(v_1, v_2) \leq r_{v_1} < r_{v_1} + r_{v_2}.$$

This proves the result for the case $m(v_1) = m(v_2)$.

Case 2. $k_1 \neq k_2$. Without loss of generality, assume that $k_1 < k_2$. Repeat the following for $k = 0$ to α.

Case 2.1. $k < k_1$.

Case 2.1.1. There exists some $v_3 \in V(G)$ such that $n(v_3) = k$. Therefore

$$c_{1(k)}(v_1) = 2r(v_3) + r(v_1), r(v_3) + r(v_1) \text{ or } r(v_3) + r(v_1) - \delta,$$

$$c_{1(k)}(v_2) = 2r(v_3) + r(v_2), r(v_3) + r(v_2), r(v_3) + r(v_2) - \delta, 2r(v_3) \text{ or } r(v_3).$$

Hence we obtain

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

Similarly, $|c_{2(k)}(v_1) - c_{2(k)}(v_2)| \leq r_{v_1} + r_{v_2}$.

Case 2.1.2. There exists $v_3 \in V(G)$ be such that $n'(v_3)$ is defined and $n'(v_3) = k$. Then we see that

$$c_{1(k)}(v_1) = 2r(v_3) + r(v_1) \text{ or } r(v_3) + r(v_1) - \delta.$$

$$c_{1(k)}(v_2) = 2r(v_3) + r(v_2), r(v_3) + r(v_2) - \delta, 2r(v_3) \text{ or } r(v_3).$$

Hence $|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.

Case 2.2. $k = k_1$.

Case 2.2.1. $n(v_1)$ is defined. Then we have $c_{1(k)}(v_1) = 0, r(v_1)$ or $2r(v_1)$.
If $c_{1(k)}(v_1) = 0$ then $c_{1(k)}(v_2) = r(v_1), 2r(v_1)$ or $r(v_1) + r(v_2) - \delta$. Hence we have

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

If $c_{1(k)}(v_1) = r(v_1)$ then we have

$$c_{1(k)}(v_2) = r(v_1), 2r(v_1), r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2) - \delta.$$

Hence, as earlier

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

If $c_{1(k)}(v_1) = 2r(v_1)$ then

$$c_{1(k)}(v_2) = r(v_1), 2r(v_1), 2r(v_1) + r(v_2), r(v_1) + r(v_2) \text{ or } r(v_1) + r(v_2) - \delta.$$

Therefore $|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.
Similarly, we obtain

$$|c_{2(k)}(v_1) - c_{2(k)}(v_2)| < r_{v_1} + r_{v_2}.$$

Case 2.2.2. $n'(v_1)$ is defined. Then we have
$c_{1(k)}(v_1) = 0$ or $r(v_1)$ and $c_{1(k)}(v_2) = r(v_1), 2r(v_1)$ or $r(v_1)+r(v_2)-\delta$. Hence $|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.

Case 2.3. $k_1 < k < k_2$. Then there exists $v_3 \in V(G)$ such that $m(v_3) = k$.

Case 2.3.1. $n(v_3)$ is defined. Then we have $c_{1(k)}(v_1) = \frac{3}{2}r(v_3)$ and

$$c_{1(k)}(v_2) = r(v_3), 2r(v_3), 2r(v_3) + r(v_2), r(v_3) + r(v_2) \text{ or } r(v_3) + r(v_2) - \delta.$$

Case 2.3.2. $n'(v_3)$ is defined. Then we have $c_{1(k)}(v_1) = \frac{3}{2}r(v_3)$ and
$c_{1(k)}(v_2) = r(v_3), 2r(v_3), 2r(v_3) + r(v_2)$ or $r(v_3) + r(v_2) - \delta$.
Therefore, in both of the above cases, we observe that

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

Case 2.4. $k = k_2$.

Case 2.4.1. $n(v_2)$ is defined. Then $c_{1(k)}(v_1) = \frac{3}{2}r(v_2)$ and $c_{1(k)}(v_2) = 0, r(v_2)$ or $2r(v_2)$.
Therefore, we have

$$|c_{1(k)}(v_1) - c_{1(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

Similarly, we obtain

$$|c_{2(k)}(v_1) - c_{2(k)}(v_2)| < r_{v_1} + r_{v_2}.$$

Case 2.4.2. $n'(v_2)$ is defined. Then $c_{1(k)}(v_1) = \frac{3}{2}r(v_2)$ and $c_{1(k)}(v_2) = 0$ or $r(v_2)$.

Case 2.4.3. $n''(v_2)$ is defined. Then $c_{v_2(k)}(v_2) = 0$ and $c_{v_2(k)}(v_1) = r(v_2)$.
Also, for $v_l \neq v_2$ such that $n''(v_l)$ is defined, we have $c_{v_l(k)}(v_2) = r(v_l) + r(v_2) = 2r(v_2)$ and $c_{v_l(k)}(v_1) = r(v_2)$ or $r(v_1) + r(v_2)$. Hence

$$|c_{v_l(k)}(v_1) - c_{v_l(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

Case 2.5. $k > k_2$. Then there exists $v_3 \in V(G)$ such that $m(v_3) = k$.

Case 2.5.1. $n(v_3)$ is defined. Then $c_{1(k)}(v_1) = \frac{3}{2}r(v_3)$ and $c_{1(k)}(v_2) = \frac{3}{2}r(v_3)$. Hence $|c_{1(k)}(v_1) - c_{1(k)}(v_2)| = 0 < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.

Case 2.5.2. $n'(v_3)$ is defined. Then $c_{1(k)}(v_1) = \frac{3}{2}r(v_3), c_{1(k)}(v_2) = \frac{3}{2}r(v_3)$. Hence $|c_{1(k)}(v_1) - c_{1(k)}(v_2)| = 0 < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.

Case 2.5.3. $n''(v_3)$ is defined. Then we have $c_{v_l(k)}(v_1) = r(v_l)$ or $r(v_l) + r(v_1)$ and $c_{v_l(k)}(v_2) = r(v_l)$ or $r(v_l) + r(v_2)$. Hence

$$|c_{v_l(k)}(v_1) - c_{v_l(k)}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

Case 2.6. $(c_{1'}(v_1), c_{2'}(v_1)) = (0, r(v_1)), (r(v_1), 0)$ or $(r(v_1), r(v_1))$.
$(c_{1'}(v_2), c_{2'}(v_2)) = (0, r(v_2)), (r(v_2), 0), (r(v_2), r(v_2))$ or $(0, 0)$.
Therefore $|c_{1'}(v_1) - c_{1'}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$.
Also $|c_{2'}(v_1) - c_{2'}(v_2)| < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}$. Hence

$$\max\{|c_{i'}(v_1) - c_{i'}(v_2)| : i = 1, 2\} < r_{v_1} + r_{v_2}.$$

Case 2.7. Let $p_{k_{1'}}$ be associated with v_1 and $p_{k_{2'}}$ be associated with v_2.
Then, either $|p_{k_{1'}} - p_{k_{1'}}| = 0$ or

$$|p_{k_{1'}} - p_{k_{1'}}| = 2(r - \delta) < r(v_1) + r(v_2) = r_{v_1} + r_{v_2}.$$

This proves the result for the case $m(v_1) \neq m(v_2)$. Hence the result. □

The previous two lemmas essentially prove the following theorem.

Theorem 12. *For $v_1, v_2 \in V(G)$, we have*

$$v_1v_2 \in E(G) \text{ if and only if } \rho(v_1, v_2) < r_{v_1} + r_{v_2}.$$

Therefore the SIG of our mapping of $V(G)$ on the Euclidean space is isomorphic to G. In other words, G is realizable in a Euclidean space, whose dimension is fixed according to our algorithm. Next we will count the dimension of this Euclidean space.

5 The Main Result

We need the following result from [1, Corollary 9].

Lemma 13. *If G is a graph of order n with no isolated vertex. If G has an independent set of size $t > 1$, then*

$$SIG(G) \leq n - 1 - t + \lceil \log_2 t \rceil.$$

Remark 14. In Step 3 of our construction, we attach $\lceil \log_2 \alpha \rceil$ co-ordinates to each vertex. As $\alpha \leq n/2$, we attach maximum $\lceil \log_2 \lfloor \frac{n}{2} \rfloor \rceil$ co-ordinates. Since

$$\left\lceil \log_2 \left\lfloor \frac{n}{2} \right\rfloor \right\rceil \leq \left\lceil \log_2 \frac{n}{2} \right\rceil = \lceil \log_2 n - \log_2 2 \rceil = \lceil \log_2 n \rceil - 1,$$

we attach maximum $\lceil \log_2 n \rceil - 1$ co-ordinates.

Now we prove the main result of this paper.

Theorem 15. *Let G be a $K_{2,2}$-free graph with $n(\geq 2)$ vertices. If G has no isolated vertex, then*

$$SIG(G) \leq \left\lfloor \frac{3n}{4} \right\rfloor + \lceil \log_2 n \rceil + 1.$$

Proof. Let $S := \{v : v \in V(G) \text{ and } n''(v) \text{ is defined}\}$. Let $|S| = \beta$. Using our construction in Section 3 along with Remark 14, we obtain

$$SIG(G) \leq \frac{2}{3}(n-\beta) + \beta + (\lceil \log_2 n \rceil - 1) + 2 = \frac{2}{3}n + \frac{1}{3}\beta + \lceil \log_2 n \rceil + 1.$$

If $\beta = \frac{n}{4}$, then

$$SIG(G) \leq \frac{2n}{3} + \frac{n}{12} + \lceil \log_2 n \rceil + 1 = \frac{3n}{4} + \lceil \log_2 n \rceil + 1.$$

If $\beta < \frac{n}{4}$, then $\beta = \frac{n}{4} - k$, for some $k > 0$ and we have

$$SIG(G) \leq \frac{2n}{3} + \frac{n}{12} - \frac{k}{3} + \lceil \log_2 n \rceil + 1 < \frac{3n}{4} + \lceil \log_2 n \rceil + 1.$$

If $\beta > \frac{n}{4}$, then $\beta = \frac{n}{4} + k$, for some $k > 0$ and then the maximum independent set has cardinality greater than or equal to $\frac{n}{4} + k$. Let t be the cardinality of the maximum independent set of G. Then $t \geq \frac{n}{4} + k$. Also, as in Lemma 13, we have

$$SIG(G) \leq n - 1 - t + \lceil \log_2 t \rceil.$$

Therefore,

$$SIG(G) \leq \frac{3n}{4} - k - 1 + \lceil \log_2 t \rceil < \frac{3n}{4} + \lceil \log_2 n \rceil.$$

Hence, we have proved that $SIG(G) \leq \frac{3n}{4} + \lceil \log_2 n \rceil + 2$. Hence

$$SIG(G) \leq \left\lfloor \frac{3n}{4} \right\rfloor + \lceil \log_2 n \rceil + 1.$$ □

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