# CONVERGENCE OF SOLUTIONS TO SIMPLIFIED SELF-ORGANIZING TARGET-DETECTION MODEL 

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#### Abstract

We consider the initial-boundary value problem for a quasilinear parabolic equation. After constructing local solutions to the equation, we show a priori estimates for them and prove global existence of solutions. A Lyapunov function is constructed for the global solutions. Furthermore, existence of a unique stationary solution is observed for each level set $X_{l}$ (given by (4.1)), together with some characterization by a functional equation. By virtue of the Lyapunov function, we can show longtime convergence of all global solutions with initial values in $X_{l}$ to the unique stationary solution.


1 Introduction We are concerned with the initial-boundary value problem for a nonlinear diffusion equation

$$
\begin{cases}\frac{\partial u}{\partial t}=\frac{\partial^{2}}{\partial x^{2}}\left(a u+\alpha u^{2}\right)-\mu \frac{\partial}{\partial x}\left[u \frac{\partial}{\partial x}(T(x) u)\right] & \text { in } I \times(0, \infty)  \tag{1.1}\\ \frac{\partial u}{\partial x}(0, t)=\frac{\partial u}{\partial x}(1, t)=0 & \text { on }(0, \infty) \\ u(x, 0)=u_{0}(x) & \text { in } I,\end{cases}
$$

in the unit open interval $I=(0,1)$. Note that the first equation of (1.1) is rewritten as

$$
\begin{equation*}
\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[(a+G(x) u) \frac{\partial}{\partial x} u\right]-\mu \frac{\partial}{\partial x}\left[T^{\prime}(x) u^{2}\right] \tag{1.2}
\end{equation*}
$$

where

$$
G(x)=2 \alpha-\mu T(x)
$$

We introduce the model (1.1) by simplifying the following attraction-repulsion chemotaxis model:

$$
\begin{cases}\frac{\partial u}{\partial t}=a_{1} \frac{\partial^{2} u}{\partial x^{2}}-\frac{\partial}{\partial x}\left[u\left(\frac{\partial}{\partial x} \chi_{1}(v)-\frac{\partial}{\partial x} \chi_{2}(w)\right)\right] & \text { in } I \times(0, \infty), \\ \frac{\partial v}{\partial t}=a_{2} \frac{\partial^{2} v}{\partial x^{2}}+g_{1} T(x, t) u-d v & \text { in } I \times(0, \infty), \\ \frac{\partial w}{\partial t}=a_{3} \frac{\partial^{2} w}{\partial x^{2}}+g_{2} u-h w & \text { in } I \times(0, \infty),  \tag{1.3}\\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=\frac{\partial w}{\partial x}=0 & \text { on } \partial I \times(0, \infty), \\ u(x, 0)=u_{0}(x), v(x, 0)=v_{0}(x), w(x, 0)=w_{0}(x) & \text { in } I .\end{cases}
$$

[^0]Okaie et al. [18] introduced this model for considering a mobile bionanosensor network designed for target tracking in molecular environments. In their modeling, they are inspired by Keller-Segel model [5]. The network consists of bioparticles and two types of signaling molecules: attractants for a group of bioparticles to move toward targets, and repellents to spread over the environment. They obtained numerical results and indicated that they could improve target tracking performance by controlling the effects of attractants and repellents suitably. They also developed an individual-based model and demonstrated numerical results that a group of bioparticles could track moving targets [19].

Unknown functions $u=u(x, t), v=v(x, t)$, and $w=w(x, t)$ denote the density of bioparticles, the concentration of chemical attractants, and the concentration of chemical repellents, respectively, in the interval $I$ at time $t$. The bioparticles are motile in response to the gradients of $\chi_{1}(v)$ and $\chi_{2}(w)$, where $\chi_{1}(v)$ and $\chi_{2}(w)$ are sensitivity functions of bioparticles to chemical attractants and chemical repellents. The term $-\frac{\partial}{\partial x}\left[u\left(\frac{\partial}{\partial x} \chi_{1}(v)-\frac{\partial}{\partial x} \chi_{2}(w)\right)\right]$ denotes a nonlinear advection which is affected by chemical attractants and chemical repellents. Bioparticles move preferentially towards higher (resp. lower) concentration of chemical attractants (resp. repellents). The term $g_{1} T(x, t) u$ denotes that bioparticles produce chemical attractant when they meet the target $T(x, t)$. On the other hand, bioparticles always release chemical repellents by the production rate $g_{2} u$. The terms $-d v$ and $-h w$ denote decay rates of chemical attractants and repellents, respectively. The unknown functions $u, v$, and $w$ satisfy the Neumann boundary conditions at $\partial I$.

After Keller-Segel [5] first introduced a diffusion-advection model for chemotactic phenomenon, this model was developed further by many researchers (for example [1, 2, 3, 21]). Many mathematicians have studied a variety of chemotaxis models. For instance, the following model:

$$
\begin{cases}\frac{\partial u}{\partial t}=a_{1} \Delta u-\nabla \cdot\left[u \nabla \chi_{1}(v)\right] & \text { in } \Omega \times(0, \infty) \\ \frac{\partial v}{\partial t}=a_{2} \Delta v+g_{1} u-d v & \text { in } \Omega \times(0, \infty), \\ \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0 & \text { on } \partial \Omega \times(0, \infty), \\ u(x, 0)=u_{0}(x), v(x, 0)=v_{0}(x), & \text { in } \Omega,\end{cases}
$$

was first studied by Childress-Percus [15]. Here, $\Omega$ is a bounded two-dimensional domain with some regular boundary. When $\chi_{1}(v)=V_{1} v_{1}$, the global existence for initial functions having small $L_{1}(\Omega)$ norm $\left\|u_{0}\right\|_{L_{1}}$ was obtained by Ryu-Yagi [14]. On the other hand, blowup of solutions was proved in [10, 11]. In addition, the stationary problem was studied by $[13,17]$. We quote $[8,9]$ for one-dimensional problem.

Some attraction-repulsion chemotaxis models have been proposed in [7, 12]. In these years attraction-repulsion chemotaxis models were studied by many mathematicians (for example $[4,6,20]$ ). However, to the best our knowledge, few researchers have handled the model (1.3), i.e., the case where a production rate of chemical attractants depends on position $x$ and time $t$.

We want to study asymptotic behavior of solutions of (1.3). Iwasaki [16] showed the global existence of solutions and constructed exponential attractors for a non-autonomous dynamical system generated by problem (1.3). But it is very difficult to investigate the time evolution of solutions in detail. One of the reasons for this difficulty is that (1.3) is a reaction, advection and diffusion equation with three components. Furthermore, the moving target $T(x, t)$ makes it hard to investigate the stationary state. Therefore, we intend to simplify the original model.

First, we assume that the target is stationary, i.e., $T(x, t) \equiv T(x)$. Next, we assume that the sensitivity functions are of the forms $\chi_{1}(v)=V_{1} v$ and $\chi_{2}(w)=V_{2} w$, where $V_{1}$ and $V_{2}$
are some positive constants. Finally, we assume that $g_{1}$ and $d$ are sufficiently large, so that $v$ is in quasi-equilibrium state, i.e., $v=\frac{g_{1}}{d} T(x) u$. It is the same for $g_{2}$ and $h$; therefore, $w=\frac{g_{2}}{h} u$. By substituting these for $\chi_{1}(v)$ and $\chi_{2}(w)$ in the first equation of (1.3), we obtain that

$$
\frac{\partial u}{\partial t}=a_{1} \frac{\partial^{2} u}{\partial x^{2}}-\frac{\partial}{\partial x}\left[u\left(\frac{g_{1} V_{1}}{d} \frac{\partial}{\partial x}(T(x) u)-\frac{g_{2} V_{2}}{h} \frac{\partial}{\partial x} u\right)\right]
$$

that is,

$$
\frac{\partial u}{\partial t}=\frac{\partial^{2}}{\partial x^{2}}\left(a_{1} u+\frac{g_{2} V_{2}}{2 h} u^{2}\right)-\frac{g_{1} V_{1}}{d} \frac{\partial}{\partial x}\left[u \frac{\partial}{\partial x}(T(x) u)\right]
$$

Putting $a_{1}=a, \frac{g_{2} V_{2}}{2 h}=\alpha$, and $\frac{g_{1} V_{1}}{d}=\mu$, we then arrive at (1.1).
In this paper, we show the global existence of solutions of (1.1) and construct a dynamical system. Since the target $T(x)$ does not depend on time $t$, we expect that a global solution $u(t)$ converges to a stationary solution. As will be found, the norm $\|u(t)\|_{L_{1}}=\left\|u_{0}\right\|_{L_{1}}$ is conserved for every $t \in[0, \infty)$. Therefore, for each $\left\|u_{0}\right\|_{L_{1}}=l>0$, we have to consider a stationary problem in a space $X_{l}$ given by (4.1), the element $u$ of which satisfies $\|u\|_{L_{1}}=l$. The stationary problem in $X_{l}$ possesses a unique solution $\bar{u}_{l}$. Fortunately, a Lyapunov function for (1.1) will be constructed. By virtue of a Lyapunov function, we will prove that every global solution $u(t)$ with initial value $u_{0}$ satisfying $\left\|u_{0}\right\|_{L_{1}}=l$ converges to $\bar{u}_{l}$.

We assume that $a, \alpha$, and $\mu$ are positive ( $>0$ ) constants. We also assume that

$$
\begin{equation*}
T \in H_{N}^{\sigma}(I), \text { i.e., } G \in H_{N}^{\sigma}(I) \tag{1.4}
\end{equation*}
$$

with some $\sigma>5 / 2$ and there exists a positive constant $c>0$ such that

$$
\begin{equation*}
c \leq G(x) \text { in } \bar{I} \tag{1.5}
\end{equation*}
$$

Let $1 / 2<\varepsilon \leq 1$ be arbitrarily fixed. The space of initial functions is set by

$$
\begin{equation*}
\mathcal{K}=\left\{u_{0} \in H^{\varepsilon}(I) ; u_{0}(x) \geq 0 \text { in } \bar{I}\right\} . \tag{1.6}
\end{equation*}
$$

This paper is organized as follows. In Section 2, a local unique solution to (1.1) is constructed for each initial value $u_{0} \in \mathcal{K}$. In Section 3, we establish a priori estimates for local solutions to obtain the global existence of solutions. In Section 4, a dynamical system is generated from problem (1.1). Section 5 is devoted to constructing a Lyapunov function. A stationary problem for (1.1) is treated in Section 6. Finally, in Section 7, we show that each global solution converges to a corresponding stationary solution.

2 Local Solutions Let $H^{1}(I) \subset L_{2}(I) \subset H^{1}(I)^{*}$ be a triplet of spaces. Problem (1.1) is written as the Cauchy problem for an abstract equation

$$
\left\{\begin{array}{l}
\frac{d u}{d t}+A(u) u=F(u), \quad 0<t<\infty  \tag{2.1}\\
u(0)=u_{0}
\end{array}\right.
$$

in $X=H^{1}(I)^{*}$. Here, $A(u)$ is a linear operator defined for $u \in Z=H^{\varepsilon_{1}}(I)$, where $1 / 2<\varepsilon_{1}<\varepsilon$. For $u \in Z$, let us consider the sesquilinear form

$$
a\left(u ; u_{1}, u_{2}\right)=\int_{I}(a+G(x) \chi(\operatorname{Re} u)) \frac{\partial u_{1}}{\partial x} \frac{\partial \bar{u}_{2}}{\partial x} d x+\int_{I} u_{1} \bar{u}_{2} d x
$$

on $H^{1}(I)$. Here $\chi(u)$ is a smooth cutoff function such that $\chi(u)=u$ for $u \geq 0$ and $\chi(u) \equiv-\delta$ for $u \leq-\delta, \delta>0$ being some small positive constant such that $2\|G\|_{L_{\infty}} \delta \leq a$. Then, thanks to (1.4) and (1.5), the sesquilinear form $a(u ; \cdot, \cdot)$ defines sectorial operators $A(u)$ with respect to the triplet $H^{1}(I) \subset L_{2}(I) \subset H^{1}(I)^{*}$ with angles $\omega_{A(u)}<\pi / 2$.

The nonlinear operator $F: W \rightarrow X$ is given by

$$
F(u)=u-\mu \frac{\partial}{\partial x}\left[T^{\prime}(x) u^{2}\right]
$$

Here, $W=H^{\varepsilon_{2}}(I)$, where $\varepsilon_{1}<\varepsilon_{2}<\varepsilon$.
Let $0<R<\infty$, and let $A(u)$ be defined for $u \in K_{R}=\left\{u \in Z ;\|u\|_{Z}<R\right\}$. We can see that the spectrum $\sigma(A(u))$ is contained in a fixed open sectorial domain, i.e.,

$$
\begin{equation*}
\sigma(A(u)) \subset \Sigma_{\omega_{R}}=\left\{\lambda \in \mathbb{C} ;|\arg \lambda|<\omega_{R}\right\}, \quad u \in K_{R} \tag{2.2}
\end{equation*}
$$

with some angle $\omega_{A(u)}<\omega_{R}<\pi / 2$, and the resolvent satisfies

$$
\begin{equation*}
\left\|(\lambda-A(u))^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{M_{R}}{|\lambda|}, \quad \lambda \notin \Sigma_{\omega_{R}}, \quad u \in K_{R} \tag{2.3}
\end{equation*}
$$

with a constant $M_{R} \geq 1$. Furthermore,

$$
\begin{equation*}
\mathcal{D}(A(u)) \equiv H^{1}(I), \quad u \in K_{R} \tag{2.4}
\end{equation*}
$$

Let us set $Y=H^{\varepsilon_{0}}(I)$ with a third exponent $\varepsilon_{0}$ chosen so that $1 / 2<\varepsilon_{0}<\varepsilon_{1}$. Thanks to the assumption of $\chi(u)$, it holds that

$$
\begin{aligned}
& \|\chi(\operatorname{Re} u)-\chi(\operatorname{Re} v)\|_{L_{\infty}} \\
& \quad=\left\|\int_{0}^{1} \chi^{\prime}(\theta \operatorname{Re} u+(1-\theta) \operatorname{Re} v) d \theta \cdot(\operatorname{Re} u-\operatorname{Re} v)\right\|_{L_{\infty}} \\
& \quad \leq C\left(\|u\|_{L_{\infty}}+\|v\|_{L_{\infty}}\right)\|u-v\|_{L_{\infty}} \\
& \quad \leq C_{R}\|u-v\|_{Y}, \quad u, v \in K_{R} .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \|[A(u)-A(v)] \tilde{u}\|_{X} \\
& =\left\|\frac{\partial}{\partial x}\left[G(x)(\chi(\operatorname{Re} u)-\chi(\operatorname{Re} v)) \frac{\partial \tilde{u}}{\partial x}\right]\right\|_{X} \\
& \leq C\|G\|_{L_{\infty}}\|\chi(\operatorname{Re} u)-\chi(\operatorname{Re} v)\|_{L_{\infty}}\|\tilde{u}\|_{H^{1}} \\
& \leq C_{R}\|G\|_{L_{\infty}}\|u-v\|_{Y}\|\tilde{u}\|_{H^{1}}, \quad \tilde{u} \in H^{1}(I), \quad u, v \in K_{R},
\end{aligned}
$$

i.e.,

$$
\begin{align*}
& \left\|[A(u)-A(v)] A(v)^{-1}\right\|_{\mathcal{L}(X)} \\
& \quad \leq C_{R}\|G\|_{L_{\infty}}\|u-v\|_{Y}, \quad u, v \in K_{R} \tag{2.5}
\end{align*}
$$

The nonlinear operator $F$ satisfies

$$
\begin{align*}
& \|F(u)-F(v)\|_{X} \\
& \quad \leq\|u-v\|_{X}+\left\|\frac{\partial}{\partial x}\left[T^{\prime}(x)(u+v)(u-v)\right]\right\|_{X} \\
& \quad \leq C\left[1+\|T\|_{H^{1}}\left(\|u\|_{Z}+\|v\|_{Z}\right)\right]\|u-v\|_{W}, \quad u, v \in W \tag{2.6}
\end{align*}
$$

We see that $a\left(u ; u_{1}, u_{2}\right)$ is a symmetric form, i.e., $a\left(u ; u_{1}, u_{2}\right)=\overline{a\left(u ; u_{2}, u_{1}\right)}$. Therefore, $A(u)$ is a positive definite self-adjoint operator of $H^{1}(I)^{*}$. Then, we note that, for any $3 / 4<\theta \leq 1, \mathcal{D}\left(A(u)^{\theta}\right)=\left[H^{1}(I)^{*}, H^{1}(I)\right]_{\theta}=H^{2 \theta-1}(I)$ with a norm equivalence (cf. [22, Chapter 16]). Thus, by setting $\tilde{\alpha}=\left(1+\varepsilon_{0}\right) / 2, \tilde{\beta}=\left(1+\varepsilon_{1}\right) / 2$, and $\tilde{\eta}=\left(1+\varepsilon_{2}\right) / 2$, we see that, for any $u \in K_{R}, \mathcal{D}\left(A(u)^{\tilde{\alpha}}\right)=Y, \mathcal{D}\left(A(u)^{\tilde{\beta}}\right)=Z$, and $\mathcal{D}\left(A(u)^{\tilde{\eta}}\right)=W$ with the estimates

$$
\left\{\begin{array}{lll}
\|\tilde{u}\|_{Y} \leq D_{1}\left\|A(u)^{\tilde{\alpha}} \tilde{u}\right\|_{X}, & \tilde{u} \in \mathcal{D}\left(A(u)^{\tilde{\alpha}}\right), & u \in K_{R},  \tag{2.7}\\
\|\tilde{u}\|_{Z} \leq D_{2}\left\|A(u)^{\tilde{\beta}} \tilde{u}\right\|_{X}, & \tilde{u} \in \mathcal{D}\left(A(u)^{\tilde{\beta}}\right), & u \in K_{R}, \\
\|\tilde{u}\|_{W} \leq D_{3}\left\|A(u)^{\tilde{\eta}} \tilde{u}\right\|_{X}, & \tilde{u} \in \mathcal{D}\left(A(u)^{\tilde{q}}\right), & u \in K_{R},
\end{array}\right.
$$

$D_{i}>0(i=1,2,3)$ being some constants. The initial value $u_{0} \in K_{R} \cap \mathcal{K}$ satisfies

$$
\begin{equation*}
u_{0} \in \mathcal{D}\left(A\left(u_{0}\right)^{\tilde{\gamma}}\right) \tag{2.8}
\end{equation*}
$$

if $\tilde{\gamma}$ is taken as $\tilde{\gamma}=(1+\varepsilon) / 2$. The exponents satisfy the relations

$$
\begin{equation*}
\frac{3}{4}<\tilde{\alpha}<\tilde{\beta}<\tilde{\eta}<\tilde{\gamma}<1 \tag{2.9}
\end{equation*}
$$

Theorem 2.1. For each $u_{0} \in K_{R} \cap \mathcal{K}$, there exists a unique local solution to (2.1) in the function space:

$$
\begin{equation*}
0 \leq u \in \mathcal{C}\left(\left(0, T_{u_{0}}\right] ; H^{1}(I)\right) \cap \mathfrak{C}\left(\left[0, T_{u_{0}}\right] ; H^{\varepsilon}(I)\right) \cap \mathfrak{C}^{1}\left(\left(0, T_{u_{0}}\right] ; H^{1}(I)^{*}\right) \tag{2.10}
\end{equation*}
$$

where $T_{u_{0}}$ is determined by the norm $\left\|u_{0}\right\|_{H^{\varepsilon}}$ alone.
Proof. Let us apply a general theorem to construct local solutions to (2.1). Conditions $(2.2) \sim(2.9)$ imply that the conditions of [22, Theorem 5.6] are satisfied. Therefore, for any $u_{0} \in K_{R} \cap \mathcal{K}$, there exists an interval $\left[0, T_{u_{0}}\right]$ such that (2.1) possesses a unique local solution in the function space:

$$
\begin{equation*}
u \in \mathcal{C}\left(\left(0, T_{u_{0}}\right] ; H^{1}(I)\right) \cap \mathcal{C}\left(\left[0, T_{u_{0}}\right] ; H^{\varepsilon}(I)\right) \cap \mathcal{C}^{1}\left(\left(0, T_{u_{0}}\right] ; H^{1}(I)^{*}\right) \tag{2.11}
\end{equation*}
$$

where $T_{u_{0}}$ is determined by the norm $\left\|u_{0}\right\|_{H^{\varepsilon}}$ alone.
Let us show the nonnegativity of local solutions of (2.1). For $u_{0} \in K_{R} \cap \mathcal{K}$, let $u(t)$ be the local solution of (2.1) constructed above in (2.11).

Let us first verify that $u(t)$ is real valued. Indeed, the complex conjugate $\overline{u(t)}$ of $u(t)$ is also a local solution of (2.1) with the same initial value $u_{0}$. Therefore, the uniqueness of solution implies that $\overline{u(t)}=u(t)$; hence, $u(t)$ is real valued.

Let $H(u)$ be a $\mathcal{C}^{1,1}$ cutoff function given by $H(u)=u^{2} / 2$ for $-\infty<u<0$ and $H(u) \equiv 0$ for $0 \leq u<\infty$. Since $u \in \mathcal{C}\left(\left(0, T_{u_{0}}\right] ; H^{1}(I)\right) \cap \mathcal{C}^{1}\left(\left(0, T_{u_{0}}\right] ; H^{1}(I)^{*}\right)$, we see that $\psi(t)=$ $\int_{I} H(u(t)) d x$ is continuously differentiable with the derivative

$$
\psi^{\prime}(t)=\left\langle H^{\prime}(u(t)), u^{\prime}(t)\right\rangle_{H^{1} \times H^{1 *}}, \quad 0<t \leq T_{u_{0}}
$$

where $\langle\cdot, \cdot\rangle_{H^{1} \times H^{1 *}}$ is a duality product of $\left\{H^{1}(I), H^{1}(I)^{*}\right\}$. Therefore, we have

$$
\begin{aligned}
\psi^{\prime}(t)= & \left\langle H^{\prime}(u(t)), \frac{\partial}{\partial x}\left[(a+G(x) \chi(u(t))) \frac{\partial}{\partial x} u(t)\right]\right\rangle_{H^{1} \times H^{1 *}} \\
& -\mu\left\langle H^{\prime}(u(t)), \frac{\partial}{\partial x}\left[T^{\prime}(x) u(t)^{2}\right]\right\rangle_{H^{1} \times H^{1 *}}
\end{aligned}
$$

Thanks to the assumption of $\chi(\cdot)$, we see that

$$
\begin{aligned}
& \left\langle H^{\prime}(u(t)), \frac{\partial}{\partial x}\left[(a+G(x) \chi(u(t))) \frac{\partial}{\partial x} u(t)\right]\right\rangle_{H^{1} \times H^{1 *}} \\
& \quad \leq-\frac{a}{2} \int_{I}\left|\frac{\partial}{\partial x} H^{\prime}(u(t))\right|^{2} d x
\end{aligned}
$$

and

$$
\begin{aligned}
& -\mu\left\langle H^{\prime}(u(t)), \frac{\partial}{\partial x}\left[T^{\prime}(x) u(t)^{2}\right]\right\rangle_{H^{1} \times H^{1 *}} \\
& \quad \leq \frac{a}{2} \int_{I}\left|\frac{\partial}{\partial x} H^{\prime}(u(t))\right|^{2} d x+C\left\|H^{\prime}(u(t))\right\|_{L_{2}}^{2}\|T\|_{H^{2}}^{2}
\end{aligned}
$$

Therefore, we obtain that

$$
\psi^{\prime}(t) \leq C\|T\|_{H^{2}}^{2} \psi(t), \quad 0<t \leq T_{u_{0}}
$$

so,

$$
\psi(t) \leq \psi(0) \exp \left(C\|T\|_{H^{2}}^{2} t\right), \quad 0<t \leq T_{u_{0}}
$$

Then, $\psi(0)=0$ implies $\psi(t) \equiv 0$. Thus, $u(t) \geq 0$ for every $0<t \leq T_{u_{0}}$.
Since $u(t) \geq 0$, it holds that $\chi(\operatorname{Re} u(t))=u(t)$; this then means that the local solution of (2.1) is regarded as a local solution to the original problem (1.1).

3 Global Solution For $u_{0} \in \mathcal{K}$, let $u$ denote any local solution of (2.1) on $\left[0, T_{u}\right]$ in the function space:

$$
\begin{equation*}
0 \leq u \in \mathcal{C}\left(\left(0, T_{u}\right] ; H^{1}(I)\right) \cap \mathcal{C}\left(\left[0, T_{u}\right] ; H^{\varepsilon}(I)\right) \cap \mathcal{C}^{1}\left(\left(0, T_{u}\right] ; H^{1}(I)^{*}\right) \tag{3.1}
\end{equation*}
$$

We then show the following a priori estimates.
Proposition 3.1. There exists a continuous increasing function $p(\cdot)$ such that, for any local solution $u$ of (2.1) in (3.1) with initial value $u_{0} \in \mathcal{K}$, it holds that

$$
\begin{equation*}
\|u(t)\|_{H^{1}} \leq p\left(\left\|u_{0}\right\|_{H^{1}}\right), \quad 0 \leq t \leq T_{u} . \tag{3.2}
\end{equation*}
$$

Proof. In the proof, the notations $C$ and $p(\cdot)$ stand for some constants and some continuous increasing functions, respectively, which are determined by the initial constants and $\|T\|_{H^{\sigma}}$ (see (1.4)) and by $I$ in a specific way in each occurrence. In the following, we divide the proof into four steps.

Step 1. Let us integrate the first equation of (1.1) in $I$. Then, obviously,

$$
\frac{d}{d t}\|u\|_{L_{1}}=0
$$

i.e.,

$$
\begin{equation*}
\|u(t)\|_{L_{1}}=\left\|u_{0}\right\|_{L_{1}}, \quad 0 \leq t \leq T_{u} \tag{3.3}
\end{equation*}
$$

Step 2. Multiply the equation (1.2) by $2 u$ and integrate the product in $I$. Then,

$$
\frac{d}{d t}\|u\|_{L_{2}}^{2}+2 \int_{I}(a+G(x) u)\left|\frac{\partial u}{\partial x}\right|^{2} d x=-2 \mu \int_{I} u \frac{\partial}{\partial x}\left[T^{\prime}(x) u^{2}\right] d x
$$

Here,

$$
\begin{aligned}
-2 \mu \int_{I} u \frac{\partial}{\partial x}\left[T^{\prime}(x) u^{2}\right] d x & =\frac{2}{3} \mu \int_{I}\left[\frac{\partial}{\partial x} u^{3}\right] T^{\prime}(x) d x \\
& =-\frac{2}{3} \mu \int_{I} u^{3} T^{\prime \prime}(x) d x \\
& \leq \zeta\|u\|_{L_{4}}^{4}+C_{\zeta}\left\|T^{\prime \prime}\right\|_{L_{4}}^{4} \\
& \leq \zeta\|u\|_{H^{1}}^{2}\|u\|_{L_{1}}^{2}+C_{\zeta} \\
& \leq \zeta\|u\|_{H^{1}}^{2}+C_{\zeta} p\left(\left\|u_{0}\right\|_{L_{1}}\right)
\end{aligned}
$$

with any $\zeta>0$. Therefore, we get

$$
\frac{d}{d t}\|u\|_{L_{2}}^{2}+\|u\|_{L_{2}}^{2} \leq p\left(\left\|u_{0}\right\|_{L_{1}}\right)
$$

i.e.,

$$
\begin{equation*}
\|u(t)\|_{L_{2}}^{2} \leq e^{-t}\left\|u_{0}\right\|_{L_{2}}^{2}+p\left(\left\|u_{0}\right\|_{L_{1}}\right), \quad 0 \leq t \leq T_{u} \tag{3.4}
\end{equation*}
$$

Step 3. In this step, we shall use the notation

$$
P_{1}\left(u_{0}\right)=p\left(\left\|u_{0}\right\|_{L_{2}}\right)
$$

Multiply the equation (1.2) by $4 u^{3}$ and integrate the product in $I$. Then,

$$
\frac{d}{d t}\|u\|_{L_{4}}^{4}+12 \int_{I}(a+G(x) u) u^{2}\left|\frac{\partial u}{\partial x}\right|^{2} d x=-\frac{12}{5} \mu \int_{I} u^{5} T^{\prime \prime}(x) d x
$$

Here,

$$
12 \int_{I}(a+G(x) u) u^{2}\left|\frac{\partial u}{\partial x}\right|^{2} d x=3 a\left\|\frac{\partial}{\partial x} u^{2}\right\|_{L_{2}}^{2}+\frac{48}{25} \int_{I} G(x)\left|\frac{\partial}{\partial x} u^{\frac{5}{2}}\right|^{2} d x
$$

and

$$
\begin{aligned}
-\frac{12}{5} \mu \int_{I} u^{5} T^{\prime \prime}(x) d x & \leq C\left\|u^{5}\right\|_{L_{1}}\left\|T^{\prime \prime}\right\|_{L_{\infty}} \leq C\left\|u^{\frac{5}{2}}\right\|_{L_{2}}^{2} \\
& \leq \zeta_{1}\left\|\frac{\partial}{\partial x} u^{\frac{5}{2}}\right\|_{L_{2}}^{2}+C_{\zeta_{1}}\left\|u^{\frac{5}{2}}\right\|_{L_{1}}^{2} \\
& \leq \zeta_{1}\left\|\frac{\partial}{\partial x} u^{\frac{5}{2}}\right\|_{L_{2}}^{2}+C_{\zeta_{1}}\|u\|_{L_{2}}^{3}\|u\|_{L_{4}}^{2} \\
& \leq \zeta_{1}\left\|\frac{\partial}{\partial x} u^{\frac{5}{2}}\right\|_{L_{2}}^{2}+C_{\zeta_{1}} P_{1}\left(u_{0}\right)\left\|u^{2}\right\|_{L_{2}} \\
& \leq \zeta_{1}\left(\left\|\frac{\partial}{\partial x} u^{\frac{5}{2}}\right\|_{L_{2}}^{2}+\left\|\frac{\partial}{\partial x} u^{2}\right\|_{L_{2}}^{2}\right)+C_{\zeta_{1}} P_{1}\left(u_{0}\right)
\end{aligned}
$$

with any $\zeta_{1}>0$. Therefore,

$$
\frac{d}{d t}\|u\|_{L_{4}}^{4}+\|u\|_{L_{4}}^{4}+a\left\|u^{2}\right\|_{H^{1}}^{2}+c\left\|\frac{\partial}{\partial x} u^{\frac{5}{2}}\right\|_{L_{2}}^{2} \leq P_{1}\left(u_{0}\right)
$$

i.e.,

$$
\begin{equation*}
\|u(t)\|_{L_{4}}^{4} \leq e^{-t}\left\|u_{0}\right\|_{L_{4}}^{4}+P_{1}\left(u_{0}\right), \quad 0 \leq t \leq T_{u} \tag{3.5}
\end{equation*}
$$

As well,

$$
\begin{equation*}
\int_{s}^{t}\left\|u(\tau)^{2}\right\|_{H^{1}}^{2} d \tau \leq P_{1}\left(u_{0}\right)\left[(t-s)+\left\|u_{0}\right\|_{L_{4}}^{4}\right], \quad 0 \leq s<t \leq T_{u} \tag{3.6}
\end{equation*}
$$

Step 4. In this step, we shall use the notation

$$
P_{2}\left(u_{0}\right)=p\left(\left\|u_{0}\right\|_{L_{4}}\right)
$$

By regarding the local solution $u \in \mathcal{C}\left(\left(0, T_{u}\right] ; H^{1}(I)\right)$ as a known function, (1.1) is handled as a nonautonomous abstract evolution equation of the form $[22,(3.61)]$ in $L_{2}(I)$. Then, applying [22, Theorem 3.9], we can assume that $u \in \mathcal{C}\left(\left(0, T_{u}\right] ; H_{N}^{2}(I)\right) \cap \mathcal{C}^{1}\left(\left(0, T_{u}\right] ; L_{2}(I)\right)$. By virtue of this fact, the inner product of (1.2) with $2 A(u) u$ in $L_{2}(I)$ makes a sense, so that we get

$$
\begin{aligned}
& \frac{d}{d t}\left\|A(u)^{\frac{1}{2}} u\right\|_{L_{2}}^{2}+2\|A(u) u\|_{L_{2}}^{2} \\
& \quad=2\left\|A(u)^{\frac{1}{2}} u\right\|_{L_{2}}^{2}-2 \mu\left\langle\frac{\partial}{\partial x}\left[T^{\prime}(x) u^{2}\right], A(u) u\right\rangle_{H^{1} \times H^{1 *}}
\end{aligned}
$$

Here,

$$
\left\|A(u)^{\frac{1}{2}} u\right\|_{L_{2}}^{2} \leq\|u\|_{L_{2}}\|A(u) u\|_{L_{2}} \leq \zeta_{2}\|A(u) u\|_{L_{2}}^{2}+C_{\zeta_{2}}\|u\|_{L_{2}}^{2}
$$

and

$$
\begin{aligned}
-\mu\left\langle\frac{\partial}{\partial x}\left[T^{\prime}(x) u^{2}\right], A(u) u\right\rangle_{H^{1} \times H^{1 *}} & =-\mu\left(\frac{\partial}{\partial x}\left[T^{\prime}(x) u^{2}\right], A(u) u\right)_{L_{2}} \\
& \leq C\left\|T^{\prime} u^{2}\right\|_{H^{1}}\|A(u) u\|_{L_{2}} \\
& \leq \zeta_{2}\|A(u) u\|_{L_{2}}^{2}+C_{\zeta_{2}}\left\|u^{2}\right\|_{H^{1}}^{2}
\end{aligned}
$$

with any $\zeta_{2}>0$. Therefore, we obtain that

$$
\frac{d}{d t}\left\|A(u)^{\frac{1}{2}} u\right\|_{L_{2}}^{2}+\left\|A(u)^{\frac{1}{2}} u\right\|_{L_{2}}^{2}+\|A(u) u\|_{L_{2}}^{2} \leq C\left\|u^{2}\right\|_{H^{1}}^{2}+P_{2}\left(u_{0}\right)
$$

Thanks to (3.6), we conclude that

$$
\left\|A(u(t))^{\frac{1}{2}} u(t)\right\|_{L_{2}}^{2} \leq e^{-t}\left\|A\left(u_{0}\right) u_{0}\right\|_{L_{2}}^{2}+P_{2}\left(u_{0}\right), \quad 0 \leq t \leq T_{u}
$$

i.e.,

$$
\|u(t)\|_{H^{1}}^{2} \leq C e^{-t}\left\|u_{0}\right\|_{H^{1}}^{2}+P_{2}\left(u_{0}\right), \quad 0 \leq t \leq T_{u}
$$

We have in this way established the desired a priori estimate (3.2).
Thanks to Proposition 3.1, we conclude the global existence of solutions. Indeed, for any initial value $u_{0} \in \mathcal{K}$, there exists a nonnegative local solution at least on an interval $\left[0, T_{u_{0}}\right]$. Let $0<t_{1}<T_{u_{0}}$ and $u_{1}=u\left(t_{1}\right)$. Then, $u_{1} \in \mathcal{K} \cap H^{1}(I)$. We next consider problem (2.1) but with the initial time $t_{1}$ and with the initial value $u_{1}$ in the ball $K_{R_{1}}$ of
$Z$, where $R_{1}=p\left(\left\|u_{1}\right\|_{H^{1}}\right)$. Proposition 3.1 ensures that any local solution starting from $u_{1}$ stays at any time in $K_{R_{1}}$. In addition, any local solution $v$ on $\left[t_{1}, T_{v}\right]$ starting from $u_{1}$ can be extended over an interval $\left[t_{1}, T_{v}+\tau\right]$ as local solution, $\tau$ being dependent only on $\sup _{0 \leq t \leq T_{v}}\|v(t)\|_{H^{\varepsilon}}$ and hence being independent of the extreme time $T_{v}$. This means that problem $(2.1)$ on $\left[t_{1}, \infty\right)$ with the initial value $u_{1}$ possesses a unique global solution. We have thus deduced that, for any initial value $u_{0} \in \mathcal{K}$, there exists a unique global solution $u$ to (2.1) in the function space:

$$
\begin{equation*}
0 \leq u \in \mathcal{C}\left((0, \infty) ; H^{1}(I)\right) \cap \mathcal{C}\left([0, \infty) ; H^{\varepsilon}(I)\right) \cap \mathcal{C}^{1}\left((0, \infty) ; H^{1}(I)^{*}\right) \tag{3.7}
\end{equation*}
$$

Note that we can apply the maximal principle of parabolic equations (cf. [23, Chapter 3]), so that

$$
\begin{equation*}
u(x, t)>0 \text { in } \bar{I} \times(0, \infty) \tag{3.8}
\end{equation*}
$$

4 Dynamical system Let us verify that problem (1.1) defines a dynamical system. For each $l>0$, we set $\mathcal{K}_{l}=\left\{u_{0} \in \mathcal{K} ;\left\|u_{0}\right\|_{L_{1}}=l\right\}$. We already know that, for any $u_{0} \in \mathcal{K}_{l}$, (1.1) possesses a unique global solution $u\left(t ; u_{0}\right)$ in (3.7). Let us set $S(t) u_{0}=u\left(t ; u_{0}\right)$. On account of (3.3) and (3.8), for a fixed $t_{0}>0$,

$$
\begin{equation*}
X_{l}=S\left(t_{0}\right) \mathcal{K}_{l} \tag{4.1}
\end{equation*}
$$

is the subset of $\left\{u \in H^{1}(I) ; u(x)>0\right.$ in $\left.\bar{I},\|u\|_{L_{1}}=l\right\}$. The uniqueness of solutions implies that for every $t \geq 0, S(t) \mathcal{X}_{l}=S(t) S\left(t_{0}\right) \mathcal{K}_{l}=S\left(t+t_{0}\right) \mathcal{K}_{l}=S\left(t_{0}\right) S(t) \mathcal{K}_{l} \subset S\left(t_{0}\right) \mathcal{K}_{l}=\mathcal{X}_{l}$. This means that $S(t)$ is a nonlinear semigroup acting on $X_{l}$.

Let us show that $S(t)$ is continuous on $X_{l}$.
Proposition 4.1. Let $0<\tilde{R}<\infty$ and $\tilde{K}_{\tilde{R}}=\left\{u \in X_{l} ;\|u\|_{H^{1}}<\tilde{R}\right\}$ be an open ball of $X_{l}$. Then, $S(t)$ satisfies

$$
\begin{align*}
& \left\|S(t) u_{0}-S(t) v_{0}\right\|_{H^{1}} \leq L_{p(\tilde{R})}^{n+1}\left\|u_{0}-v_{0}\right\|_{H^{1}}^{\tilde{\delta}^{n+1}} \\
& \quad t \in\left[n t_{p(\tilde{R})},(n+1) t_{p(\tilde{R})}\right] ; n=0,1,2, \ldots ; u_{0}, v_{0} \in \tilde{K}_{\tilde{R}} \tag{4.2}
\end{align*}
$$

with the exponent $\tilde{\delta}=(1-\tilde{\eta}) /(1-\tilde{\alpha})$ and some constant $L_{p(\tilde{R})}>0$, where $p(\cdot)$ is the same continuous increasing function as in (3.2).
Proof. Since $H^{1}(I)$ is continuously embedded in $Z$, there exists some constant $\tilde{C}$ such that $\|\cdot\|_{Z} \leq \tilde{C}\|\cdot\|_{H^{1}}$. Therefore, for $u_{0} \in \tilde{K}_{\tilde{R}}$, it holds that $\left\|u_{0}\right\|_{Z}<\tilde{C} \tilde{R}$. So, [22, Corollary 5.4] is applicable for the solutions $S(t) u_{0}, u_{0} \in \tilde{K}_{\tilde{R}}$, to observe that

$$
\begin{aligned}
& t^{\tilde{\eta}-\tilde{\alpha}}\left\|S(t) u_{0}-S(t) v_{0}\right\|_{W}+t^{\tilde{\beta}-\tilde{\alpha}}\left\|S(t) u_{0}-S(t) v_{0}\right\|_{Z} \\
& \quad+\left\|S(t) u_{0}-S(t) v_{0}\right\|_{Y} \leq C_{\tilde{R}}\left\|u_{0}-v_{0}\right\|_{H^{1}}, \quad 0<t \leq t_{\tilde{R}}, \quad u_{0}, v_{0} \in \tilde{K}_{\tilde{R}}
\end{aligned}
$$

On the basis of this estimate, in the same way as [22, Propositions 6.6 and 6.7], we can obtain the desired inequality (4.2).

We define the mapping $G:[0, \infty) \times \tilde{K}_{\tilde{R}} \rightarrow X_{l}$ as $G\left(t, u_{0}\right)=S(t) u_{0}$. Estimate (4.2) means that $G(t, \cdot)$ is Hölder continuous on the ball $\tilde{K}_{\tilde{R}}$ and the Hölder exponent is uniform in $t$ on any finite time interval. As shown in (3.7), for each $u_{0} \in \tilde{K}_{\tilde{R}}, G\left(\cdot, u_{0}\right)$ is continuous from $[0, \infty)$ to $H^{1}(I)$. Therefore, problem (1.1) determines a dynamical system $\left(S(t), x_{l}, H^{1}(I)\right)$. Note that the asymptotic behavior of every global solution $u\left(t ; u_{0}\right)$ starting from $u_{0} \in \mathcal{K}_{l}$ is reduced to that of a trajectory in $\left(S(t), X_{l}, H^{1}(I)\right)$.

5 Lyapunov function In this section, let us construct a Lyapunov function for (1.1). Let $u$ be a global solution of (1.1) in the function space (3.7). As $u$ is positive due to (3.8), the first equation of (1.1) is written as

$$
\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[u \frac{\partial}{\partial x}(a \log u+G(x) u)\right] .
$$

Consider the duality product of this equation with $(a \log u+G(x) u)$ in $H^{1}(I) \times H^{1}(I)^{*}$. Then,

$$
\frac{d}{d t} \int_{I}\left[a(u \log u-u)+\frac{G(x)}{2} u^{2}\right] d x \leq-\int_{I} u\left|\frac{\partial}{\partial x}(a \log u+G(x) u)\right|^{2} d x
$$

Let $0<s<t<\infty$. Integrating this inequality in $[s, t]$, we obtain that

$$
\left[\int_{I}\left[a(u(\tau) \log u(\tau)-u(\tau))+\frac{G(x)}{2} u(\tau)^{2}\right] d x\right]_{\tau=s}^{\tau=t} \leq 0
$$

If we set

$$
\Phi(u)=\int_{I}\left[a(u \log u-u)+\frac{G(x)}{2} u^{2}\right] d x
$$

then $\Phi(u(t)) \leq \Phi(u(s))$. This means that $\Phi$ is a Lyapunov function for (1.1).
We show a property of $\Phi$ which is used in Section 7. Let us set $H_{+}^{1}(I)=\{u \in$ $H^{1}(I) ; u(x)>0$ in $\left.\bar{I}\right\}$. Then, $H_{+}^{1}(I)$ is an open set of $H^{1}(I)$. For every $u \in H_{+}^{1}(I)$, we see that

$$
\begin{equation*}
\Phi(u+h)-\Phi(u)-\langle a \log u+G(x) u, h\rangle_{H^{1} \times H^{1 *}}=\|h\|_{H^{1 *}} R(h), \tag{5.1}
\end{equation*}
$$

here $R(h)$ is defined for $h \in H^{1}(I)$ such that $u+h \in H_{+}^{1}(I)$ and satisfies $R(h) \rightarrow 0$ as $\|h\|_{H^{1}} \rightarrow 0$. Indeed, we verify that

$$
\begin{aligned}
\int_{I} & {[(u+h) \log (u+h)-(u+h)] d x-\int_{I}[u \log u-u] d x-\langle\log u, h\rangle_{H^{1} \times H^{1 *}} } \\
& =\int_{I}\left[\int_{0}^{1} \log (u+\theta h) d \theta\right] h d x-\int_{I}(\log u) h d x \\
& =\int_{I}\left[\int_{0}^{1} \log \left(1+\theta \frac{h}{u}\right) d \theta\right] h d x \\
& \leq \int_{I}\left[\int_{0}^{1} \theta \frac{h}{u} d \theta\right] h d x \\
& =\frac{1}{2} \int_{I} \frac{h^{2}}{u} d x \\
& \leq \frac{1}{2 \min _{x \in \bar{I}}\{u(x)\}}\|h\|_{H^{1 *}}\|h\|_{H^{1}}
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{I} G(x) \frac{(u+h)^{2}}{2} d x-\int_{I} G(x) \frac{u^{2}}{2} d x-\langle G(x) u, h\rangle_{H^{1} \times H^{1 *}} \\
& \quad=\int_{I} G(x) \frac{h^{2}}{2} d x \\
& \quad \leq \frac{\|G\|_{L_{\infty}}}{2}\|h\|_{H^{1 *}}\|h\|_{H^{1}} .
\end{aligned}
$$

So, (5.1) is valid.


Fig. 1: Graphs of $f(y, C)$ and $g(y)$.

6 Stationary solutions In this section, we investigate the stationary solutions of (1.1). As shown in Section 4, problem (1.1) determines a dynamical system ( $\left.S(t), X_{l}, H^{1}(I)\right)$. Therefore, when the initial function $u_{0}$ of (1.1) is taken from $\mathcal{K}_{l}$, we have to consider a corresponding stationary problem

$$
\begin{cases}\frac{d^{2}}{d x^{2}}\left(a \bar{u}+\alpha \bar{u}^{2}\right)-\mu \frac{d}{d x}\left[\bar{u} \frac{d}{d x}(T(x) \bar{u})\right]=0 & \text { in } I,  \tag{6.1}\\ \frac{d \bar{u}}{d x}(0)=\frac{d \bar{u}}{d x}(1)=0, & \text { in } \bar{I},\end{cases}
$$

in $H^{1}(I)$.
Theorem 6.1. For each $l>0$, the stationary problem (6.1) possesses a unique solution $\bar{u}_{l}(x)$. Moreover, there exists a bijection between $l \in(0, \infty)$ and $C_{l} \in \mathbb{R}$ and $\bar{u}_{l}(x)$ is characterized by the functional equation

$$
\bar{u}_{l}(x)=\exp \left(\frac{C_{l}-G(x) \bar{u}_{l}(x)}{a}\right) \text { in } \bar{I} .
$$

The proof of Theorem 6.1 relies on the following two lemmas (see Fig. 1).
Lemma 6.1. Let $x_{0} \in \bar{I}$ be fixed. Then, for each $C \in \mathbb{R}$, a transcendental equation with respect to $y$ :

$$
\begin{equation*}
y=\exp \left(\frac{C-G\left(x_{0}\right) y}{a}\right) \tag{6.2}
\end{equation*}
$$

possesses a unique positive solution.
Proof. Let us put $f(y, C)=y \exp (-C / a)$ and $g(y)=\exp \left(-G\left(x_{0}\right) y / a\right)$. Since $a>0$ and $G\left(x_{0}\right)>0$, we can observe that there exists a unique $y(C)>0$ satisfying $f(y(C), C)=$ $g(y(C))$. This $y(C)$ is the solution of (6.2).

Lemma 6.2. Let $x_{0} \in \bar{I}$ be fixed. Then the mapping

$$
y: C \in \mathbb{R} \mapsto y(C) \in(0, \infty)
$$

where $y(C)$ is the solution of (6.2) with $C$, is a strictly increasing continuous function satisfying

$$
y(-\infty)=0 \text { and } y(\infty)=\infty
$$

Proof. When $-\infty<C<C^{\prime}<\infty$, we obviously see that $f(y, C)>f\left(y, C^{\prime}\right)$ for all $y>$ 0 . Since $f(y, C)$ and $g(y)$ are continuous functions with respect to $y$, we observe that $y(C)<y\left(C^{\prime}\right)$ and $y(C) \rightarrow y\left(C^{\prime}\right)$ as $C \rightarrow C^{\prime}$. Furthermore, $\exp -(C / a)$, i.e., the slope of $f(y, C)$, converges to $\infty$ (resp. 0 ) as $C \rightarrow-\infty$ (resp. $C \rightarrow \infty$ ). Thus, $y(-\infty)=0$ and $y(\infty)=\infty$.

Proof of Theorem 6.1. The first equation of (6.1) is written as

$$
\frac{d}{d x}\left[\bar{u} \frac{d}{d x}(a \log \bar{u}+G(x) \bar{u})\right]=0 \text { in } I .
$$

Considering the duality product of this equation with $(a \log \bar{u}+G(x) \bar{u})$ in $H^{1}(I) \times H^{1}(I)^{*}$, we obtain that

$$
\int_{I} \bar{u}\left|\frac{d}{d x}(a \log \bar{u}+G(x) \bar{u})\right|^{2} d x=0
$$

Therefore, $\bar{u}$ satisfies

$$
a \log \bar{u}(x)+G(x) \bar{u}(x)=C \text { in } \bar{I}
$$

where $C \in \mathbb{R}$ is some constant, i.e.,

$$
\begin{equation*}
\bar{u}(x)=\exp \left(\frac{C-G(x) \bar{u}(x)}{a}\right) \text { in } \bar{I} . \tag{6.3}
\end{equation*}
$$

On account of Lemma 6.1, we verify the existence and uniqueness of $\bar{u}(x)$ satisfying (6.3).
Let $\bar{u}^{C}$ denote the solution of (6.3) with $C$. Thanks to Lemma 6.2, we know that the mapping

$$
L: C \in \mathbb{R} \mapsto L(C)=\left\|\bar{u}^{C}\right\|_{L_{1}} \in(0, \infty)
$$

is a bijection. Thus, there exists a certain $C_{l}$ such that $\left\|\bar{u}^{C_{l}}\right\|_{L_{1}}=l$. This $\bar{u}^{C_{l}}$ is the very solution $\bar{u}_{l}$ of (6.1).

Let us show some information about stationary solutions. On account of (1.4), we know that $G^{\prime}(\cdot)$ and $G^{\prime \prime}(\cdot)$ are continuous functions. Due to (6.3), the first derivative of $\bar{u}(\cdot)$ satisfies

$$
\bar{u}^{\prime}(x)=-\frac{G^{\prime}(x) \bar{u}(x)+G(x) \bar{u}^{\prime}(x)}{a} \exp \left(\frac{C-G(x) \bar{u}(x)}{a}\right) \text { in } \bar{I} .
$$

We observe from this equation that $\bar{u}^{\prime}(\cdot)$ is a continuous function and

$$
\bar{u}^{\prime}\left(x_{0}\right)=0 \text { at } x_{0} \in \bar{I} \Longleftrightarrow G^{\prime}\left(x_{0}\right)=0 \text { at } x_{0} \in \bar{I}
$$

i.e.,

$$
\begin{equation*}
\bar{u}^{\prime}\left(x_{0}\right)=0 \text { at } x_{0} \in \bar{I} \Longleftrightarrow T^{\prime}\left(x_{0}\right)=0 \text { at } x_{0} \in \bar{I} \tag{6.4}
\end{equation*}
$$

Furthermore, the second derivative of $\bar{u}(\cdot)$ satisfies

$$
\begin{aligned}
\bar{u}^{\prime \prime}(x)= & \exp \left(\frac{C-G(x) \bar{u}(x)}{a}\right) \\
& \times\left[-\frac{G^{\prime \prime}(x) \bar{u}(x)+2 G^{\prime}(x) \bar{u}^{\prime}(x)+G(x) \bar{u}^{\prime \prime}(x)}{a}\right. \\
& \left.+\left(\frac{G^{\prime}(x) \bar{u}(x)+G(x) \bar{u}^{\prime}(x)}{a}\right)^{2}\right] \text { in } \bar{I} .
\end{aligned}
$$

This equation and (6.4) also imply that $\bar{u}^{\prime \prime}(\cdot)$ is a continuous function and

$$
\begin{aligned}
\bar{u}^{\prime}\left(x_{0}\right) & =0 \text { and } \bar{u}^{\prime \prime}\left(x_{0}\right)>0 \text { at } x_{0} \in \bar{I} \\
& \Longleftrightarrow G^{\prime}\left(x_{0}\right)=0 \text { and } G^{\prime \prime}\left(x_{0}\right)<0 \text { at } x_{0} \in \bar{I},
\end{aligned}
$$

i.e.,

$$
\begin{align*}
\bar{u}^{\prime}\left(x_{0}\right) & =0 \text { and } \bar{u}^{\prime \prime}\left(x_{0}\right)>0 \text { at } x_{0} \in \bar{I} \\
& \Longleftrightarrow T^{\prime}\left(x_{0}\right)=0 \text { and } T^{\prime \prime}\left(x_{0}\right)>0 \text { at } x_{0} \in \bar{I} . \tag{6.5}
\end{align*}
$$

From above all, we conclude that $\bar{u}(\cdot) \in \mathcal{C}^{2}([0,1] ; \mathbb{R})$. Moreover, $\bar{u}(\cdot)$ and $T(\cdot)$ have the same local minimum points and local maximum points.

7 Convergence to stationary solutions We are now in a position to prove the convergence of global solutions.

Theorem 7.1. For each $l>0$, every global solution $S(t) u_{0}$ of (1.1) in (3.7) with initial value $u_{0} \in \mathcal{K}_{l}$ converges to $\bar{u}_{l}(x)$ in $H^{1}(I)$.

In the proof, we shall use the following proposition.
Proposition 7.1. For each $u_{0} \in \mathcal{K}_{l}$, let $\omega\left(u_{0}\right)=\left\{v \in \mathcal{X}_{l} ; \exists t_{n} \nearrow \infty\right.$ s.t. $\left\|S\left(t_{n}\right) u_{0}-v\right\|_{H^{1}} \rightarrow$ $0\}$ denote the $\omega$-limit set of the trajectory $S(t) u_{0}$. Then, $\omega\left(u_{0}\right)=\left\{\bar{u}_{l}\right\}$, where $\bar{u}_{l}$ is a unique solution of (6.1).

Proof. At first, since $u \log u-u \geq-1$ for all $u \geq 0$, we obtain that

$$
\Phi(u)=\int_{I}\left[a(u \log u-u)+\frac{G(x)}{2} u^{2}\right] d x \geq-a, \quad \forall u \in X_{l}
$$

Let $v \in \omega\left(u_{0}\right) .\left\|S\left(t_{n}\right) u_{0}-v\right\|_{H^{1}} \rightarrow 0$ implies that

$$
\Phi(v)=\lim _{t_{n} \rightarrow \infty} \Phi\left(S\left(t_{n}\right) u_{0}\right)=\inf _{0 \leq t<\infty} \Phi\left(S(t) u_{0}\right),
$$

i.e., $\Phi(v)$ is constant on $\omega\left(u_{0}\right)$. Note that $S(t) \omega\left(u_{0}\right)=\omega\left(u_{0}\right)$ for every $0 \leq t<\infty$. Thus, $\Phi(S(t) v)$ is constant for $0 \leq t<\infty$. We know that $S(\cdot) v \in \mathcal{E}\left([0, \infty) ; H^{1}(I)\right) \cap$ $\mathcal{C}^{1}\left([0, \infty) ; H^{1}(I)^{*}\right)$. Then, we see that

$$
\begin{equation*}
\left.\frac{d}{d t} \Phi(S(t) v)\right|_{t=0}=\left.\left\langle a \log v+G(x) v, \frac{d}{d t} S(t) v\right\rangle_{H^{1} \times H^{1 *}}\right|_{t=0} \tag{7.1}
\end{equation*}
$$

Indeed, for sufficiently small $\Delta t>0$, we obtain from (5.1) that

$$
\begin{aligned}
& \Phi(S(\Delta t) v)-\Phi(v)-\langle a \log v+G(x) v, S(\Delta t) v-v\rangle_{H^{1} \times H^{1 *}} \\
& \quad=\|S(\Delta t) v-v\|_{H^{1 *}} R(S(\Delta t) v-v)
\end{aligned}
$$

Dividing this equality by $\Delta t$, we have

$$
\begin{aligned}
& \frac{\Phi(S(\Delta t) v)-\Phi(v)}{\Delta t}-\left\langle a \log v+G(x) v, \frac{S(\Delta t) v-v}{\Delta t}\right\rangle_{H^{1} \times H^{1 *}} \\
& \quad=\left\|\frac{S(\Delta t) v-v}{\Delta t}\right\|_{H^{1 *}} R(S(\Delta t) v-v)
\end{aligned}
$$

Since $S(\Delta t) v \rightarrow v$ in $H^{1}(I)$ and $\left.\frac{S(\Delta t) v-v}{\Delta t} \rightarrow \frac{d}{d t} S(t) v\right|_{t=0}$ in $H^{1}(I)^{*}$ as $\Delta t \rightarrow 0,(7.1)$ is verified. Then, we see that

$$
\begin{aligned}
0 & =\left.\frac{d}{d t} \Phi(S(t) v)\right|_{t=0} \\
& =\left\langle a \log v+G(x) v, \frac{\partial}{\partial x}\left[v \frac{\partial}{\partial x}(a \log v+G(x) v)\right]\right\rangle_{H^{1} \times H^{1 *}} \\
& =-\left\|\sqrt{v} \frac{\partial}{\partial x}(a \log v+G(x) v)\right\|_{L_{2}}^{2} \\
& \leq-\min _{x \in \bar{I}}\{v(x)\}\left\|\frac{\partial}{\partial x}(a \log v+G(x) v)\right\|_{L_{2}}^{2}
\end{aligned}
$$

Therefore, $v \in \omega\left(u_{0}\right)$ implies that

$$
a \log v+G(x) v=C \text { in } \bar{I}
$$

i.e., $v$ is a solution of (6.1). By Theorem 6.1, we obtain that $\omega\left(u_{0}\right)=\left\{\bar{u}_{l}\right\}$.

Proof of Theorem 7.1. We verify that

$$
\inf _{v \in \omega\left(u_{0}\right)}\left\|S(t) u_{0}-v\right\|_{H^{1}} \rightarrow 0 \text { as } t \rightarrow \infty
$$

Indeed, suppose the contrary; then there exists a sequence of time $t_{n} \nearrow \infty$ such that $\Phi\left(S\left(t_{n}\right) u_{0}\right)$ does not converge to $\Phi(v), v \in \omega\left(u_{0}\right)$, which is a contradiction. As shown in Proposition 7.1, $\omega\left(u_{0}\right)$ consists of $\bar{u}_{l}$. Therefore, we obtain that

$$
\left\|S(t) u_{0}-\bar{u}_{l}\right\|_{H^{1}} \rightarrow 0 \text { as } t \rightarrow \infty
$$

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