ON A NONLOCAL BIHARMONIC MEMS EQUATION WITH THE NAVIER BOUNDARY CONDITION

Tosiya Miyasita

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Abstract. We study a biharmonic nonlocal MEMS equation. It arises in the Micro-Electro Mechanical System(MEMS) devices. First we establish the local solution and extend it globally in time by the use of the energy. Next, we consider the dynamical properties. The dynamical system has an absorbing set and a global attractor. Finally we prove the convergence of the global solution to a stationary solution.

1 Introduction We consider the following biharmonic nonlocal MEMS equation:

$$(1)\qquad \begin{cases} u_{tt} + u_t + \Delta^2 u = G(\beta, \gamma, \nabla u)\, \Delta u + \frac{\lambda}{(1-u)^\sigma} I(\sigma, \chi, u) & x \in \Omega,\ t > 0, \\ u = \Delta u = 0 & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x) & x \in \Omega, \\ u_t(x, 0) = u_1(x) & x \in \Omega, \end{cases}$$

where $\lambda > 0$, $\beta > 0$, $\gamma > 0$, $\chi > 0$, $\sigma \geq 2$, $\Omega \subset \mathbb{R}^n$ for $n \in \mathbb{N}$ is a bounded domain with smooth boundary $\partial\Omega$,

$$G(\beta, \gamma, \nabla u) = \beta \int_\Omega |\nabla u|^2\, dx + \gamma,$$

$$I(\sigma, \chi, u) = \frac{1}{(H(\sigma, \chi, u))^\sigma} \qquad \text{and} \qquad H(\sigma, \chi, u) = 1 + \chi \int_\Omega \frac{dx}{(1-u)^{\sigma-1}}.$$

If the solution $u(x, t)$ of (1) reaches 1 at some point in Ω in finite time $t = T_q$, the right-hand side of (1) becomes infinite, which leads to the singularity. In this case, the solution $u(x, t)$ is said to quench in finite time $t = T_q$ and T_q is called the quenching time of the solution. This equation has been considered in [2, 4] and is a natural extension of MEMS equation [7, 8, 20, 23]. The MEMS (Micro-Electro Mechanical System) equation arises in the study of the MEMS devices which are often utilized to combine electronics with micro-size mechanical devices. They can be modelled as the dynamic deflection of an elastic membrane inside this system and arise in the accelerometers for airbag deployment in automobiles, in the ink jet printer heads, in the optical switches, in the chemical sensors and so on.
In [2], the authors establish the stationary solution with Steklov and Dirichlet boundary condition by the implicit function theorem [25]. They construct the stationary solution $u \in H^4(\Omega) \cap H_0^1(\Omega)$ of (1) provided that the diameter of Ω is sufficiently small. In [4], the authors consider the periodic solution of (1) by [25]. In the limiting case $\chi = 0$, there is supposed to be no capacitor in the circuit, which is studied in [13] with $\beta = 0$ and $\sigma = 2$. The author derives the results of existence, convergence to the stationary solution and exponential decay of the global solution. On the other hand, he deals with the quenching of the solution. The aim of this paper is to investigate the dynamical properties to the biharmonic nonlocal

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problem (1). For the second order nonlocal equation, see [10, 11, 12, 16, 17, 19, 21, 22, 24]. First, we obtain the theorem concerned with the local existence of the solution. Throughout this paper, the definition of the function spaces and their norms is presented in Section 2.

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ with $n = 1, 2, 3$. We denote $X \equiv H^2(\Omega) \cap H_0^1(\Omega)$, $D \equiv X \times L^2(\Omega)$ and $H \equiv L^2(\Omega) \times H^{-2}(\Omega)$. For any $\lambda > 0$, $\beta > 0$, $\gamma > 0$, $\chi > 0$, $\sigma \geq 2$ and $\phi_0 \equiv \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in D$ with*

$$\|u_0\|_C < 1 - \delta$$

for some $\delta \in (0,1)$, there exists a unique solution of (1) with

$$\phi \equiv \begin{pmatrix} u \\ u_t \end{pmatrix} \in C\left([0,T); D\right) \cap C^1\left([0,T); H\right)$$

for sufficiently small $T > 0$, where T depends only on λ, β, γ, χ, σ, Ω, (u_0, u_1) and δ. The solution u can be continued as long as $\|u(\,\cdot\,,t)\|_C < 1$. Here, $\|\cdot\|_C$ denotes the standard $C(\overline{\Omega})$ norm defined in Section 2.

To establish the global solution, we define the energies by

$$\mathcal{E}(\phi(t)) = \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{1}{2}\int_\Omega (\Delta u)^2\,dx + \frac{\beta}{4}\left(\int_\Omega |\nabla u|^2\,dx\right)^2 + \frac{\gamma}{2}\int_\Omega |\nabla u|^2\,dx$$

and

$$\mathcal{E}_0 \equiv \frac{1}{2}\int_\Omega u_1^2\,dx + \frac{1}{2}\int_\Omega (\Delta u_0)^2\,dx + \frac{\beta}{4}\left(\int_\Omega |\nabla u_0|^2\,dx\right)^2 + \frac{\gamma}{2}\int_\Omega |\nabla u_0|^2\,dx,$$

respectively. Then

$$E(\phi(t)) \equiv \mathcal{E}(\phi(t)) + \frac{\lambda}{(\sigma-1)^2\chi}\left(H\left(\sigma,\chi,u\right)\right)^{1-\sigma}$$

is a Lyapunov function for (1), which plays an important role in proving the global existence and dynamical properties of the solution. We impose the smallness condition on the parameter $\lambda > 0$ and initial energy $\mathcal{E}_0$. To state the condition, we define

$$\lambda^* \equiv \frac{(\sigma-1)^2\chi a}{2}\left(\frac{2^{\sigma-1} + \chi|\Omega|}{2^{\sigma-1}}\right)^{\sigma-1}$$

for any fixed $\gamma > 0$, $\chi > 0$, $\sigma \geq 2$ and $\Omega \subset \mathbb{R}^n$, where $a > 0$ depends only on γ and Ω and is defined in Section 4. Moreover we define

$$\mathcal{E}_0^* \equiv \frac{a}{2} - \frac{\lambda}{(\sigma-1)^2\chi}\left(\frac{2^{\sigma-1}}{2^{\sigma-1} + \chi|\Omega|}\right)^{\sigma-1} > 0$$

for these fixed constants and any $0 < \lambda < \lambda^*$. Then we have the next theorem on the global existence of the solution.

Theorem 2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ with $n = 1, 2, 3$. For any $\beta > 0$, $\gamma > 0$, $\chi > 0$ and $\sigma \geq 2$, let $\lambda < \lambda^*$ be fixed arbitrarily. For all $\kappa \in (0, \mathcal{E}_0^*)$, there exists $\delta_0 \in (0,1)$ such that for any $\phi_0 \in D$ with*

$$\|u_0\|_C < 1 - \delta_0$$

and

$$\mathcal{E}_0 < \mathcal{E}_0^* - \kappa,$$

(1) *has a global solution satisfying*

$$\phi \in C\left([0,\infty); D\right) \cap C^1\left([0,\infty); H\right)$$

and

$$\left\|u(\,\cdot\,,t)\right\|_C < 1 - \delta_0$$

for all $t \geq 0$. *Here,* δ_0 *depends only on* λ, γ, χ, σ, κ *and* Ω.

Next theorem is on the regularity of the solution obtained in Theorem 2. To state the theorem, we define Y by

$$Y = \left\{u \in H^4(\Omega) \cap H_0^1(\Omega) \mid \Delta u = 0 \text{ on } \partial\Omega\right\}.$$

If $n \leq 3$, the Sobolev embedding $H^2(\Omega) \subset C(\overline{\Omega})$ holds. Hence we note that $\Delta u = 0$ on $\partial\Omega$ makes sense in this paper. Now we denote $E \equiv Y \times X$. Then we have the following:

Theorem 3 *Under the same hypotheses as Theorem 2, for any* $\phi_0 \in E$, *there exists a unique global solution of* (1) *with*

$$\phi \in C\left([0,\infty); E\right) \cap C^1\left([0,\infty); D\right) \cap C^2\left([0,\infty); H\right).$$

We define

$$Z_{\delta_0} \equiv \left\{ \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in D \mid \left\|u^1\right\|_C < 1 - \delta_0 \right\}$$

and consider the nonlinear semigroup $S(t) : Z_{\delta_0} \to Z_{\delta_0}$ by

$$S(t)\phi_0 = \phi(t).$$

In the fourth theorem, we establish an absorbing set to show that $S(t)$ has a global attractor in Z_{δ_0}.

Theorem 4 *In addition to the same hypotheses as Theorem 2, let*

$$\mathcal{E}_0 + \frac{\lambda}{(\sigma-1)^2 \chi} < \frac{\gamma K_1}{2\beta K_2} \tag{2}$$

hold, where $K_1 > 0$ *and* $K_2 > 0$ *depend only on* γ *and* Ω *and are defined in Lemma 4. Then the dynamical system* $S(t) : Z_{\delta_0} \to Z_{\delta_0}$ *possesses an absorbing set* $\mathcal{B} \subset Z_{\delta_0}$. *The omega limit set* $\mathcal{A} = \omega(\mathcal{B})$ *of* $\mathcal{B}$ *is a global attractor in* Z_{δ_0}.

To argue the behaviour as $t \to +\infty$, we introduce the set $\mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma}$ of stationary solution by

$$\mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma} = \left\{\eta \in Z_{\delta_0} \mid \eta = \eta(x) \text{ is a stationary solution for (1)}\right\}.$$

In [2], they construct the stationary solution by the implicit function theorem for the small domain. We also find the stationary solution without imposing any smallness condition on Ω. We derive the following theorem on dynamical properties of $S(t)$.

Theorem 5 *Under the same hypotheses as Theorem 4, the omega limit set $\omega(\phi_0)$ is invariant, non-empty, compact and connected in Z_{δ_0}. Moreover $\omega(\phi_0) \subset \mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma} \times \{0\}$. In particular,*

$$\mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma} \neq \emptyset$$

for $\lambda \in (0, \underline{\lambda}^*)$, where

$$\underline{\lambda}^* \equiv \min\left(\lambda^*, \frac{\gamma\chi\left(\sigma-1\right)^2 K_1}{2\beta K_2}\right).$$

We prove that the omega limit set is composed of a single point in Z_{δ_0}.

Theorem 6 *Under the same hypotheses as Theorem 4, there exists $\eta \in \mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma}$ such that the omega limit set is composed of a single point in Z_{δ_0} with*

$$\omega(\phi_0) = (\eta, 0)$$

and

$$\lim_{t\to+\infty}\left(\|u(\,\cdot\,,t)-\eta\|_X + \|u_t(\,\cdot\,,t)\|_2\right) = 0. \tag{3}$$

This paper is organized as follows: In Section 2, we recall the facts about Sobolev space and dynamical system. We introduce the existence theorem [2] of stationary solution. In Section 3, we establish the local solution by the contraction mapping theorem. In Section 4, we extend the local solution to the global one for small parameters and initial values. Moreover we study the regularity of the global solution. In Section 5, we consider the dynamical properties. By the existence of the Lyapunov function, we can treat the omega limit set and global attractor. In Section 6, by the Lojasiewicz-Simon inequality, we show that the omega limit set is composed of a single point. In an appendix, we prove the Lojasiewicz-Simon inequality. This kind of inequalities is proven in many situations. In this paper, we treat the case with nonlocal term.

2 Preliminaries First, we introduce the notations of function spaces and the Sobolev embedding theorems. In this paper, $C(\overline{\Omega})$ denotes the space of all continuous functions in $\overline{\Omega}$ with the norm

$$\|u\|_C = \sup_{x\in\overline{\Omega}} |u(x)|$$

for $u \in C(\overline{\Omega})$. For $1 \leq p \leq +\infty$, we denote the usual Sobolev space in Ω by $W^{s,p}(\Omega)$ and in particular write $W^{s,2}(\Omega) = H^s(\Omega)$. $H^s_0(\Omega)$ is defined as the closure of the set $\mathcal{D}(\Omega)$ in the space $H^s(\Omega)$, where we denote by $\mathcal{D}(\Omega)$ the space of all infinitely differentiable functions on Ω with compact supports. $H^{-s}(\Omega)$ is defined as the dual space of $H^s_0(\Omega)$ equipped with the norm

$$\|u\|_{H^{-s}} = \sup_{w\in H^s_0(\Omega),\|w\|_{H^s_0}\leq 1} \left|\int_\Omega uw\,dx\right|.$$

$(\ ,\ )$ and $(\ ,\ )_{H^{-s}}$ denote the inner product in $L^2(\Omega)$ and $H^{-s}(\Omega)$, respectively. According to [1, 3] , we adopt the norm in $H^1_0(\Omega)$, X and Y as

$$\|u\|_{H^1_0} = \|\nabla u\|_2\,, \quad \|u\|_X = \|\Delta u\|_2 \quad \text{and} \quad \|u\|_Y = \left\|\Delta^2 u\right\|_2,$$

respectively. Here, $\|\cdot\|_p$ denotes the standard L^p norm in Ω with $p \in [1,\infty]$. We define

$$\|\phi\|_D = \left(\left\|u^1\right\|_X^2 + \left\|u^2\right\|_2^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|\phi\|_E = \left(\left\|u^1\right\|_Y^2 + \left\|u^2\right\|_X^2\right)^{\frac{1}{2}}$$

for $\phi = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$. We introduce the embedding inequalities [1].

Lemma 1 *Let $n = 1$. For $u \in H_0^1(\Omega)$, we have*

$$\|u\|_C \le C_S \|u\|_{H_0^1},$$

where $C_S > 0$ depends only on Ω.

Lemma 2 *Let $n = 2, 3$. For $u \in X$, we have*

$$\|u\|_C \le C_S \|u\|_X,$$

where $C_S > 0$ depends only on Ω.

Lemma 3 *For $u \in H_0^1(\Omega)$, we have*

$$\|u\|_2 \le C_P \|u\|_{H_0^1},$$

where $C_P > 0$ depends only on Ω.

Henceforth we shall adopt universal notations C_S and C_P to denote these constants for the case $n = 1, 2, 3$.
We introduce the theorem of existence of the global attractor. For other basic notions and results, see [26, 28]. Let Z be Banach space and $S(t)$ be a continuous semigroup on Z. The semigroup $S(t)$ is said to be uniformly compact if for every bounded set $B \subset Z$, there exists t_0 such that $\cup_{t \ge t_0} S(t)B$ is relatively compact in Z.

Theorem 7 (Theorem 1.1 in [26]) *Let $S(t)$ be a continuous semigroup on Banach space Z. We assume that it can be decomposed into $S(t) = S_1(t) + S_2(t)$, where $S_1(t)$ is uniformly compact for large $t > 0$ and $S_2(t)$ is continuous from Z to Z satisfying the following condition: For any bounded set $B \subset Z$,*

$$\sup_{\phi_0 \in B} \|S_2(t)\phi_0\|_Z \to 0$$

as $t \to \infty$. We also assume that there exist an open set $\mathcal{U}$ and absorbing set $\mathcal{B} \subset \mathcal{U}$. Then the omega limit set $\mathcal{A} = \omega(\mathcal{B})$ of $\mathcal{B}$ is a global attractor in $\mathcal{U}$ for $S(t)$.

Finally we mention the existence of stationary solution. We consider the corresponding elliptic equation

$$\left\{\begin{array}{ll} \Delta^2\eta = G(\beta,\gamma,\nabla\eta)\,\Delta\eta + \frac{\lambda}{(1-\eta)^\sigma} I(\sigma,\chi,\eta) & x \in \Omega, \\ \eta = \Delta\eta - d\frac{\partial\eta}{\partial\nu} = 0 & x \in \partial\Omega, \end{array}\right. \tag{4}$$

where $d \in [0,+\infty]$, ν is the outer unit normal vector and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Then the existence of the stationary solution is guaranteed by the implicit function theorem [25].

Theorem 8 (Theorem 1 in [2]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ with $n \leq 7$. For any $\lambda > 0$, $\beta > 0$, $\gamma > 0$, $\chi > 0$ and $\sigma \geq 2$, there exist $\overline{\lambda} > 0$ and $d_0 > 0$ such that* (4) *possesses a solution $\eta \in H^4(\Omega) \cap H_0^1(\Omega)$ for all $\lambda \in (0, \overline{\lambda})$ provided that one of the following holds*:
Steklov boundary condition: $0 \leq d < d_0$
or
Dirichlet boundary condition: $d = +\infty$ *and Ω is a ball and the diameter of Ω is sufficiently small.*

The relation between $\overline{\lambda}$ and $\underline{\lambda}^*$ in Theorem 5 is not clear.

3 Local existence We consider the linear wave equation

$$\left\{ \begin{array}{ll} w_{tt} + w_t + Aw = 0 & x \in \Omega,\ t > 0, \\ w = \Delta w = 0 & x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x) & x \in \Omega, \\ w_t(x,0) = w_1(x) & x \in \Omega, \end{array} \right. \tag{5}$$

where

$$Aw = \Delta^2 w - \gamma \Delta w$$

and derive the decay estimate of the solution. Next we construct the time local solution of (1) by the contraction mapping theorem. We omit the detail of the computations. See [13, 18, 21].

Lemma 4 (Proposition 4.3.4 in [15] and (3.5) in [13]) *For any $\psi_0 \equiv \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \in D$, there exists a unique solution*

$$\psi \equiv \begin{pmatrix} w \\ w_t \end{pmatrix} \in C\left([0,\infty); D\right) \cap C^1\left([0,\infty); H\right)$$

of (5). *Moreover, we have*

$$\|\psi\|_D \leq K_2 \|\psi_0\|_D \, e^{-K_1 t},$$

where $K_1 > 0$ and $K_2 > 0$ depend only on γ and Ω.

Proof of Theorem 1. To deal with the nonlinear term with the singularity, we modify $1/(1-u)$ and $I(\sigma, \chi, u)$ by

$$F_\delta(u) = \left\{ \begin{array}{ll} \frac{1}{1-u} & u \leq 1 - \frac{\delta}{2}, \\ \frac{4}{\delta} & u \geq 1 - \frac{\delta}{4} \end{array} \right.$$

and

$$I_\delta(\sigma, \chi, u) = \frac{1}{\left(H_\delta(\sigma, \chi, u)\right)^\sigma} \quad \text{with} \quad H_\delta(\sigma, \chi, u) = 1 + \chi \int_\Omega F_\delta\left(u(x)\right)^{\sigma-1} dx,$$

where we continue $F_\delta(u)$ suitably in the range $(1 - \delta/2, 1 - \delta/4)$ so that we assume that F_δ is positive, bounded and sufficiently smooth. Under the abstract setting

$$\phi = \begin{pmatrix} u \\ u_t \end{pmatrix}, \quad \phi_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i_d \\ A & i_d \end{pmatrix}$$

and

$$J_\delta(u) = \begin{pmatrix} 0 \\ \beta\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u + \lambda F_\delta(u)^\sigma I_\delta(\sigma,\chi,u) \end{pmatrix},$$

we transform (1) into the modified equation

$$\phi_t + B\phi = J_\delta(u)$$

and consider the corresponding integral equation

$$\phi = e^{-Bt}\phi_0 + \int_0^t e^{-B(t-s)} J_\delta(u(s))\, ds. \tag{6}$$

We construct the local solution by the contraction mapping theorem. Taking $l = \|\phi_0\|_D$, we set

$$X_T \equiv \left\{\phi \in C([0,T];D) \mid \|\phi\|_{X_T} \le 2K_2 l\right\},$$

where T is a positive constant to be determined later. Here in the space X_T, the norm is equipped with

$$\|\phi\|_{X_T} = \sup_{t\in[0,T]} \|\phi(\,\cdot\,,t)\|_D .$$

For $\phi \in X_T$, we define the mapping $V(t)$ on D by the right-hand side of (6), that is,

$$V(t)\phi = e^{-Bt}\phi_0 + \int_0^t e^{-B(t-s)} J_\delta(u(s))\, ds.$$

Then we can show that V is a contraction mapping from X_T into itself for small $T > 0$.

Lemma 5 (Cf. Lemmas 2 and 3 in [21]) *If $T < \tau$, then V is a contraction mapping from X_T into X_T, where $\tau > 0$ is a constant determined only by λ, β γ, χ, σ, Ω, l and δ.*

By Lemma 5, (6) has a unique time local solution $\phi \in C([0,T);D) \cap C^1([0,T);H)$. If the solution of (6) begins with $\|u_0\|_C < 1-\delta$ and satisfies $\|u(\,\cdot\,,t)\|_C \le 1-\delta/2$ for all $t > 0$, then u is a solution of (1). Otherwise there is a finite time $T_0 > 0$ at which $\max_{x\in\overline{\Omega}} u(x,T_0) = 1-\delta/2$. We choose $\delta_1 \in (0,\delta)$ and apply the contraction mapping theorem to (6) with δ replaced by δ_1. We may extend $u(x,t)$ uniquely to an interval $(0,T_0')$ with $T_0 < T_0'$ such that $\|u(\,\cdot\,,t)\|_C \le 1-\delta_1/2$ for all $t \in [0,T_0')$. Since we can take $\delta_1 \in (0,\delta)$ arbitrarily small, $u(x,t)$ is a solution of (1) on $\overline{\Omega}\times[0,T_0')$ as long as $\|u(\,\cdot\,,t)\|_C < 1$. □

4 Global existence In this section, we shall show that the local solution can be continued up to $t = +\infty$. We introduce the Lyapunov function to obtain the necessary estimates and extend it globally in time. The idea is from [17]. At first, in order to introduce the lemma, we set

$$a = \begin{cases} \frac{\gamma}{C_S^2} & \text{for } n = 1, \\ \frac{1}{C_S^2} & \text{for } n = 2,3, \end{cases} \qquad b = \frac{2\lambda}{(\sigma-1)^2\chi} \quad \text{and} \quad c = \chi|\Omega|$$

and define

$$g(x) = ax^2 + b\left\{\frac{(1-x)^{\sigma-1}}{(1-x)^{\sigma-1}+c}\right\}^{\sigma-1}$$

for $-1 \le x \le 1$, where C_S is the constant defined in Lemmas 1 and 2. Let

$$G(x) = g(x) - 2\mathcal{E}_0^* - g(-1) + g(1) + 2\kappa$$

for $0 \le x \le 1$.

Lemma 6 *Under the same hypotheses as Theorem 2, there exists a zero $x_0 \in (0,1)$ of $G(x)$, where x_0 depends only on λ, γ, χ, σ, κ and Ω.*

Proof. Since we have

$$g(-1) = a + b\left(\frac{2^{\sigma-1}}{2^{\sigma-1}+c}\right)^{\sigma-1}, \quad g(0) = b\left(\frac{1}{1+c}\right)^{\sigma-1} \quad \text{and} \quad g(1) = a$$

and

$$h(x) = \left(\frac{x}{x+c}\right)^{\sigma-1}$$

is increasing for $x \geq 0$, a simple computation yields

$$G(0) = 2\left(\kappa - \mathcal{E}_0^*\right) + b\left(h(1) - h(2^{\sigma-1})\right) < 0$$

by the hypotheses $0 < \kappa < \mathcal{E}_0^*$ and $\sigma \geq 2$. On the other hand, we have

$$G(1) = a - b\left(\frac{2^{\sigma-1}}{2^{\sigma-1}+c}\right)^{\sigma-1} - 2\mathcal{E}_0^* + 2\kappa = 2\kappa > 0.$$

Thus the intermediate theorem guarantees at least one zero in $(0,1)$. Henceforth, we denote the least zero by

$$x_0 = 1 - \delta_0$$

with $\delta_0 \in (0,1)$. □

Proof of Theorem 2. For (1), we have the Lyapunov function

$$E(\phi(t)) \equiv \mathcal{E}(\phi(t)) + \frac{\lambda}{(\sigma-1)^2\chi}\left(H\left(\sigma,\chi,u\right)\right)^{1-\sigma}$$

for $t \in [0,T)$ and set

$$E_0 \equiv E(\phi_0) = \mathcal{E}_0 + \frac{\lambda}{(\sigma-1)^2\chi}\left(H\left(\sigma,\chi,u_0\right)\right)^{1-\sigma},$$

where T is the maximal existence time of the solution determined in Section 3. In fact, we obtain

$$\frac{d}{dt}E(\phi(t)) = -\int_\Omega u_t^2\,dx \leq 0,$$

which implies that

$$E(\phi(t)) \leq E(\phi(t)) + \int_0^t\int_\Omega u_t^2\,dx\,ds = E_0. \tag{7}$$

Now we estimate $E(\phi(t))$ and E_0 as follows:

$$2E(\phi(t)) \geq \|u\|_X^2 + \gamma\|u\|_{H_0^1}^2 + \frac{b}{\left(1 + c\left(\frac{1}{1-\|u\|_C}\right)^{\sigma-1}\right)^{\sigma-1}} \geq g(\|u\|_C)$$

by Lemmas 1 and 2 and

$$2E_0 < 2\mathcal{E}_0 + b\left(H\left(\sigma,\chi,-1\right)\right)^{1-\sigma} \leq 2\mathcal{E}_0^* + b\left(\frac{2^{\sigma-1}}{2^{\sigma-1}+c}\right)^{\sigma-1} - 2\kappa$$

due to $-1 < u_0$. Then the energy inequality (7) yields

$$G(\|u(t)\|_C) = g(\|u(t)\|_C) - 2\mathcal{E}_0^* - g(-1) + g(1) + 2\kappa < 0$$

for all $t \in [0, T)$. By Lemma 6 and $\|u_0\|_C < 1 - \delta_0$,

$$\|u(t)\|_C < 1 - \delta_0 \tag{8}$$

holds for all $t \in [0, T)$. Owing to the energy (7), we have

$$\gamma \|u(t)\|_{H_0^1}^2 + \|u(t)\|_X^2 + \|u_t(t)\|_2^2 \leq 2E_0 \tag{9}$$

for all $t \in [0, T)$. We note that

$$E_0 < \mathcal{E}_0^* + \frac{\lambda^*}{(\sigma - 1)^2 \chi} - \kappa < \frac{a}{2} + \frac{a}{2} \left(\frac{2^{\sigma-1} + \chi |\Omega|}{2^{\sigma-1}} \right)^{\sigma-1}. \tag{10}$$

Hence the right-hand side depends only on γ, χ, σ and Ω and is independent of $\|\phi_0\|_D$ and T. Finally (8) and (9) are valid for all $t \geq 0$, which ensures that the solution exists globally in time. Since $L^2(\Omega) \subset H^{-2}(\Omega)$, we have

$$u_{tt} = -u_t - \Delta^2 u + G(\beta, \gamma, \nabla u)\Delta u + \frac{\lambda}{(1-u)^\sigma} I(\sigma, \chi, u) \in H^{-2}(\Omega)$$

and $\phi \in C([0, \infty); D) \cap C^1([0, \infty); H)$. □

Proof of Theorem 3. In this proof, by L we denote the universal positive constants which depend only on λ, β, γ, χ, σ, κ, Ω and $\|\phi_0\|_E$. We define $(u_1)_t$ by

$$(u_1)_t = -u_1 - \Delta^2 u_0 + G(\beta, \gamma, \nabla u_0)\Delta u_0 + \frac{\lambda}{(1-u_0)^\sigma} I(\sigma, \chi, u_0) \in L^2(\Omega).$$

First by differentiating $\|u_{tt}\|_2^2$ with respect to t, we obtain

$$\frac{d}{dt}\int_\Omega u_{tt}^2\, dx = -2\int_\Omega u_{tt}^2\, dx - \frac{d}{dt}\int_\Omega (\Delta u_t)^2\, dx + 2I_1 + 2I_2. \tag{11}$$

The definition and estimation of I_1 and I_2 are as follows. (8), (9), the Hölder and Young inequalities yield

$$\begin{aligned} I_1 &\equiv \int_\Omega u_{tt}\Big(G(\beta, \gamma, \nabla u)\Delta u\Big)_t\, dx \\ &= -2\beta \int_\Omega \Delta u u_t\, dx \int_\Omega u_{tt}\Delta u\, dx - G(\beta, \gamma, \nabla u)\int_\Omega \nabla u_{tt} \cdot \nabla u_t\, dx \\ &\leq \frac{1}{4}\int_\Omega u_{tt}^2\, dx + L\int_\Omega u_t^2\, dx - \frac{1}{2}G(\beta, \gamma, \nabla u)\frac{d}{dt}\int_\Omega |\nabla u_t|^2\, dx \end{aligned} \tag{12}$$

and

$$I_2 \equiv \int_\Omega u_{tt}\left(\frac{\lambda}{(1-u)^\sigma} I(\sigma, \chi, u)\right)_t dx \leq \frac{1}{4}\int_\Omega u_{tt}^2\, dx + L\int_\Omega u_t^2\, dx, \tag{13}$$

respectively. Eventually integrating (11) over $(0,t)$ with respect to t together with (12) and (13), we obtain

$$\begin{aligned}
&\int_0^t \int_\Omega u_{tt}^2 \, dx \, ds + \int_\Omega \left(u_{tt}^2 + (\Delta u_t)^2 \right) dx + G(\beta, \gamma, \nabla u) \int_\Omega |\nabla u_t|^2 \, dx \\
&\le \|(u_1)_t\|_2^2 + \|u_1\|_X^2 + G(\beta, \gamma, \nabla u_0) \|u_1\|_{H_0^1}^2 \\
&\quad + \int_0^t \frac{d}{dt}\Big(G(\beta, \gamma, \nabla u) \Big) \left(\int_\Omega |\nabla u_t|^2 \, dx \right) ds + L \int_0^t \int_\Omega u_t^2 \, dx \, ds.
\end{aligned}$$

Since the fourth integral term in the right-hand side yields

$$\begin{aligned}
\frac{d}{dt}\Big(G(\beta, \gamma, \nabla u) \Big) \int_\Omega |\nabla u_t|^2 \, dx &= 2\beta \int_\Omega \Delta u u_t \, dx \int_\Omega \Delta u_t u_t \, dx \\
&\le L \int_\Omega u_t^2 \, dx + \beta \int_\Omega u_t^2 \, dx \int_\Omega (\Delta u_t)^2 \, dx,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^t \int_\Omega u_{tt}^2 \, dx \, ds + \int_\Omega \left(u_{tt}^2 + (\Delta u_t)^2 \right) dx + G(\beta, \gamma, \nabla u) \int_\Omega |\nabla u_t|^2 \, dx \\
&\le L + \beta \int_0^t \left(\int_\Omega u_t^2 \, dx \right) \left(\int_\Omega (\Delta u_t)^2 \, dx \right) ds
\end{aligned}$$

and in particular

$$\int_\Omega (\Delta u_t)^2 \, dx \le L + \beta \int_0^t \left(\int_\Omega u_t^2 \, dx \right) \left(\int_\Omega (\Delta u_t)^2 \, dx \right) ds.$$

We apply the Gronwall inequality (Lemma 2.1.1 in [15]) to derive

$$\int_\Omega (\Delta u_t)^2 \, dx \le L \exp\left(\beta \int_0^{+\infty} \int_\Omega u_t^2 \, dx \, ds \right) \le L,$$

which implies that $u \in Y$, $u_t \in X$ and $u_{tt} \in L^2(\Omega)$ due to

$$\Delta^2 u = -u_{tt} - u_t + G(\beta, \gamma, \nabla u) \Delta u + \frac{\lambda}{(1-u)^\sigma} I(\sigma, \chi, u) \in L^2(\Omega).$$

□

5 Global attractor First, we show that the orbit $\cup_{t \ge 0} \phi(t)$ is contained in some absorbing set in Z_{δ_0}. Hence this fact leads us to the existence of a global attractor by Theorem 7 in Section 2. Next, we consider the properties of $\omega(\phi_0)$. We show that $(\eta, 0) \in \omega(\phi_0)$ for some $\eta \in \mathcal{S}^\lambda_{\beta,\gamma,\chi,\sigma}$. In other words, there exist $\eta \in \mathcal{S}^\lambda_{\beta,\gamma,\chi,\sigma}$ and $t_n \to +\infty$ such that

$$\lim_{n \to +\infty} \Big(\|u(\,\cdot\,, t_n) - \eta\|_X + \|u_t(\,\cdot\,, t_n)\|_2 \Big) = 0. \tag{14}$$

In [2], the authors establish the stationary solution $\eta \in Y$. However they impose the smallness condition on Ω. See Theorem 8 in this paper. In this section, the Lyapunov

function plays an important role in the argument.
For a solution u of (1) obtained in Theorem 2, we denote by v a solution of

$$\begin{cases} v_{tt} + v_t + Av = \beta \left(\int_\Omega |\nabla u|^2 \, dx \right) \Delta v & x \in \Omega,\ t > 0, \\ v = \Delta v = 0 & x \in \partial\Omega,\ t > 0, \\ v(x,0) = u_0(x) & x \in \Omega, \\ v_t(x,0) = u_1(x) & x \in \Omega. \end{cases} \tag{15}$$

Let $\psi = \begin{pmatrix} v \\ v_t \end{pmatrix}$ and $S_2(t)\phi_0 = \psi(t)$. From now on, we show that the semigroup S_2 has a decaying property. First, ψ satisfies

$$\psi = e^{-Bt}\phi_0 + \int_0^t e^{-B(t-s)} P\left(u(s), v(s)\right) \, ds,$$

where

$$P\left(u(t), v(t)\right) = \begin{pmatrix} 0 \\ \beta \left(\int_\Omega |\nabla u|^2 \, dx \right) \Delta v \end{pmatrix}.$$

We set

$$K_3 \equiv K_1 - \frac{2\beta E_0 K_2}{\gamma},$$

where $K_1 > 0$ and $K_2 > 0$ are constants defined in Lemma 4 and derive

$$K_3 > K_1 - \frac{2\beta K_2}{\gamma} \left(\mathcal{E}_0 + \frac{\lambda}{(\sigma - 1)^2 \chi} \right) > 0$$

provided that (2) holds.

Lemma 7 *Under the same hypotheses as Theorem 4, for any $\phi_0 \in D$, there exists a unique solution*

$$\psi \in C\left([0,\infty); D\right) \cap C^1\left([0,\infty); H\right)$$

of (15). *Moreover, we have*

$$\|\psi(t)\|_D \le K_2 e^{-K_3 t} \|\phi_0\|_D .$$

Proof. Thanks to Lemma 4, we have

$$\|\psi\|_D \le K_2 \|\phi_0\|_D e^{-K_1 t} + \frac{2\beta E_0 K_2}{\gamma} \int_0^t e^{-K_1(t-s)} \|\psi\|_D \, ds$$

by the use of (9) and

$$e^{K_1 t} \|\psi(t)\|_D \le K_2 \|\phi_0\|_D + (K_1 - K_3) \int_0^t e^{K_1 s} \|\psi(s)\|_D \, ds.$$

The Gronwall inequality yields

$$\|\psi(t)\|_D \le K_2 e^{-K_3 t} \|\phi_0\|_D .$$

□

Next, in order to prove uniformly compactness, we introduce the lemmas. Their proofs are similar to that of Theorem 3. Henceforth we shall adopt universal notations $M > 0$ to denote the various constants which depend only on λ, β, γ, χ, σ, κ, Ω and $\|\phi_0\|_D$.

Lemma 8 *The solution ψ obtained in Lemma 7 satisfies*

$$\|v_t\|_2^2 + \|v\|_X^2 + \gamma \|v\|_{H_0^1}^2 + 2\int_0^t \|v_t\|_2^2 \, ds \le M.$$

Proof. We have

$$\left(\|v_t\|_2^2 + \|v\|_X^2 + \gamma \|v\|_{H_0^1}^2\right)_t + 2\|v_t\|_2^2 = -\beta \|u\|_{H_0^1}^2 \left(\|v\|_{H_0^1}^2\right)_t$$

and integrate this equation with respect to t to get

$$\begin{aligned}
&\|v_t\|_2^2 + \|v\|_X^2 + \gamma \|v\|_{H_0^1}^2 + 2\int_0^t \|v_t\|_2^2 \, ds \\
\le\ & M\|\phi_0\|_D^2 + M\|\phi_0\|_D^4 + \beta \int_0^t \left(\|u\|_{H_0^1}^2\right)_t \|v\|_{H_0^1}^2 \, ds \\
\le\ & M + M\int_0^t \|u\|_X \|u_t\|_2 \|\psi\|_D^2 \, ds \\
\le\ & M + M\|\phi_0\|_D^2 \int_0^t e^{-2K_3 s} \, ds \\
\le\ & M
\end{aligned}$$

by (9). □

Next, let w be a solution of

$$(16) \qquad \begin{cases} w_{tt} + w_t + Aw = \beta\left(\int_\Omega |\nabla u|^2 \, dx\right)\Delta w + \frac{\lambda}{(1-u)^\sigma} I(\sigma, \chi, u) & x \in \Omega,\ t > 0, \\ w = \Delta w = 0 & x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_t(x,0) = 0 & x \in \Omega. \end{cases}$$

We set $\xi = \begin{pmatrix} w \\ w_t \end{pmatrix}$ and $S_1(t)\phi_0 = \xi(t)$. Then we have

$$\phi(t) = \xi(t) + \psi(t) \quad \text{and} \quad S(t) = S_1(t) + S_2(t).$$

Lemma 9 (16) *possesses a unique solution*

$$\xi \in C([0,\infty); E) \cap C^1([0,\infty); D) \cap C^2([0,\infty); H).$$

Proof. First of all, we note that

$$\|w\|_X = \|u - v\|_X \le \|u\|_X + \|v\|_X \le M$$

by (9) and Lemma 8 and that

$$\int_0^\infty \|w_t\|_2^2 \, ds = \int_0^\infty \|u_t - v_t\|_2^2 \, ds \le 2\int_0^\infty \|u_t\|_2^2 \, ds + 2\int_0^\infty \|v_t\|_2^2 \, ds \le M$$

by (7) and Lemma 8. In the same computations as (12) and (13) in the proof of Theorem 3, we have

$$\frac{d}{dt}\int_\Omega w_{tt}^2\,dx = -2\int_\Omega w_{tt}^2\,dx - \frac{d}{dt}\int_\Omega (\Delta w_t)^2\,dx + 2I_3 + 2I_4,$$

where I_3 and I_4 are defined and computed similarly as follows:

$$\begin{aligned} I_3 &\equiv \int_\Omega w_{tt}\Big(G(\beta,\gamma,\nabla u)\,\Delta w\Big)_t\,dx \\ &\le \frac{1}{4}\int_\Omega w_{tt}^2\,dx + M\int_\Omega u_t^2\,dx - \frac{1}{2}G(\beta,\gamma,\nabla u)\frac{d}{dt}\int_\Omega |\nabla w_t|^2\,dx \end{aligned}$$

and

$$I_4 \equiv \int_\Omega w_{tt}\left(\frac{\lambda}{(1-u)^\sigma}I(\sigma,\chi,u)\right)_t\,dx \le \frac{1}{4}\int_\Omega w_{tt}^2\,dx + M\int_\Omega u_t^2\,dx.$$

Hence we obtain

$$\begin{aligned} &\int_\Omega w_{tt}^2\,dx + \frac{d}{dt}\int_\Omega \left(w_{tt}^2 + (\Delta w_t)^2\right)dx + G(\beta,\gamma,\nabla u)\frac{d}{dt}\int_\Omega |\nabla w_t|^2\,dx \\ &\le\; M\int_\Omega u_t^2\,dx. \end{aligned}$$

We integrate this inequality to derive

$$\begin{aligned} &\int_0^t\int_\Omega w_{tt}^2\,dx\,ds + \int_\Omega\left(w_{tt}^2 + (\Delta w_t)^2\right)dx + G(\beta,\gamma,\nabla u)\int_\Omega |\nabla w_t|^2\,dx \\ &\le\; M + 2\beta\int_0^t\left(\int_\Omega \Delta u u_t\,dx\right)\left(\int_\Omega \Delta w_t w_t\,dx\right)ds \\ &\le\; M + M\int_0^t\left(\int_\Omega w_t^2\,dx\right)\left(\int_\Omega (\Delta w_t)^2\,dx\right)ds \end{aligned}$$

and apply the Gronwall inequality to obtain

$$\int_\Omega (\Delta w_t)^2\,dx \le M\exp\left(M\int_0^t\int_\Omega w_t^2\,dx\,ds\right) \le M,$$

which yields $w \in Y$, $w_t \in X$ and $w_{tt} \in L^2(\Omega)$. □

Proof of Theorem 4. Since λ and $\mathcal{E}_0$ are restricted to the hypotheses in Theorem 2 and (2), it is obvious that $S(t)$ has an absorbing set owing to (10) and

$$\|\phi\|_D^2 \le 2E(\phi(t)) \le 2E_0.$$

S_2 has a decaying property from Lemma 7. Lemma 9 implies that $\|\xi(t)\|_E$ is bounded for all $t > 0$. The inclusion $E \subset D$ is compactly embedded. Hence S_1 is uniformly compact. Thus we apply Theorem 7 to $S(t)$ to complete the proof. □

Proof of Theorem 5. Following the argument of Lemma 7.6.2 in [15] and originally [27] along with the proof of Theorem 4, we can prove that the orbit $\cup_{t\ge 0}\phi(t)$ is relatively compact in Z_{δ_0}. Hence, the omega limit set $\omega(\phi_0)$ is invariant, non-empty, compact and connected in Z_{δ_0} by Theorem 5.1.8 in [15]. Moreover by Theorem 7.6.1 in [15] together with both the existence of Lyapunov function $E(\phi)$ and the precompactness of the orbit, we have

$$\lim_{t\to+\infty} \|u_t(\,\cdot\,,t)\|_2 = 0.$$

Finally we reach

$$\omega(\phi_0) = \left\{(\eta,0) \mid \text{there exist } \eta\in\mathcal{S}^{\lambda}_{\beta,\gamma,\chi,\sigma} \text{ and } t_n\to\infty \text{ such that (14) holds}\right\}.$$

□

6 Omega limit set We discuss the convergence of the global solution $u(\,\cdot\,,t)$ to the stationary solution η in the norm of X. We conclude that the omega limit set is composed of a stationary solution η in Z_{δ_0} with (3). For $z=u-\eta$, the method in [14, 15] is valid by existence of Lyapunov function and the precompactness of the orbit in X as proven in Section 5. In the limiting case $\chi=0$ and $\beta=0$, the same conclusion is obtained in [13].

Proof of Theorem 6. In this proof, by N_i for $i\in\mathbb{N}$, we denote the positive constant which depends only on the constants λ, β, γ, χ, σ, κ, Ω and $\|\eta\|_X$. Changing variable $z=u-\eta$, we consider

$$\begin{cases} z_{tt}+z_t+Az = f(\beta,z,\eta)+g(\lambda,\sigma,\chi,z,\eta) & x\in\Omega,\ t>0,\\ z=\Delta z=0 & x\in\partial\Omega,\ t>0,\\ z(x,0)=u_0(x)-\eta(x) & x\in\Omega,\\ z_t(x,0)=u_1(x) & x\in\Omega \end{cases}$$

and obtain

$$\lim_{n\to+\infty}\Big(\|z(\,\cdot\,,t_n)\|_X+\|z_t(\,\cdot\,,t_n)\|_2\Big)=0$$

instead of (1) and (14), where

$$f(\beta,z,\eta)=\beta\left(\int_\Omega|\nabla(z+\eta)|^2\,dx\right)\Delta(z+\eta)-\beta\left(\int_\Omega|\nabla\eta|^2\,dx\right)\Delta\eta$$

and

$$g(\lambda,\sigma,\chi,z,\eta)=\frac{\lambda}{(1-(z+\eta))^\sigma}I(\sigma,\chi,z+\eta)-\frac{\lambda}{(1-\eta)^\sigma}I(\sigma,\chi,\eta),$$

respectively. For the sake of simplicity, we shall write

$$f=f(z,\eta)=f(\beta,z,\eta) \quad\text{and}\quad g=g(z,\eta)=g(\lambda,\sigma,\chi,z,\eta).$$

Defining

$$\begin{aligned} F(z) &= \frac12\int_\Omega\Big((\Delta z)^2+\gamma|\nabla z|^2\Big)\,dx+\frac{\beta}{4}\left(\int_\Omega|\nabla(z+\eta)|^2\,dx\right)^2\\ &\quad-\frac{\beta}{4}\left(\int_\Omega|\nabla\eta|^2\,dx\right)^2+\beta\left(\int_\Omega|\nabla\eta|^2\,dx\right)\left(\int_\Omega z\Delta\eta\,dx\right)\\ &\quad+\lambda\int_\Omega\frac{z}{(1-\eta)^\sigma}\,dx\,I(\sigma,\chi,\eta)\\ &\quad+\frac{\lambda}{(\sigma-1)^2\chi}\Big(H(\sigma,\chi,z+\eta)^{1-\sigma}-H(\sigma,\chi,\eta)^{1-\sigma}\Big) \end{aligned}$$

and

$$G(t)=\frac{1}{2}\int_{\Omega} z_t^2\,dx+F(z(t))+\varepsilon\Big(Az-f\left(z,\eta\right)-g\left(z,\eta\right),z_t\Big)_{H^{-2}},$$

where $\varepsilon>0$ is a small constant to be determined later, we have

$$\frac{d}{dt}F(z(t))=-\frac{1}{2}\frac{d}{dt}\int_{\Omega} z_t^2\,dx-\int_{\Omega} z_t^2\,dx$$

and

$$\begin{aligned}G'(t)=&-\int_{\Omega} z_t^2\,dx-\varepsilon\Big(Az-f\left(z,\eta\right)-g\left(z,\eta\right),z_t\Big)_{H^{-2}}\\&-\varepsilon\left\|Az-f\left(z,\eta\right)-g\left(z,\eta\right)\right\|_{H^{-2}}^2\\&+\varepsilon\Big(Az_t-f_z\left(z,\eta\right)z_t-g_z\left(z,\eta\right)z_t,z_t\Big)_{H^{-2}},\end{aligned}$$

where f_z and g_z are linearized operators from $L^2(\Omega)$ to $H^{-2}(\Omega)$ given by

$$f_z\left(z,\eta\right)w=2\beta\left(\int_{\Omega}\nabla(z+\eta)\cdot\nabla w\,dx\right)\Delta(z+\eta)+\beta\left(\int_{\Omega}\left|\nabla(z+\eta)\right|^2\,dx\right)\Delta w$$

and

$$\begin{aligned}g_z\left(z,\eta\right)w=&\frac{\lambda\sigma w}{\left(1-(z+\eta)\right)^{\sigma+1}}I\left(\sigma,\chi,z+\eta\right)\\&-\frac{\lambda\sigma\left(\sigma-1\right)\chi}{\left(1-(z+\eta)\right)^{\sigma}}H\left(\sigma,\chi,z+\eta\right)^{-\sigma-1}\int_{\Omega}\frac{w}{\left(1-(z+\eta)\right)^{\sigma}}\,dx,\end{aligned}$$

respectively. Then the Young and Hölder inequalities yield

$$\begin{aligned}G'(t)\ \le\ &-\left\|z_t\right\|_2^2-\frac{\varepsilon}{2}\left\|Az-f-g\right\|_{H^{-2}}^2+\frac{\varepsilon}{2}\left\|z_t\right\|_{H^{-2}}^2\\&+\varepsilon\left\|z_t\right\|_2^2+\varepsilon\gamma\left\|z_t\right\|_2\left\|z_t\right\|_{H^{-2}}-\varepsilon\Big(f_zz_t+g_zz_t,z_t\Big)_{H^{-2}}\\\le\ &\left(\varepsilon N_1-1\right)\left\|z_t\right\|_2^2-\varepsilon\Big(f_zz_t+g_zz_t,z_t\Big)_{H^{-2}}-\frac{\varepsilon}{2}\left\|Az-f-g\right\|_{H^{-2}}^2.\end{aligned}$$

Since we estimate the linearized operators as

$$\left|\Big(f_z\left(z,\eta\right)w,w\Big)_{H^{-2}}\right|\le\beta\Big(2\left\|z+\eta\right\|_X\left\|z+\eta\right\|_2+\left\|z+\eta\right\|_{H_0^1}^2\Big)\left\|w\right\|_2\left\|w\right\|_{H^{-2}}$$

and

$$\left|\Big(g_z\left(z,\eta\right)w,w\Big)_{H^{-2}}\right|\le\frac{N_2\left\|w\right\|_2^2}{\left(1-\left\|z+\eta\right\|_C\right)^{\sigma+1}}+\frac{N_2\left\|w\right\|_2^2}{\left(1-\left\|z+\eta\right\|_C\right)^{2\sigma}},$$

respectively, thanks to (8) and (9), we can take sufficiently small $\varepsilon>0$ so that the following estimate is valid:

$$\begin{aligned}G'(t)\ \le\ &\left(\varepsilon N_3-1\right)\left\|z_t\right\|_2^2-\frac{\varepsilon}{2}\left\|Az-f-g\right\|_{H^{-2}}^2\\\le\ &-2N_4\Big(\left\|z_t\right\|_2^2+\left\|Az-f-g\right\|_{H^{-2}}^2\Big)\\\le\ &-N_4\Big(\left\|z_t\right\|_2+\left\|Az-f-g\right\|_{H^{-2}}\Big)^2\end{aligned}\tag{17}$$

for $t\ge0$. Hence since $G(t)$ is non-increasing in $t\ge0$ and $(0,0)\in\omega(u_0-\eta,u_1)$, we have $G(t)\to0$ as $t\to+\infty$ and $G(t)>0$ for $t\ge0$. As in [13, 14], we can prove the following type of Lojasiewicz-Simon inequality:

Lemma 10 (Theorems 2.2 in [14] and 11.2.7 in [15]) *There exist $\theta \in (0, \frac{1}{2})$ and $\rho > 0$ such that for all $z \in X$ with $\|z\|_X < \rho$, we have*

$$|F(t)|^{1-\theta} \le \|Az - f(z,\eta) - g(z,\eta)\|_{H^{-2}}.$$

This inequality is proven in the same argument as Theorems 2.2 in [14] and 11.2.7 in [15]. For the sake of completeness, we give a sketch of a proof in an appendix. The proof of Theorem 6 is also similar to that of Theorem 1.2 in [14]. For all $t \ge 0$, we have

$$\begin{aligned} -\frac{d}{dt}(G(t))^\theta &= -\theta (G(t))^{\theta-1} G'(t) \\ &\ge \theta N_4 (G(t))^{\theta-1} \Big(\|z_t\|_2 + \|Az - f - g\|_{H^{-2}} \Big)^2 \end{aligned} \tag{18}$$

by (17). Now that $\lim_{t\to+\infty} \|z_t\|_2 = 0$ holds, there exists sufficiently large $T > 0$ such that we may suppose that $\|z_t\|_2 \le 1$ as long as $t \ge T$. Noting that $1/2 < 1-\theta < 1$ and that $1 < (1-\theta)/\theta$ for $\theta \in (0, 1/2)$, we have

$$\begin{aligned} (G(t))^{1-\theta} &\le \frac{1}{2^{1-\theta}} \|z_t\|_2^{2(1-\theta)} + |F(t)|^{1-\theta} + \varepsilon^{1-\theta} \|Az - f - g\|_{H^{-2}}^{1-\theta} \|z_t\|_{H^{-2}}^{1-\theta} \\ &\le \frac{1}{2^{1-\theta}} \|z_t\|_2^{2(1-\theta)} + |F(t)|^{1-\theta} \\ &\quad + \varepsilon^{1-\theta}(1-\theta) \|Az - f - g\|_{H^{-2}} + \varepsilon^{1-\theta}\theta \|z_t\|_{H^{-2}}^{\frac{1-\theta}{\theta}} \\ &\le N_5 \Big(\|z_t\|_2 + |F(t)|^{1-\theta} + \|Az - f - g\|_{H^{-2}} \Big) \end{aligned} \tag{19}$$

for $t \ge T$. For any $0 < \xi < \rho$, there exists $N \in \mathbb{N}$ such that $t_n > T$ satisfying

$$\|z(\cdot, t_n)\|_2 < \frac{1}{2}\xi, \quad (G(t_n))^\theta < \frac{\theta N_4}{4N_5}\xi \quad \text{and} \quad \|z(\cdot, t_N)\|_X < \frac{1}{2}\xi$$

for all $n \ge N$. Let

$$\bar{t} = \sup\{t \ge t_N \mid \|z(\cdot, s)\|_X < \rho \text{ for all } s \in [t_N, t]\}.$$

Hence for all $t \in [t_N, \bar{t}]$, (18) becomes

$$-\frac{d}{dt}(G(t))^\theta \ \ge\ \frac{\theta N_4}{2N_5}\Big(\|z_t\|_2 + \|Az - f - g\|_{H^{-2}} \Big) \ge \frac{\theta N_4}{2N_5} \|z_t\|_2$$

owing to

$$(G(t))^{1-\theta} \le 2N_5 \Big(\|z_t\|_2 + \|Az - f - g\|_{H^{-2}} \Big)$$

by (19) together with Lemma 10. By integrating this inequality over $[t_N, \bar{t}]$, we obtain

$$\int_{t_N}^{\bar{t}} \|z_t(\cdot, s)\|_2 \, ds \le \frac{2N_5}{\theta N_4}(G(t_N))^\theta < \frac{1}{2}\xi.$$

Claim 1

$$\bar{t} = +\infty$$

holds.

Proof. If $\bar{t} < +\infty$, we have

$$\begin{aligned} \left\|z(\,\cdot\,,\bar{t})\right\|_2 &= \int_{t_N}^{\bar{t}} \left(\frac{d}{dt} \|z(\,\cdot\,,s)\|_2 \right) ds + \|z(\,\cdot\,,t_N)\|_2 \\ &\leq \int_{t_N}^{\bar{t}} \left(\|z(\,\cdot\,,s)\|_2^{-1} \int_\Omega |z(x,s)|\,|z_t(x,s)|\ dx \right) ds + \|z(\,\cdot\,,t_N)\|_2 \\ &\leq \int_{t_N}^{\bar{t}} \|z_t(\,\cdot\,,s)\|_2\ ds + \|z(\,\cdot\,,t_N)\|_2 \\ &< \xi. \end{aligned}$$

From the compactness of $z(t)$ in X, we can choose $\xi > 0$ sufficiently small to obtain $\left\|z(\,\cdot\,,\bar{t})\right\|_X < \rho$, which contradicts the definition of $\bar{t}$. □

Since the claim is shown,

$$\lim_{t\to+\infty} \|z(\,\cdot\,,t)\|_2 \leq \int_{t_N}^{+\infty} \|z_t(\cdot\ ,s)\|_2\ ds + \|z(\,\cdot\,,t_N)\|_2 < \xi,$$

which implies the convergence of z in X. □

A Proof of Lemma 10 In this section, we sketch the proof of the Lojasiewicz-Simon inequality. If we establish Lemma 11, we can follow the argument in [14, 15]. As in our problem, the Lojasiewicz-Simon inequality can be applicable to the convergence problem in infinite dimensions We remark that the lemma is also proven by [5, 6] and applied to [9].

Sketch of the proof of Lemma 10. Let

$$\mathcal{M}z = Az - f\,(z,\eta) - g\,(z,\eta)\,.$$

We prepare an orthogonal projection. We denote the i-th eigenpair of A by (μ_i, φ_i), where $\{\varphi_i\}_{i\in\mathbb{N}}$ is a set of orthonormal eigenfunctions in $L^2(\Omega)$. We define by W_k the vector space spanned by $\varphi_1, \varphi_2, \ldots, \varphi_k$. Let

$$Q_k : L^2(\Omega) \to W_k$$

be the orthogonal projection onto W_k. For all $k \in \mathbb{N}$, we have

$$\begin{aligned} (Az + \mu_k Q_k z, z) &= \frac{1}{2}\,(Az,z) + \frac{1}{2}\,(Az,z) + \mu_k\,(Q_k z, z) \\ &\geq \frac{1}{2}\,\|z\|_X^2 + \frac{\gamma}{2}\,\|z\|_{H_0^1}^2 + \frac{\mu_k}{2}\,\|z - Q_k z\|_2^2 + \mu_k\,\|Q_k z\|_2^2 \\ &\geq \frac{1}{2}\,\|z\|_X^2 + \frac{\gamma}{2}\,\|z\|_{H_0^1}^2 + \frac{\mu_k}{2}\left(\|z - Q_k z\|_2^2 + \|Q_k z\|_2^2\right) \\ &\geq \frac{1}{2}\,\|z\|_X^2 + \frac{\gamma}{2}\,\|z\|_{H_0^1}^2 + \frac{\mu_k}{4}\,\|z\|_2^2\,. \end{aligned}$$

Putting

$$\mathcal{L} = A - f_z\,(0,\eta) - g_z\,(0,\eta) + \mu_k Q_k,$$

we obtain

$$\begin{aligned}(\mathcal{L}z,z) &\geq \frac{1}{2}\|z\|_X^2+\frac{\gamma}{2}\|z\|_{H_0^1}^2+\frac{\mu_k}{4}\|z\|_2^2+\beta\|\eta\|_{H_0^1}^2\|z\|_{H_0^1}^2\\ &\quad +2\beta\left(\int_\Omega \nabla z\cdot\nabla\eta\,dx\right)^2-\lambda\sigma I(\sigma,\chi,\eta)\int_\Omega\frac{1}{(1-\eta)^{\sigma+1}}z^2\,dx\\ &\quad +\lambda\sigma(\sigma-1)\chi H(\sigma,\chi,\eta)^{-\sigma-1}\left(\int_\Omega\frac{z}{(1-\eta)^\sigma}\,dx\right)^2\\ &\geq \frac{1}{2}\|z\|_X^2+\left(\frac{\mu_k}{4}-\frac{\lambda\sigma}{(1-\|\eta\|_C)^{\sigma+1}}\right)\|z\|_2^2.\end{aligned}$$

Now we take k so large that the inequality

$$\mu_k>\frac{4\lambda\sigma}{(1-\|\eta\|_C)^{\sigma+1}}$$

is satisfied. Hence $\mathcal{L}$ is coercive and bijective from X to $H^{-2}(\Omega)$. Let

$$\mathcal{N}=\mu_kQ_k+\mathcal{M}.$$

$\mathcal{N}$ is a C^1 diffeomorphism in the neighbourhood of $0\in X$ to $H^{-2}(\Omega)$ and its derivative at 0 is $\mathcal{L}$. The inversion theorem implies the following lemma:

Lemma 11 (Lemmas 2.6 in [14] and 11.2.8 in [15]) *There exist a neighbourhood V_1 of 0 in X, V_2 of 0 in $H^{-2}(\Omega)$, $C_1>0$ and $C_2>0$ such that*

$$\|\mathcal{M}z-\mathcal{M}w\|_{H^{-2}}\leq C_1\|z-w\|_X$$

for all z, $w\in V_1$ and

$$\left\|\mathcal{N}^{-1}(f)-\mathcal{N}^{-1}(g)\right\|_X\leq C_2\|f-g\|_{H^{-2}}$$

for all f, $g\in V_2$.

Owing to Lemma 11, the same estimates as in [14, 15] hold, which yields the conclusion. □

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Communicated by *Atsushi Yagi*

Tosiya Miyasita
250-201 Imamichi-cho, Kyoto 605-0042,
Japan
sk109685@mail.doshisha.ac.jp