# ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO RACETRACK MODEL IN SPATIAL ECONOMY

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Abstract. We are concerned with the racetrack model which has been presented by Krugman et al.[9] in the study of spatial economics. They intended to describe the movement of labors and goods along a racetrack where dense economic regions are continuously distributed and to investigate geographical structures provoked automatically by the superposition of specific economic laws. Tabata et al.[26, 22, 23] have already studied similar but more general models mainly from the view point of mathematical analysis by constructing the global solutions, although they did not handle the very racetrack model.

The objective of the present paper is to explore asymptotic behavior of the global solutions to the racetrack model. After constructing global solutions, we shall show that, as $t \to \infty$, every solution tends to a stationary solution of the model. We shall then demonstrate in somewhat heuristic way that the stationary solution is given by either a homogeneous solution on the racetrack or a concentrated solution in which distribution of the manufacturing is a sum of Dirac delta functions. As the homogeneous stationary solution is proved to be always unstable, the asymptotic limit of the global solution is in general a concentrated solution, thereby there remains only a finite number of regions on the racetrack that possess the manufacturing sector.

These results may suggest that in the racetrack model the spatial economic state tends to a discretely concentrated state. According to numerical results, the number and the location of the spikes of concentration seem to have a certain freedom, evidently depending on the initial state. The maximum number of spikes is, however, clearly controlled by the transportation cost and the preference for variety of goods.

## 1 Introduction.

We are concerned with the racetrack model that has been presented by Fujita, Krugman and Venables [9] in 1999 for the study of spatial economy. The model equations are given by

$$
(1.1)\quad \begin{cases}
w(t,x) = \left[\displaystyle\int_S \{\mu\lambda(t,y)w(y,t) + (1-\mu)\phi(y)\}G(t,y)^{\sigma-1}e^{-(\sigma-1)\tau|x-y|}dy\right]^{\frac{1}{\sigma}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (t,x)\in[0,\infty)\times S, \\
G(t,x) = \left[\displaystyle\int_S \lambda(t,y)w(t,y)^{1-\sigma}e^{-(\sigma-1)\tau|x-y|}dy\right]^{\frac{1}{1-\sigma}} \quad \text{for } (t,x)\in[0,\infty)\times S, \\
\omega(t,x) = w(t,x)G(t,x)^{-\mu} \quad \text{for } (t,x)\in[0,\infty)\times S, \\
\dfrac{\partial\lambda}{\partial t}(t,x) = \gamma\left[\omega(t,x) - \displaystyle\int_S \omega(t,y)\lambda(t,y)dy\right]\lambda(t,x) \quad \text{for } (t,x)\in[0,\infty)\times S, \\
\lambda(0,x) = \lambda_0(x) \quad \text{for } x\in S.
\end{cases}
$$

Here, $S$ is a circumference on which economic regions exist continuously and $x$ is a spatial variable varying on $S$. The functions $w(t,x)$, $G(t,x)$ are unknown functions denoting, respectively, manufacturing wage, price index at time $t\in[0,\infty)$ and at a region $x\in S$. The function $\lambda(t,x)$ is an unknown function such that $\mu\lambda(t,x)$ denotes population density of manufacturing workers at time $t\in[0,\infty)$ at $x\in S$. The function $\phi(x)$ is a given function such that $(1-\mu)\phi(x)$ denotes population density of agricultural workers on $S$. The function $|x-y|$ denotes a symmetric distance between $x$, $y \in S$ along the circumference $S$. The exponent $0 < \mu \le 1$ denotes a ratio of the manufacturing workers on $S$ to the total number of (manufacturing and agricultural) workers. Meanwhile $\sigma > 1$ stands for an index of preference for manufacturing goods, and $\tau > 0$ stands for a parameter concerning the transportation cost. Finally $\gamma > 0$ is some constant.

When $\lambda(t,x)$ is given at any time $t$, $w(t,x)$ and $G(t,x)$ at that moment are automatically determined as instant economic equilibria through the nonlinear integral transformations on $S$ given by the first and second equations. The third equation shows that real wage $\omega(t,x)$ is given by a ratio of $w(t,x)$ divided by $G(t,x)^\mu$. The fourth one is an evolution equation for $\lambda(t,x)$ that indicates that $\lambda(t,x)$ increases (or declines) at a rate $\gamma[\omega(t,x) - \int_S \omega(t,y)\lambda(t,y)dy]$. The initial density of $\lambda$ is given by $\lambda_0(x)$.

In their study of spatial economy, Fujita, Krugman and Venables are interested in understanding theoretically the economic activities that are performed by homogeneous regions laid in the two-dimensional space. They study how the various economic factors (including general equilibrations, preference for manufacturing goods, transportation costs, mobility of manufacturing workers and increasing returns) create the spatial inhomogeneity spontaneously. One can then find a similar mechanism of self-organization that certain structures of regions emerge through simple interactions acting among homogeneous constituents that was first penetrated by Alan Turing [32] in the study of cell differentiation. Starting with a two-region model (see [12] and confer also [31, 29]), they extend their models to a three-region model or a multi-region model which are all, however, 0-dimensional models. As a one-dimensional model, they consider an economy in which homogeneous regions are laid continuously on a circumference and call it the racetrack model.

The objective of the present paper is to develop mathematical researches for (1.1). In the first part, we shall formulate the equations of (1.1) as a system of two integral equations in the space $\mathcal{C}(S)$ of continuous functions on $S$ for $w(t,x)$ and $G(t,x)$ and an evolution equation in the space $L^1(S)$ of $L^1$-functions on $S$ for $\lambda(t,x)$, and then shall construct a global solution. As generally known, integral equations have uniqueness of solutions only in special cases. In the present case, too, we can guarantee existence and uniqueness of the global solution only under somewhat restrictive conditions for the exponents. In this part, we shall use the similar techniques devised by Tabata and Eshima and their collaborators [26, 22, 23] for studying the analogous models to (1.1) but in two or multi-dimensional domains. The second part of paper will be devoted to studying longtime behavior of the global solution $w(t,x)$, $G(t,x)$ and $\lambda(t,x)$. Firstly, they are proved to have $\omega$-limits, namely, there exists an increasing time sequence $t_n \to \infty$ such that $\lambda(t_n,x)$ are convergent to some limit functions $\overline{\lambda}(x)$ in a weak* topology. Therefore, $\overline{\lambda}(x)$ may lie no longer in $L^1(S)$ but can be an element of the space $\mathcal{M}(S)$ of all measures on $S$. Naturally $\overline{\lambda}(x)$ are expected to be a stationary solution of the equations of (1.1). Secondly, when $\mu = 1$, we can claim that, if $\big(\lambda(x), w(x), G(x)\big)$ is a stationary solution to (1.1) such that $\lambda(x) \in \mathcal{M}(S)$ and $w(x)$, $G(x) \in \mathcal{C}(S)$, then either $\lambda(x)$ is constant on $S$ or $\lambda(x)$ is a sum of Dirac delta functions on $S$. Moreover, it is proved that any constant stationary solution $\overline{w}$, $\overline{G}$ and $\overline{\lambda}$ is unstable. These analytical results ultimately suggest that the global solution to (1.1) converges as $t \to \infty$ to a stationary solution $\overline{w}(x)$, $\overline{G}(x)$ and $\overline{\lambda}(x)$ in which $\overline{\lambda}(x) = \sum_{k=1}^K \alpha_k \delta_{x_k}(x)$ is a finite sum of Dirac measures $\delta_{x_k}(x) \in \mathcal{M}(S)$ with suitable centers $x_k \in S$ and weights $\alpha_k > 0$ and $\overline{w}(x)$ and $\overline{G}(x)$ are functions in $\mathcal{C}(S)$ satisfying the integral equations of (1.1). All the numerical examples support this fact. The number $K$ and the location $x_k$ of regions at which the manufacturing workers concentrate depend on the initial function $\lambda_0(x)$. In the meantime, the maximum number of concentrated regions on $S$ depends on the exponents $\sigma$ and $\tau$.

As for 0-dimensional models, namely, discrete space models, there are already several published papers. We here quote Castoro-Correia da Silva-Mossay [2], Combes-Mayer-Thisse [3], Currie-Kubin [4], Fujita-Thisse [8], Forslid-Ottaviano [7], Ikeda-Akamatsu-Kono [10], Ioan-Ioan [11], Lange-Quaas [14], Lanaspa-Sanz [13], Leite-Castro-Correia-da-Silva [15], Maffezzoli-Trionfetti [16], Mossay [18], Robert-Nicoud [20], Sidorov-Zhelobodko [21], Tabata-Eshima [24], Tabata-Eshima-Kiyonari-Takagi [25], Tabata-Eshima-Sakai [27], and Tabata-Eshima-Sakai [28]. Tabuchi-Thisse [30] considered an extended model which consists of 0-dimensional manufacturing and one-dimensional agriculture. As for one-dimensional model, only few papers were published. Picard-Tabuchi [19], Fabinger [6] treated some related models to (1.1), together with 0-dimensional ones. Recently Tabata, Eshima and their collaborators published a series of papers [26, 22, 23] concerning two or multi-dimensional models, where regions are laid in a bounded domain of two or multi-dimensional space. In [26] they studied the equilibrating problem for $w(x)$ and $G(x)$ of (1.1) under the situation that some $\lambda(x)$ is known. In [22, 23] they considered the evolutional problem in the case that $\mu = 1$ and $0 < \mu < 1$, respectively, and constructed a global solution under suitable conditions for the exponents. In [22] they succeeded also in showing longtime convergence of the solution that, for some initial function $\lambda_0(x)$, $\lambda(t, x)$ tends to a Dirac delta function.

This paper is organized as follows. In Section 2, the modeling of the racetrack model will be reviewed. The model is formulated as an initial value problem for an abstract evolution equation in Section 3, and existence and uniqueness of global solutions is proved in Section 4. Section 5 is devoted to studying asymptotic behavior of the solutions. First, a dynamical system is constructed, and existence of the weak* $\omega$-limit set is proved. Next, through heuristic discussions, we will explore details about weak* $\omega$-limit. Stability of homogeneous stationary solutions is also discussed here. Section 6 is devoted to showing numerical results.

## 2 Review of Modeling.

In this section, we want to review the modeling of (1.1) according to M. Fujita, P. Krugman, and A. Venables [9, Chapter 4].

### 2.1 Settings.

Let us first list all the settings of model.

1. Economic regions are distributed on a whole circumference $S$.

2. The economy is composed of two sectors, i.e., manufacturing and agriculture. The manufacturing sector is imperfectly competitive and produces a large variety of differentiated goods. The agricultural sector is perfectly competitive and produces a single, homogeneous variety of goods.

3. Manufacturing workers and agricultural workers are the only factor of production for each sector. Manufacturing workers offer their labor to manufacturing firms and earn a nominal wage in return at each position. Manufacturing workers and manufacturing firms can move along $S$ freely. Agricultural workers engage in production of agricultural goods and earn a wage in return, but they settle down in their position and cannot move. The total number of workers is fixed.

4. Workers consume all goods produced in the economy as consumers. Goods produced at a position are consumed in the same position, or they are transported along $S$ and consumed in any other different regions.

**2.2 Assumptions.** Under these settings, the following conditions are assumed.

1. Each manufacturing firm chooses its price to maximize the profit taking the nominal wages and the price indices on $S$ to be given.

2. Technology of manufacturing firms is assumed to be increasing return, i.e., the more goods are produced, the less becomes an average cost of goods per unit. The technology is the same for all varieties and at all regions. That is, firm's only input is the labor, and the required labor input is uniquely determined in accordance with the production quantity of a variety, regardless of varieties and regions.

3. The maximized profit is 0 for all manufacturing firms.

4. The number of varieties of goods is innumerably infinite.

5. Each variety of manufactured goods is produced only in one region by a single manufacturing firm. Therefore, the number of manufacturing firms is the same as the number of available varieties.

6. The goods which are produced at a region have a uniform price at any region. The price of goods produced at $x \in S$ and sold at $y \in S$ is denoted by $p(x, y)$, with abbreviation $p(x, x) = p(x)$.

7. The price of agricultural goods is assumed to be $p(x, y) \equiv 1$ for all $x, y \in S$.

8. The total number of the manufacturing workers is fixed and is denoted by $\mu$, where $\mu \in (0, 1)$. A function $\lambda(x)$ is defined so that $\mu\lambda(x)$ denotes density of manufacturing workers at $x \in S$, i.e., $\int_S \lambda(x)dx = 1$. All manufacturing workers at $x \in S$ earn a uniform nominal wage $w(x)$.

9. Manufacturing workers move toward regions that offer higher real wages away from those of lower real wages. The real wage is defined in a usual manner.

10. The total number of the agricultural workers is $1 - \mu$. A function $\phi(x)$ is defined so that $(1 - \mu)\phi(x)$ denotes density of agricultural workers at $x \in S$, i.e., $\int_S \phi(x)dx = 1$. Their wage is assumed to be fixed to 1 at every region.

11. Transportation cost is incurred by transporting of manufactured goods between regions. The transportation cost is assumed to be given by the "iceberg form". That is, to transport one unit of goods from $x$ to $y$ along $S$, $T(x, y) \geq 1$ times units of the goods must be shipped. Thereby, $p(x, y) = T(x, y)p(x)$. Furthermore, we assume that $T(x, y) = e^{\tau|x-y|}$, where $\tau$ is a positive exponent and $|x - y|$ is a shorter distance along $S$ between $x$ and $y$. (By contrast, transportation of agricultural goods is costless, remember (7).)

12. Consumer's satisfaction obtained from consumption of goods is expressed by a utility function. Consumers intend to buy manufactured goods and agricultural goods to maximize their utility function. All consumers are assumed to have a unified utility function. They have a common preference for varieties of goods which is represented through an index $\sigma > 1$. As $\sigma$ decreases, the desire to consume a larger variety of manufactured goods increases.

13. For goods and labor, the demand and the supply are always balancing. More precisely, their demand and supply meet instantly in an equilibrium much faster than workers move on $S$.

**2.3 Mathematical Modeling.** Under these settings and assumptions, let us survey modeling of (1.1). The modeling is composed of three steps. First, each consumption and production activity is described without explicitly referring to the interactions between regions. Next, interactions among regions are taken into account. Finally, dynamics of the change of $\lambda(x)$ is formulated.

*2.3.1 Local equilibration.* Let us focus our attention on an arbitrarily fixed region, and consider economic activities at this region. From Assumption 4, let the varieties of manufactured goods be parameterized by $i \in [0, n]$, where $n$ is a fixed number. Let $m(i)$ be the consumption of $i$-th variety, and $p(i)$ be the price of $i$-th variety.

Consider the consumer's activity. Their utility functions mentioned in Assumption 12 is assumed to be uniformly given by

$$U = \left\{ \int_0^n m(i)^{\frac{\sigma-1}{\sigma}} di \right\}^{\frac{\sigma\mu}{\sigma-1}} A^{1-\mu}, \tag{2.1}$$

where $A$ stands for the consumption of agricultural goods. Here, as mentioned in Assumption 12 $\sigma$ stands for their uniform preference to varieties. The consumer maximizes (2.1) subject to the budget constraint

$$p^A A + \int_0^n p(i)m(i)di = Y,$$

where $p^A$ stands for the price of agricultural goods and $Y$ is income of the consumer. Solving this maximization problem for $U$ with given $p^A$ and $p(i)$, we have

$$\begin{aligned} A &= \frac{(1-\mu)Y}{p^A}, \\ m(j) &= \mu Y p(j)^{-\sigma} G^{\sigma-1}. \end{aligned} \tag{2.2}$$

Here, $G$ has been defined by

$$G = \left\{ \int_0^n p(i)^{1-\sigma} di \right\}^{\frac{1}{1-\sigma}}, \tag{2.3}$$

and is called the price index at this region.

On the other hand, consider the manufacturing firms. According to Assumption 2, their technology is uniformly described by

$$l^M = F + c^M Q, \tag{2.4}$$

where $l^M$ is labor input required to produce a quantity $Q$ of a variety. Here, $F > 0$ stands for a fixed cost and $c^M > 0$ stands for a marginal cost at the position.

*2.3.2 Global equilibration.* We now focus our attention on interactions among regions. We consider activities that take place among regions.

Let the number of varieties produced at $x$ be denoted by $n(x)$. By applying Assumptions 6 and 11 to (2.3), the price index at $x$ is written as

$$G(x) = \left[ \int_S n(y) \left( p(y) e^{\tau|x-y|} \right)^{1-\sigma} dy \right]^{\frac{1}{1-\sigma}}, \tag{2.5}$$

where the integration is carried out along $S$. Let $Y(y)$ denote income at region $y$. By (2.2), consumption demand at $y$ for a product manufactured at $x$ is given by

$$\mu Y(y)\left(p(x)e^{\tau|x-y|}\right)^{-\sigma}G(y)^{\sigma-1}, \tag{2.6}$$

where $|x-y|$ is a distance along $S$. By Assumptions 11 and 13, the total sales $Q(x)$ of a product manufactured at $x$ must equal

$$Q(x)=\mu\int_S Y(y)\left(p(x)e^{\tau|x-y|}\right)^{-\sigma}G(y)^{\sigma-1}e^{\tau|x-y|}dy. \tag{2.7}$$

Consider next manufacturing firms at $x$. From (2.4) and (2.7), the firm's profit $\Pi(x)$ must be

$$\Pi(x)=p(x)Q(x)-w(x)\left(F+c^MQ(x)\right). \tag{2.8}$$

Substitute $Q(x)$ in (2.7) into (2.8), and maximize $\Pi(x)$ under the conditions that $w(x)$ and $G(x)$ are known on $S$ (as explained in Assumption 1). Then, we see that

$$p(x)=\frac{c^M\sigma}{\sigma-1}w(x). \tag{2.9}$$

Consequently,

$$\Pi(x)=w(x)\left[\frac{Q(x)c^M}{\sigma-1}-F\right].$$

But this profit must be zero by Assumption 3. Thereby, we see that the output $Q(x)$ is equal to

$$Q(x)\equiv Q=\frac{F(\sigma-1)}{c^M}$$

and the input $l(x)$ is equal to

$$l(x)\equiv l=F+c^MQ=F\sigma.$$

Both $Q$ and $l$ are uniform to all manufacturing firms on $S$.

It then follows from (2.7) that

$$Q=\mu p(x)^{-\sigma}\int_S Y(y)e^{-(\sigma-1)\tau|x-y|}G(y)^{\sigma-1}dy.$$

Then, from (2.9),

$$w(x)=\frac{\sigma-1}{\sigma c^M}\left[\frac{\mu}{Q}\int_S Y(y)e^{-(\sigma-1)\tau|x-y|}G(y)^{\sigma-1}dy\right]^{\frac{1}{\sigma}}. \tag{2.10}$$

Let us here verify that $n(x)$ and $\lambda(x)$ are proportional. Indeed, Assumption 5 means that $n(x)$ is equal to the number of manufacturing firms at $x$. Therefore, as in Assumption 8, we have

$$n(x)=\frac{\mu\lambda(x)}{l}=\frac{\mu\lambda(x)}{F\sigma}. \tag{2.11}$$

As a consequence, substituting $p(x)$ in (2.9) and this $n(x)$ into (2.5), the price index is now described as

$$G(x) = \left[ \frac{\mu}{F\sigma} \left( \frac{\sigma c^M}{\sigma - 1} \right)^{1-\sigma} \int_S \lambda(y) w(y)^{1-\sigma} e^{-(\sigma-1)\tau|x-y|} dy \right]^{\frac{1}{1-\sigma}}. \tag{2.12}$$

According to [9], it is allowed to choose units of measurement in such a way that $c^M = (\sigma - 1)/\sigma$ and $F = \mu/\sigma$ are the case. Then, we verify that (2.12) becomes the second equation of (1.1). In addition, since

$$Y(x) = \mu\lambda(x)w(x) + (1 - \mu)\phi(x)$$

by Assumptions 8 and 10, we verify that (2.10) is nothing more than the first equation of (1.1).

*2.3.3 Evolution equation for $\lambda(x)$.* Manufacturing workers at $x$ migrate in response to differences between real wage of $x$ and those of other regions as assumed in Assumption 9. The real wage is defined as the nominal wage deflated by the "cost-of-living index", i.e., $G(x)^\mu (p^A)^{1-\mu}$. Since $p^A = 1$ (by Assumption 7), the real wage is given by the third equation of (1.1). As the averaged real wage is $\int_S \omega(x)\lambda(x)dx$, the fourth equation of (1.1) naturally expresses motion of manufacturing workers at $x$.

**3 Mathematical Formulation.** In this section, we will formulate the problem (1.1) as the initial value problem for an abstract evolution equation in a Banach space.

**3.1 Function Spaces.** Let us set up function spaces for $w$ and $\lambda$.

Let $\mathcal{C}(S)$ be the Banach space of all continuous functions on $S$ equipped with the norm $\|u\|_{\mathcal{C}} := \max_{x\in S} |u(x)|$ for $u \in \mathcal{C}(S)$. We consider a subset of $\mathcal{C}(S)$ given by

$$\mathcal{C}_+(S) := \{u \in \mathcal{C}(S) | u > 0 \text{ on } S\}.$$

It is reasonable to consider that the nominal wage is positive and is distributed on $S$ continuously. So we assume that $w$ belongs to $\mathcal{C}_+(S)$ at any time. Meanwhile, let $L^1(S)$ be the Banach space of all integrable functions on $S$ equipped with the norm $\|u\|_{L^1} := \int_S |u(x)|dx$ for $u \in L^1(S)$. We consider a subset of $L^1(S)$ given by

$$L^1_{\mathcal{M}}(S) := \left\{ u \in L^1(S) \middle| u \geq 0 \text{ on } S, \int_S u(x)dx = 1 \right\}.$$

As the population density is a nonnegative function on $S$ and is normalized to have a unit $L^1$ -norm, we consider that $\lambda$ belongs to $L^1_{\mathcal{M}}(S)$ at any time.

Actually, both the nominal wage and the population density are functions from the half real line $[0, \infty)$ with values in $\mathcal{C}_+(S)$ and $L^1_{\mathcal{M}}(S)$, respectively. Let $X$ denote one of $\mathcal{C}(S)$ and $L^1(S)$. Then, $\mathcal{C}([0, \infty); X)$ is the space of all continuous functions on $[0, \infty)$ with values in $X$, and $\mathcal{C}_b([0, \infty); X)$ is the space of all uniformly bounded continuous functions on $[0, \infty)$. We equip $\mathcal{C}_b([0, \infty); X)$ with a norm

$$\|f\|_{\mathcal{C}_b} := \sup_{t\in[0,\infty)} e^{-t} \|f(t)\|_X. \tag{3.1}$$

When $B$ is a bounded subset of $X$, the space of all continuous functions with values in $B$ is denoted similarly by $\mathcal{C}([0, \infty); B)$. Clearly, $\mathcal{C}([0, \infty); B)$ is a subset of $\mathcal{C}_b([0, \infty); X)$.

**3.2 Formulation.** Let $\phi \in L^1_{\mathcal{M}}(S)$ be given. We want to formulate equations (1.1) as equations in a Banach space. In the following, the constant $\gamma > 0$ of (1.1) is fixed to be 1 without loss of generality.

We first formulate the first equation in (1.1) as a fixed point problem in $\mathcal{C}(S)$. To do so, we introduce the operator $\Phi : \mathcal{C}_+(S) \times L^1_{\mathcal{M}}(S) \to \mathcal{C}_+(S)$ as

$$\Phi(w,\lambda)(x) = \left[\int_S \frac{\mu\lambda(y)w(y) + (1-\mu)\phi(y)}{\int_S \lambda(z)w(z)^{1-\sigma} e^{-(\sigma-1)\tau|y-z|}dz} e^{-(\sigma-1)\tau|x-y|}dy\right]^{\frac{1}{\sigma}}.$$

Then, the first equation of (1.1) is written as

$$w = \Phi(w,\lambda), \qquad w \in \mathcal{C}_+(S), \quad \lambda \in L^1_{\mathcal{M}}(S). \tag{3.2}$$

Second, we formulate the fourth equation in (1.1) as an ordinary differential equation in $L^1(S)$. To do so, we introduce the operators $G$, $\omega : \mathcal{C}_+(S) \times L^1_{\mathcal{M}}(S) \to \mathcal{C}_+(S)$ as

$$\begin{cases} G(w,\lambda)(x) = \left[\int_S \lambda(y)w(y)^{1-\sigma}e^{-(\sigma-1)\tau|x-y|}dy\right]^{\frac{1}{1-\sigma}}, \\ \omega(w,\lambda)(x) = w(x)\left[G(w,\lambda)(x)\right]^{-\mu}. \end{cases} \tag{3.3}$$

These operators correspond to the second and the third equations of (1.1), respectively. Using (3.3), we define the operator $\Psi : \mathcal{C}_+(S) \times L^1_{\mathcal{M}}(S) \to L^1(S)$ as

$$\Psi(w,\lambda)(x) = \left[\omega(w,\lambda)(x) - \int_S \omega(w,\lambda)(y)\lambda(y)dy\right]\lambda(x). \tag{3.4}$$

Then, the fourth equation in (1.1) is written as

$$\frac{d\lambda}{dt}(t) = \Psi(w(t),\lambda(t)).$$

Therefore, for each $\phi \in L^1_{\mathcal{M}}(S)$, the problem (1.1) is formulated as the initial value problem for an abstract evolution equation

$$\begin{cases} w(t) = \Phi(w(t),\lambda(t)), & 0 \le t < \infty, \\ \dfrac{d\lambda}{dt}(t) = \Psi(w(t),\lambda(t)), & 0 \le t < \infty, \\ \lambda(0) = \lambda_0. \end{cases} \tag{3.5}$$

in a product Banach space

$$\mathcal{C}(S) \times L^1(S) = \left\{(w,\lambda)\middle| w \in \mathcal{C}(S),\ \lambda \in L^1(S)\right\}.$$

The initial function $\lambda_0$ is taken from $L^1_{\mathcal{M}}(S)$.

**4 Existence Results.** In this section, we construct a global solution for (3.5) by using the analogous arguments in Tabata et al.[26, 22, 23]. This section consists of two subsections. In Subsection 4.1, the fixed point problem (3.2) is handled for each fixed $\lambda$. Based on the results, a local solution is constructed in Subsection 4.2, and a global solution is constructed in Subsection 4.3.

**4.1 Fixed Point Problem** (3.2)**.** Let us introduce notations in this section. For real numbers $0 < r_1 < r_2$, we consider a bounded closed subset of $\mathcal{C}_+(S)$ determined by

$$\mathcal{C}_{r_1,r_2}(S) := \{u \in \mathcal{C}_+(S) | r_1 \leq u \leq r_2 \text{ on } S\}.$$

We define an operator $\chi : L^1_{\mathcal{M}}(S) \times L^1_{\mathcal{M}}(S) \to \mathcal{C}_+(S)$ as

$$\chi(\phi, \lambda)(x) := \int_S \frac{\phi(y) e^{-(\sigma-1)\tau|x-y|}}{\int_S \lambda(z) e^{-(\sigma-1)\tau|y-z|} dz} dy.$$

For simplicity, let us denote $\chi(\lambda, \lambda)$ by $\chi(\lambda)$.

Since $|x - y| \leq \pi$ for any $x, y \in S$, it is easy to see that

$$e^{-(\sigma-1)\tau\pi} \leq e^{-(\sigma-1)\tau|x-y|} \leq 1, \quad x, y \in S. \tag{4.1}$$

Then, $\chi(\phi, \lambda)$ satisfies

$$\max_{x \in S}[\chi(\phi, \lambda)(x)] \leq e^{(\sigma-1)\tau\pi}, \quad \phi, \lambda \in L^1_{\mathcal{M}}(S), \tag{4.2}$$

$$\min_{x \in S}[\chi(\phi, \lambda)(x)] \geq e^{-(\sigma-1)\tau\pi}, \quad \phi, \lambda \in L^1_{\mathcal{M}}(S). \tag{4.3}$$

*4.1.1 Problem* (3.2) *for the general exponents*

**Theorem 4.1.** *Let $\lambda \in L^1_{\mathcal{M}}(S)$ satisfy*

$$\max_{x \in S}[\chi(\lambda)(x)] < 1/\mu. \tag{4.4}$$

*Then, put $a_\lambda$ and $b_\lambda$ as*

$$\begin{cases} a_\lambda = \dfrac{(1-\mu)\min_{x \in S}[\chi(\phi, \lambda)(x)]}{1 - \mu \min_{x \in S}[\chi(\lambda)(x)]}, \\ b_\lambda = \dfrac{(1-\mu)\max_{x \in S}[\chi(\phi, \lambda)(x)]}{1 - \mu \max_{x \in S}[\chi(\lambda)(x)]}, \end{cases} \tag{4.5}$$

*respectively. Then,* (3.2) *has at least one solution $w$ in $\mathcal{C}_{a_\lambda, b_\lambda}(S)$.*

*Proof.* Let $\lambda$ satisfy (4.4). We want to show that $\Phi(\cdot, \lambda)$ is a compact operator from $M = \mathcal{C}_{a_\lambda, b_\lambda}(S)$ into itself, and to apply the Schauder fixed point theorem to $\Phi(\cdot, \lambda)$.

It is clear that $\Phi(\cdot, \lambda)$ maps $M$ into itself. In fact, for $w \in M$,

$$\begin{aligned} \Phi(w, \lambda) &\leq \left[\mu b_\lambda^\sigma \max_x[\chi(\lambda)(x)] + (1-\mu) b_\lambda^{\sigma-1} \max_x[\chi(\phi, \lambda)(x)]\right]^{\frac{1}{\sigma}} \\ &= \left[\mu b_\lambda^\sigma \max_x[\chi(\lambda)(x)] + b_\lambda^\sigma \left(1 - \mu \max_x[\chi(\lambda)(x)]\right)\right]^{\frac{1}{\sigma}} \\ &= b_\lambda. \end{aligned} \tag{4.6}$$

Here, the second equality follows immediately from the definition of $b_\lambda$ (see (4.5)). Similarly, it is clear to see that $\Phi(w, \lambda) \geq a_\lambda$.

The operator $\Phi(\cdot, \lambda)$ is observed to be compact as follows. For $u \in \Phi(M, \lambda)$, let $u = \Phi(w, \lambda)$ with $w \in M$. For each $x_1, x_2 \in S$, put

$$X_i = [\Phi(w, \lambda)(x_i)]^\sigma, \ (i = 1, 2).$$

Then, we have

$$
\begin{aligned}
|u(x_1) - u(x_2)| &= \left| X_1^{\frac{1}{\sigma}} - X_2^{\frac{1}{\sigma}} \right| \\
&= \left| \int_0^1 \frac{\partial}{\partial \theta} \left[\theta X_1 + (1-\theta) X_2\right]^{\frac{1}{\sigma}} d\theta \right| \\
&\le \frac{1}{\sigma} \int_0^1 \left[\theta X_1 + (1-\theta) X_2\right]^{\frac{1}{\sigma}-1} d\theta \cdot |X_1 - X_2| \\
&\le \frac{1}{\sigma} a_\lambda^{1-\sigma} |X_1 - X_2| \\
&\le \left( \frac{1}{\sigma} a_\lambda^{1-\sigma} \mu b_\lambda^\sigma + (1-\mu) b_\lambda^{\sigma-1} e^{(\sigma-1)\tau\pi} \right) \\
&\quad \times \max_{y \in S} \left| e^{-(\sigma-1)\tau|x_1 - y|} - e^{-(\sigma-1)\tau|x_2-y|} \right| .
\end{aligned}
\tag{4.7}
$$

This shows that $\Phi(M, \lambda)$ is equicontinuous, i.e.,

$$\forall \epsilon > 0, \exists \delta > 0, \forall u \in \Phi(M, \lambda), |x_1 - x_2| < \delta \Rightarrow |u(x_1) - u(x_2)| < \epsilon.$$

It is obvious that $\Phi(M, \lambda)$ is uniformly bounded, i.e.,

$$\exists K \ge 0,\ \forall u \in \Phi(M, \lambda),\ \|u\|_{\mathcal{C}} \le K.$$

Therefore, $\Phi(M, \lambda)$ is a relatively compact subset by the Arzelà-Ascoli theorem. Thus, $\Phi(\cdot, \lambda)$ turns out to be a compact operator.

Hence, thanks to the Schauder fixed point theorem, (3.2) has at least one solution . □

Under (4.4), Theorem 4.1 guarantees existence of solution to (3.2) in $\mathcal{C}_{a_\lambda, b_\lambda}(S)$. As a matter of fact, when (4.4) holds true, any solution to (3.2) must be in $\mathcal{C}_{a_\lambda, b_\lambda}(S)$.

**Theorem 4.2.** *Under* (4.4)*, any solution* $w \in \mathcal{C}_+(S)$ *to* (3.2) *satisfies* $a_\lambda \le \min_{x \in S} w(x)$*, and* $\max_{x\in S} w(x) \le b_\lambda$*, i.e.,* $w \in \mathcal{C}_{a_\lambda, b_\lambda}(S)$*.*

*Proof.* By estimating the maximum value of $w$, we have

$$
\begin{aligned}
\left( \max_{x \in S} w(x) \right)^\sigma &= \left( \max_{x\in S} \Phi(w, \lambda) \right)^\sigma \\
&\le \mu \left( \max_{x \in S} w(x) \right)^\sigma \max_{x \in S} [\chi(\lambda)(x)] \\
&\quad + (1-\mu) \left( \max_{x\in S} w(x) \right)^{\sigma - 1} \max_{x \in S} [\chi(\phi, \lambda)(x)].
\end{aligned}
$$

By solving this inequality for $\max_{x\in S} w(x)$, we obtain $\max_{x \in S} w(x) \le b_\lambda$. It is similar for $a_\lambda \le \min_{x\in S} w(x)$. □

Uniqueness of the solution is described in the following theorem.

**Theorem 4.3.** *Let* $\lambda \in L^1_{\mathcal{M}}(S)$ *satisfy* (4.4)*. In addition, let* $\lambda$ *satisfy*

$$\chi(\lambda)(x) < \frac{\sigma}{\mu} \left( \frac{a_\lambda}{b_\lambda} \right)^{\sigma - 1} \left\{ 1 - \frac{\sigma - 1}{\sigma} \left( \frac{a_\lambda}{b_\lambda} \right)^{1 - 2\sigma} \right\}. \tag{4.8}$$

*Then, the solution to* (3.2) *obtained by theorem 4.1 is unique in* $\mathcal{C}_+(S)$*.*

*Proof.* We already know that $\Phi(\cdot,\lambda)$ maps $\mathcal{C}_{a_\lambda,b_\lambda}(S)$ into itself. Therefore, it suffices to show that $\Phi(\cdot,\lambda)$ becomes a contraction mapping under (4.8).

Let $D_w\Phi(w,\lambda)$ denote the Fréchet derivative of $\Phi(w,\lambda)$ with respect to $w$. Then, for $h\in\mathcal{C}(S)$, we see that

$$
\begin{aligned}
&(D_w\Phi(w,\lambda)h)(x)\\
&=\frac{1}{\sigma}\Phi(w,\lambda)^{1-\sigma}\int_S\frac{\mu\lambda(y)h(y)e^{-(\sigma-1)\tau|x-y|}}{\int_S\lambda(z)w(z)^{1-\sigma}e^{-(\sigma-1)\tau|y-z|}dz}dy\\
&+\frac{\sigma-1}{\sigma}\Phi(w,\lambda)^{1-\sigma}\int_S\frac{\mu\lambda(y)w(y)\int_S\lambda(z)w(z)^{-\sigma}h(z)e^{-(\sigma-1)\tau|y-z|}dz}{\left\{\int_S\lambda(z)w(z)^{1-\sigma}e^{-(\sigma-1)\tau|y-z|}dz\right\}^2}e^{-(\sigma-1)\tau|x-y|}dy\\
&+\frac{\sigma-1}{\sigma}\Phi(w,\lambda)^{1-\sigma}\int_S\frac{(1-\mu)\phi(y)\int_S\lambda(z)w(z)^{-\sigma}h(z)e^{-(\sigma-1)\tau|y-z|}dz}{\left\{\int_S\lambda(z)w(z)^{1-\sigma}e^{-(\sigma-1)\tau|y-z|}dz\right\}^2}e^{-(\sigma-1)\tau|x-y|}dy.
\end{aligned}
$$

We estimate the operator norm of $D_w\Phi(w,\lambda)$ given by

$$\|D_w\Phi(w,\lambda)\|_{\mathrm{op}}:=\sup_{\|h\|_{\mathcal{C}}=1}\|D_w\Phi(w,\lambda)h\|_{\mathcal{C}}.$$

In view of Theorem 4.2, it is observed that

$$
\begin{aligned}
\|D_w\Phi(w,\lambda)\|_{\mathrm{op}}\le{}&\frac{\mu}{\sigma}\left(\frac{a_\lambda}{b_\lambda}\right)^{1-\sigma}\max_{x\in S}[\chi(\lambda)(x)]\\
&+\frac{\mu(\sigma-1)}{\sigma}\left(\frac{a_\lambda}{b_\lambda}\right)^{1-2\sigma}\max_{x\in S}[\chi(\lambda)(x)]\\
&+\frac{\sigma-1}{\sigma}\frac{a_\lambda^{1-2\sigma}}{b_\lambda^{2(1-\sigma)}}(1-\mu)\max_{x\in S}[\chi(\phi,\lambda)(x)].
\end{aligned}
\tag{4.9}
$$

Then, noting that

$$(1-\mu)\max_{x\in S}[\chi(\phi,\lambda)(x)]=b_\lambda\left(1-\mu\max_{x\in S}[\chi(\lambda)(x)]\right),$$

we verify that (4.8) implies $\|D_w\Phi(w,\lambda)\|_{\mathrm{op}}<1$. □

*4.1.2 Problem* (3.2) *for the exponents satisfying* (4.10)

**Theorem 4.4.** *Assume that the exponents in* (1.1) *satisfy the relation*

$$e^{(\sigma-1)\tau\pi}<1/\mu. \tag{4.10}$$

*We put $a>0$ and $b>0$ as*

$$
\begin{aligned}
a&=\frac{1-\mu}{e^{(\sigma-1)\tau\pi}-\mu}<1,\\
b&=\frac{1-\mu}{e^{-(\sigma-1)\tau\pi}-\mu}>1,
\end{aligned}
\tag{4.11}
$$

*respectively. Then, for any $\lambda\in L^1_{\mathcal{M}}(S)$,* (3.2) *has at least one solution $w$ in $\mathcal{C}_{a,b}(S)$.*

*Proof.* It is obvious that (4.10) implies that (4.4) holds for any $\lambda\in L^1_{\mathcal{M}}(S)$. It is also obvious that $a\le a_\lambda, b_\lambda\le b$ for any $\lambda\in L^1_{\mathcal{M}}(S)$. Therefore, $\Phi(\cdot,\lambda)$ is a compact operator from $\mathcal{C}_{a,b}(S)$ into itself and has at least one fixed point thanks to Theorem 4.1. □

Uniqueness of solutions is obtained by the following theorem.

**Theorem 4.5.** *In addition to* (4.10)*, if the exponents in* (1.1) *satisfy*

$$e^{(\sigma-1)\tau\pi} < \left\{ \frac{\mu}{\sigma}\left(\frac{a}{b}\right)^{1-\sigma} + \frac{\mu(\sigma-1)}{\sigma}\left(\frac{a}{b}\right)^{1-2\sigma} + \frac{(\sigma-1)(1-\mu)}{\sigma}\frac{a^{1-2\sigma}}{b^{2(1-\sigma)}} \right\}^{-1}, \tag{4.12}$$

*then, for any* $\lambda \in L^1_{\mathcal{M}}(S)$, (3.2) *possesses a unique solution* $w \in \mathcal{C}_+(S)$.

*Proof.* In the same way as for the proof of Theorem 4.3, we can estimate the operator norm of $D_w\Phi(w,\lambda)$ to verify that $\Phi(\cdot,\lambda)$ is a contraction mapping for any $\lambda \in L^1_{\mathcal{M}}(S)$. □

**Remark.** Notice that $a \to 1$ and $b \to 1$ if $\sigma \to 1$ or $\tau \to 0$. So, we see that the right-hand side of (4.12) converges to $\sigma/(\sigma+\mu-1) > 1$. On the other hand, the left-hand side converges to 1 if $\sigma \to 1$ or $\tau \to 0$. Therefore, there exist $\sigma$ and $\tau$ which satisfy (4.12).

The following proposition gives upper and lower bounds for $G(w,\lambda)$ and $\omega(w,\lambda)$ when $(w,\lambda)$ varies in $\mathcal{C}_{a,b}(S) \times L^1_{\mathcal{M}}(S)$.

**Proposition 4.1.** *We have the estimates*

$$a \le G(w,\lambda) \le be^{\tau\pi}, \quad (w,\lambda) \in \mathcal{C}_{a,b}(S) \times L^1_{\mathcal{M}}(S), \tag{4.13}$$

$$ab^{-\mu}e^{-\mu\tau\pi} \le \omega(w,\lambda) \le ba^{-\mu}, \quad (w,\lambda) \in \mathcal{C}_{a,b}(S) \times L^1_{\mathcal{M}}(S). \tag{4.14}$$

*Proof.* These are obvious from (3.3) and the condition that $a \le w(x) \le b$ on $S$. □

In the case when (3.2) admits a unique solution $w \in \mathcal{C}_+(S)$ for each $\lambda \in L^1_{\mathcal{M}}(S)$, we denote it by $w = \Phi_{\mathrm{f}}(\lambda)$. Then, (3.5) ultimately reduces to the Cauchy problem for an ordinary differential equation

$$\begin{cases} \dfrac{d\lambda}{dt}(t) = \Psi(\Phi_{\mathrm{f}}(\lambda(t)), \lambda(t)), & 0 \le t < \infty, \\ \lambda(0) = \lambda_0, \end{cases} \tag{4.15}$$

in $L^1(S)$ with initial value $\lambda_0 \in L^1_{\mathcal{M}}(S)$.

**4.2 Local solution for** (4.15)**.** We construct a local solution to (4.15) by applying the theory of ordinary differential equations in a Banach spaces (see [33]). To do so, we first prepare the following propositions.

**Proposition 4.2.** *Let* (4.10) *be satisfied. More strongly than* (4.12) *assume that*

$$e^{(\sigma-1)\tau\pi} < \left( \frac{\mu}{\sigma}\frac{a^{2(1-\sigma)}}{b^{2(1-\sigma)}} + \frac{\mu(\sigma-1)}{\sigma}\frac{a^{1-2\sigma}}{b^{1-2\sigma}} + \frac{(1-\mu)(\sigma-1)}{\sigma}\frac{a^{1-2\sigma}}{b^{2(1-\sigma)}} \right)^{-1}. \tag{4.16}$$

*Then, the following three estimates*

$$\|\Phi_{\mathrm{f}}(\lambda_1) - \Phi_{\mathrm{f}}(\lambda_2)\|_{\mathcal{C}} \le \mathcal{L}_1\|\lambda_1 - \lambda_2\|_{L^1}, \quad \lambda_1, \lambda_2 \in L^1_{\mathcal{M}}(S), \tag{4.17}$$

$$\|G(\Phi_{\mathrm{f}}(\lambda_1),\lambda_1) - G(\Phi_{\mathrm{f}}(\lambda_2),\lambda_2)\|_{\mathcal{C}} \le \mathcal{L}_2\|\lambda_1 - \lambda_2\|_{L^1}, \quad \lambda_1, \lambda_2 \in L^1_{\mathcal{M}}(S), \tag{4.18}$$

$$\|\omega(\Phi_{\mathrm{f}}(\lambda_1),\lambda_1) - \omega(\Phi_{\mathrm{f}}(\lambda_2),\lambda_2)\|_{\mathcal{C}} \le \mathcal{L}_3\|\lambda_1 - \lambda_2\|_{L^1}, \quad \lambda_1, \lambda_2 \in L^1_{\mathcal{M}}(S). \tag{4.19}$$

*hold true with some constants* $\mathcal{L}_1,\ \mathcal{L}_2,\ \mathcal{L}_3 > 0$.

*Proof.* It suffices to prove (4.17), because (4.18) and (4.19) are easily verified from (4.17). We have, for any $x \in S$,

$$
\begin{aligned}
|\Phi_{\mathrm{f}}(\lambda_1)(x) - \Phi_{\mathrm{f}}(\lambda_2)(x)| &= \left| [\{\Phi_{\mathrm{f}}(\lambda_1)(x)\}^{\sigma}]^{\frac{1}{\sigma}} - [\{\Phi_{\mathrm{f}}(\lambda_2)(x)\}^{\sigma}]^{\frac{1}{\sigma}} \right| \\
&\le \left| \int_0^1 \frac{\partial}{\partial\theta} \left[\theta \{\Phi_{\mathrm{f}}(\lambda_1)(x)\}^{\sigma} + (1-\theta)\{\Phi_{\mathrm{f}}(\lambda_2)(x)\}^{\sigma}\right]^{\frac{1}{\sigma}} d\theta \right| \\
&\le \frac{1}{\sigma} a^{1-\sigma} \left| \{\Phi_{\mathrm{f}}(\lambda_1)(x)\}^{\sigma} - \{\Phi_{\mathrm{f}}(\lambda_2)(x)\}^{\sigma} \right|.
\end{aligned}
\tag{4.20}
$$

Here, put

$$
\begin{aligned}
p(y) = (\mu\lambda_1(y)\Phi_{\mathrm{f}}(\lambda_1)(y) + (1-\mu)\phi(y)) \cdot \int_S \lambda_2(z)\Phi_{\mathrm{f}}(\lambda_2)(z)^{1-\sigma} e^{-(\sigma-1)\tau|y-z|} dz \\
- (\mu\lambda_2(y)\Phi_{\mathrm{f}}(\lambda_2)(y) + (1-\mu)\phi(y)) \cdot \int_S \lambda_1(z)\Phi_{\mathrm{f}}(\lambda_1)(z)^{1-\sigma} e^{-(\sigma-1)\tau|y-z|} dz, \quad y \in S,
\end{aligned}
$$

and

$$
\begin{aligned}
q(y) = \int_S \lambda_1(y)\Phi_{\mathrm{f}}^{1-\sigma}(\lambda_1)(y) e^{-(\sigma-1)\tau|y-z|} dz \\
\cdot \int_S \lambda_2(y)\Phi_{\mathrm{f}}^{1-\sigma}(\lambda_2)(y) e^{-(\sigma-1)\tau|y-z|} dz, \quad y \in S.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left|\{\Phi_{\mathrm{f}}(\lambda_1)(x)\}^{\sigma} - \{\Phi_{\mathrm{f}}(\lambda_2)(x)\}^{\sigma}\right| &= \left| \int_S \frac{p(y)e^{-(\sigma-1)\tau|x-y|}}{q(y)} dy \right| \\
&\le \int_S \left| \frac{p(y)}{q(y)} \right| dy \le b^{2(\sigma-1)} e^{2(\sigma-1)\tau\pi} \int_S |p(y)| dy.
\end{aligned}
\tag{4.21}
$$

Therefore, it follows from (4.20) and (4.21) that

$$
|\Phi_{\mathrm{f}}(\lambda_1)(x) - \Phi_{\mathrm{f}}(\lambda_2)(x)| \le \frac{b^{2(\sigma-1)} e^{2(\sigma-1)\tau\pi}}{\sigma a^{\sigma-1}} \int_S |p(y)|\, dy.
\tag{4.22}
$$

Let us next estimate $|p(y)|$ on $S$. Note that

$$
\begin{aligned}
\left| \Phi_{\mathrm{f}}(\lambda_1)^{1-\sigma} - \Phi_{\mathrm{f}}(\lambda_1)^{1-\sigma} \right| &= \int_0^1 \frac{\partial}{\partial\theta} \left[\theta\Phi_{\mathrm{f}}(\lambda_1) + (1-\theta)\Phi_{\mathrm{f}}(\lambda_2)\right]^{1-\sigma} d\theta \\
&\le (\sigma-1)a^{-\sigma} \left| \Phi_{\mathrm{f}}(\lambda_1) - \Phi_{\mathrm{f}}(\lambda_2) \right|.
\end{aligned}
\tag{4.23}
$$

Then, the estimate

$$
\begin{aligned}
|p(y)| \le (\mu\lambda_1(y)\Phi_{\mathrm{f}}(\lambda_1)(y) + (1-\mu)\phi(y)) \\
\times \left| \int_S \lambda_1(z)[\Phi_{\mathrm{f}}(\lambda_1)(z)]^{1-\sigma} e^{-(\sigma-1)\tau|y-z|} dz \right. \\
\left. - \int_S \lambda_2(z)[\Phi_{\mathrm{f}}(\lambda_2)(z)]^{1-\sigma} e^{-(\sigma-1)\tau|y-z|} dz \right| \\
+ \mu \int_S \lambda_1(z)[\Phi_{\mathrm{f}}(\lambda_1)(z)]^{1-\sigma} e^{-(\sigma-1)\tau|y-z|} dz \\
\times \left| \lambda_1(y)\Phi_{\mathrm{f}}(\lambda_1)(y) - \lambda_2(y)\Phi_{\mathrm{f}}(\lambda_2)(y) \right|, \quad y \in S,
\end{aligned}
\tag{4.24}
$$

yields

$$
(4.25)\qquad \begin{aligned}
|p(y)| \le{}& \left[\mu\left\{(\sigma-1)a^{-\sigma}b + a^{1-\sigma}\right\}\lambda_1(y) + (\sigma-1)a^{-\sigma}(1-\mu)\phi(y)\right] \\
&\qquad \times \|\Phi_{\mathrm{f}}(\lambda_1) - \Phi_{\mathrm{f}}(\lambda_2)\|_{\mathcal{C}} \\
&+ a^{1-\sigma}\left\{\mu b\lambda_1(y) + (1-\mu)\phi(y)\right\}\|\lambda_1 - \lambda_2\|_{L^1} \\
&+ \mu a^{1-\sigma} b|\lambda_1(y) - \lambda_2(y)|.
\end{aligned}
$$

By (4.22) and (4.25), we have

$$
(4.26)\qquad \|\Phi_{\mathrm{f}}(\lambda_1) - \Phi_{\mathrm{f}}(\lambda_2)\|_{\mathcal{C}} \le \alpha_1\|\Phi_{\mathrm{f}}(\lambda_1) - \Phi_{\mathrm{f}}(\lambda_2)\|_{\mathcal{C}} + \alpha_2\|\lambda_1 - \lambda_2\|_{L^1}
$$

with $\alpha_1, \alpha_2$ such that

$$
\begin{aligned}
\alpha_1 &= \frac{b^{2(\sigma-1)}e^{2(\sigma-1)\tau\pi}}{a^{2\sigma-1}\sigma}\left[\mu\left\{(\sigma-1)b + a\right\} + (\sigma-1)(1-\mu)\right], \\
\alpha_2 &= \frac{b^{2(\sigma-1)}e^{2(\sigma-1)\tau\pi}}{a^{\sigma-1}\sigma}\left\{2\mu b + 1 - \mu\right\}.
\end{aligned}
$$

Since (4.16) means that $\alpha_1 < 1$, we observe from (4.26) that (4.17) holds true with $\mathcal{L}_1 = \alpha_2/(1-\alpha_1)$. □

We are now ready to construct a local solution to (4.15).

We have to use, however, an auxiliary problem for (4.15). For a given $\tilde{\lambda} \in \mathcal{C}([0,c]; L^1_{\mathcal{M}}(S))$, let $\tilde{\Psi}(w,\lambda) : \mathcal{C}_+(S) \times L^1_{\mathcal{M}}(S) \to L^1(S)$ be an operator defined by

$$
(4.27)\qquad \tilde{\Psi}(w,\lambda) = \left[\omega(w,\tilde{\lambda})(x) - \int_S \omega(w,\tilde{\lambda})(y)\lambda(y)dy\right]\lambda(x).
$$

Consider an auxiliary problem

$$
(4.28)\qquad \begin{cases} \dfrac{d\lambda}{dt}(t) = \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}(t)), \lambda(t)), & 0 \le t < \infty, \\ \lambda(0) = \lambda_0, \end{cases}
$$

of (4.15).

**Proposition 4.3.** *Under* (4.10) *and* (4.12), *let* $\tilde{\lambda} \in \mathcal{C}([0,1]; L^1_{\mathcal{M}}(S))$ *be given. Then,* (4.28) *possesses a unique local solution* $\lambda \in \mathcal{C}^1([0,c]; L^1_{\mathcal{M}}(S))$, *provided that* $(1 \ge)\ c > 0$ *is sufficiently small, but* $c$ *being independent of the given function* $\tilde{\lambda}$.

*Proof.* We employ the usual techniques for ODEs in Banach spaces. Set a subset of $L^1(S)$ given by

$$
L^1_1(S) := \left\{ f \in L^1(S) \,\middle|\, \int_S f(x)dx = 1 \right\},
$$

and define an operator $\tilde{T} : \mathcal{C}([0,c]; L^1_1(S)) \to \mathcal{C}([0,c]; L^1(S))$ by

$$
\tilde{T}(\lambda) = \lambda_0 + \int_0^t \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}(s)), \lambda(s))ds.
$$

Using $\tilde{T}$, we rewrite (4.28) into an equivalent problem

$$
\lambda(t) = [\tilde{T}(\lambda)](t), \quad 0 \le t < \infty.
$$

It is verified that $\tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}), \lambda)$ is Lipscitz continuous with respect to $\lambda \in L^1_1(S)$. Indeed, by (4.14) and (4.27), we see that

$$\left\| \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}), \lambda_1) - \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}), \lambda_2) \right\|_{L^1} \le 3ba^{-\mu} \|\lambda_1 - \lambda_2\|_{L^1}, \quad \lambda_1, \lambda_2 \in L^1_1(S). \tag{4.29}$$

Meanwhile, $\tilde{T}$ maps $\mathcal{C}([0,c]; L^1_1(S))$ into itself. Indeed, since $\tilde{T}$ maps $\mathcal{C}([0,c]; L^1_1(S))$ into $\mathcal{C}([0,c]; L^1(S))$, it is sufficient to see that $\int_S \tilde{T}(\lambda)(x)dx = 1$. But,

$$\begin{aligned}
\int_S \tilde{T}(\lambda)(x)dx - \int_S \lambda_0(x)dx &= \int_S \int_0^t \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}(s)), \lambda(s))dsdx \\
&= \int_0^t \int_S \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}(s)), \lambda(s))dxds = 0
\end{aligned}$$

due to (4.27).

Finally, $\tilde{T}$ is a contraction mapping if $c$ is sufficiently small. Indeed, from (4.29),

$$\begin{aligned}
\left\| \tilde{T}(\lambda_1) - \tilde{T}(\lambda_2) \right\|_{\mathcal{C}([0,c];L^1)} &\le \max_{t\in[0,c]} e^{-t} \int_0^t \left\| \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}), \lambda_1) - \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}), \lambda_2) \right\|_{L^1} ds \\
&\le 3ba^{-\mu} \max_{t\in[0,c]} e^{-t} \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{L^1} ds \\
&\le 3ba^{-\mu} \max_{t\in[0,c]} e^{-t} \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{L^1} e^{-s} e^{s} ds \\
&= 3ba^{-\mu}(1 - e^{-c}) \|\lambda_1 - \lambda_2\|_X.
\end{aligned}$$

Therefore, if $c$ is sufficiently small, $\tilde{T}$ becomes a contraction mapping. Thus, (4.28) has a unique fixed point $\lambda \in \mathcal{C}^1([0,c]; L^1_1(S))$ for sufficiently small $c > 0$.

As a matter of fact, the solution $\lambda \in \mathcal{C}([0,c]; L^1_1(S))$ above is in $\mathcal{C}([0,c]; L^1_{\mathcal{M}}(S))$. In fact, it is sufficient to verify $\lambda(t) \ge 0$ on $S$ for $t \in [0,c]$. But, since

$$\lambda(t) = \lambda_0 \exp\left\{ \int_0^t \left[ \omega(w, \tilde{\lambda}(s)) - \int_S \omega(w, \tilde{\lambda}(s))(y)\lambda(s,y)dy \right] ds \right\},$$

$\lambda_0 \ge 0$ on $S$ implies $\lambda(t) \ge 0$ on $S$.

As seen above, the time $c > 0$ is independent of the given function $\tilde{\lambda}$. □

We finally construct a unique local solution to (4.15).

**Theorem 4.6.** *Under* (4.10) *and* (4.16), *for each* $\lambda_0 \in L^1_{\mathcal{M}}(S)$, *there exists a unique local solution* $\lambda \in \mathcal{C}^1([0,1]; L^1_{\mathcal{M}}(S))$ *to* (4.15), *provided that* $(1 \ge)$ $c > 0$ *is sufficiently small, but* $c$ *being independent of the initial value* $\lambda_0$.

*Proof.* By virtue of Proposition 4.3, for each $\lambda_0 \in L^1_{\mathcal{M}}(S)$, we can define an operator $F_{\lambda_0} : \tilde{\lambda} \in \mathcal{C}([0,c]; L^1_{\mathcal{M}}(S)) \mapsto \lambda \in \mathcal{C}([0,c]; L^1_{\mathcal{M}}(S))$, where $\lambda$ is the local solution of (4.28). By the definition of $F_{\lambda_0}$, it is verified that

$$[F_{\lambda_0}(\tilde{\lambda})](t) = \lambda_0 + \int_0^t \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}(s)), [F_{\lambda_0}(\tilde{\lambda})](s))ds, \quad 0 \le t \le c.$$

Since a fixed point of $F_{\lambda_0}$ is a solution to (4.15), we will prove that $F_{\lambda_0}$ is a contraction mapping from $\mathcal{C}([0,c]; L^1_{\mathcal{M}}(S))$ into itself.

For $\tilde{\lambda}_1,\ \tilde{\lambda}_2 \in \mathcal{C}([0,c]; L^1_{\mathcal{M}}(S))$,

$$
\begin{aligned}
&\left\| F_{\lambda_0}(\tilde{\lambda}_1)(t) - F_{\lambda_0}(\tilde{\lambda}_2)(t) \right\|_{L^1} \\
&\le \int_0^t \left\| \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}_1), F_{\lambda_0}(\tilde{\lambda}_1)) - \tilde{\Psi}(\Phi_{\mathrm{f}}(\tilde{\lambda}_2), F_{\lambda_0}(\tilde{\lambda}_2)) \right\|_{L^1} ds \\
&\le \int_0^t \left\| \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}_1), \tilde{\lambda}_1) F_{\lambda_0}(\tilde{\lambda}_1) - \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}_2), \tilde{\lambda}_2) F_{\lambda_0}(\tilde{\lambda}_2) \right\|_{L^1} ds \\
&\quad + \int_0^t \left\| \int_S \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}_2), \tilde{\lambda}_2) F_{\lambda_0}(\tilde{\lambda}_2) dy \cdot F_{\lambda_0}(\tilde{\lambda}_2) \right. \\
&\qquad\qquad \left. - \int_S \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}_1), \tilde{\lambda}_1) F_{\lambda_0}(\tilde{\lambda}_1) dy \cdot F_{\lambda_0}(\tilde{\lambda}_1) \right\|_{L^1} ds.
\end{aligned} \tag{4.30}
$$

Noting (4.14) and (4.19), we obtain that

$$
\begin{aligned}
&\left\| \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}_1), \tilde{\lambda}_1) F_{\lambda_0}(\tilde{\lambda}_1) - \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}_2), \tilde{\lambda}_2) F_{\lambda_0}(\tilde{\lambda}_2) \right\|_{L^1} \\
&\le ba^{-\mu} \left\| F_{\lambda_0}(\tilde{\lambda}_1)(t) - F_{\lambda_0}(\tilde{\lambda}_2)(t) \right\|_{L^1} + \mathcal{L}_3 \left\| \tilde{\lambda}_1 - \tilde{\lambda}_2 \right\|_{L^1},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \int_S \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}_2), \tilde{\lambda}_2) F_{\lambda_0}(\tilde{\lambda}_2) dy \cdot F_{\lambda_0}(\tilde{\lambda}_2) \right. \\
&\qquad \left. - \int_S \omega(\Phi_{\mathrm{f}}(\tilde{\lambda}_1), \tilde{\lambda}_1) F_{\lambda_0}(\tilde{\lambda}_1) dy \cdot F_{\lambda_0}(\tilde{\lambda}_1) \right\|_{L^1} \\
&\le 2ba^{-\mu} \left\| F_{\lambda_0}(\tilde{\lambda}_1)(t) - F_{\lambda_0}(\tilde{\lambda}_2)(t) \right\|_{L^1} + \mathcal{L}_3 \left\| \tilde{\lambda}_1(t) - \tilde{\lambda}_2(t) \right\|_{L^1}.
\end{aligned}
$$

Therefore, it follows from (4.30) that

$$
\begin{aligned}
&\left\| F_{\lambda_0}(\tilde{\lambda}_1)(t) - F_{\lambda_0}(\tilde{\lambda}_2)(t) \right\|_{L^1} \\
&\le 3ba^{-\mu} \int_0^t \left\| F_{\lambda_0}(\tilde{\lambda}_1)(s) - F_{\lambda_0}(\tilde{\lambda}_2)(s) \right\|_{L^1} ds \\
&\quad + 2\mathcal{L}_3 \int_0^t \left\| \tilde{\lambda}_1(s) - \tilde{\lambda}_2(s) \right\|_{L^1} ds, \quad 0 \le t \le c.
\end{aligned}
$$

In this way, we arrive at

$$
\left\| F_{\lambda_0}(\tilde{\lambda}_1) - F_{\lambda_0}(\tilde{\lambda}_2) \right\|_{\mathcal{C}([0,c];L^1)} \le k \left\| \tilde{\lambda}_1 - \tilde{\lambda}_2 \right\|_{\mathcal{C}([0,c];L^1)}
$$

with

$$
k = \frac{2\mathcal{L}_3(1 - e^{-c})}{1 - 3ba^{-\mu}(1 - e^{-c})}.
$$

Remember that the norm $\| \cdot \|_{\mathcal{C}([0,c];L^1)}$ was defined by (3.1).

If $c > 0$ is sufficiently small, then $k < 1$, and hence $F_{\lambda_0}$ becomes a contraction mapping of $\mathcal{C}([0,c]; L^1_{\mathcal{M}}(S))$.

As seen above, the time $c$ is independent of the initial value $\lambda_0$. $\square$

**4.3 Global solution for** (4.15)**.** We can extend the local solution of (4.15) constructed above to global one.

**Theorem 4.7.** *Under* (4.10) *and* (4.16)*, for each* $\lambda_0 \in L^1_{\mathcal{M}}(S)$*, there exists a unique global solution* $\lambda \in \mathcal{C}^1([0,\infty); L^1_{\mathcal{M}}(S))$ *to* (4.15)*.*

*Proof.* Note that the time $c > 0$ of the interval $[0, c]$ on which we construct the local solution is independent of the initial value $\lambda_0$. Then, the uniqueness of the local solution shows that the unique local solution $\lambda \in \mathcal{C}^1([0, 2c]; L^1_{\mathcal{M}}(S))$ is obtained by repeating the same argument but with the initial value $\lambda(c)$. By repeating this procedure, we finally obtain a unique global solution to (4.15) □

The global solution depends continuously on initial values.

**Theorem 4.8.** *Under* (4.10) *and* (4.16)*, let* $\lambda_1, \lambda_2 \in \mathcal{C}^1([0,\infty); L^1_{\mathcal{M}}(S))$ *be the global solutions to* (4.15) *with initial values* $\lambda_{1,0}, \lambda_{2,0} \in L^1_{\mathcal{M}}(S)$*, respectively. Then, the Lipschitz condition*

$$\|\lambda_1(t) - \lambda_2(t)\|_{L^1} \le \|\lambda_{1,0} - \lambda_{2,0}\|_{L^1} e^{\left(3ba^{-\mu} + 2\mathcal{L}_3\right)t}, \quad 0 \le t < \infty, \tag{4.31}$$

*holds true.*

*Proof.* For $i = 1, 2$, we have

$$\lambda_i(t) = \lambda_{i,0} + \int_0^t \Psi(\Phi_{\rm f}(\lambda_i(s)), \lambda_i(s))ds, \quad 0 \le t < \infty.$$

So,

$$\begin{aligned} \|\lambda_1(t) - \lambda_2(t)\|_{L^1} &\le \|\lambda_{1,0} - \lambda_{2,0}\|_{L^1} \\ &\quad + \int_0^t \|\Psi(\Phi_{\rm f}(\lambda_1(s)), \lambda_1(s)) - \Psi(\Phi_{\rm f}(\lambda_2(s)), \lambda_2(s))\|_{L^1} ds. \end{aligned} \tag{4.32}$$

The operator $\Psi(\Phi_{\rm f}(\lambda), \lambda)$ is seen to be Lipscitz continuous in $\lambda \in L^1_{\mathcal{M}}(S)$. In fact, by (3.4), we have

$$\begin{aligned} &\|\Psi(\Phi_{\rm f}(\lambda_1), \lambda_1) - \Psi(\Phi_{\rm f}(\lambda_2), \lambda_2)\|_{L^1} \le \|\omega(\Phi_{\rm f}(\lambda_1), \lambda_1)\lambda_1 - \omega(\Phi_{\rm f}(\lambda_2), \lambda_2)\lambda_2\|_{L^1} \\ &+ \left\| \int_S \omega(\Phi_{\rm f}(\lambda_2), \lambda_2)(y)\lambda_2(y)dy \cdot \lambda_2 - \int_S \omega(\Phi_{\rm f}(\lambda_1), \lambda_1)(y)\lambda_1(y)dy \cdot \lambda_1 \right\|_{L^1}. \end{aligned} \tag{4.33}$$

Then, by (4.14) and (4.19),

$$\|\Psi(\Phi_{\rm f}(\lambda_1), \lambda_1) - \Psi(\Phi_{\rm f}(\lambda_2), \lambda_2)\|_{L^1} \le \left(3ba^{-\mu} + 2\mathcal{L}_3\right) \|\lambda_1 - \lambda_2\|_{L^1}. \tag{4.34}$$

Hence,

$$\|\lambda_1(t) - \lambda_2(t)\|_{L^1} \le \|\lambda_{1,0} - \lambda_{2,0}\|_{L^1} + \int_0^t \left(3ba^{-\mu} + 2\mathcal{L}_3\right) \|\lambda_1(s) - \lambda_2(s)\|_{L^1} ds.$$

By the Gronwall's lemma, we obtain (4.31). □

**5 Asymptotic behavior of solutions.** In this section, we want to investigate the asymptotic behavior of solutions of (1.1). First, for each global solution, we construct a non-empty weak* $\omega$-limit set in the space $\mathcal{M}(S)$ of measures on $S$. Second, we observe that, if $\big(\lambda(x), w(x), G(x)\big)$ is a stationary solution to (1.1) such that $\lambda(x) \in \mathcal{M}(S)$ and $w(x)$, $G(x) \in \mathcal{C}(S)$, then either $\lambda(x)$ is constant on $S$ (hence $\big(\lambda(x), w(x), G(x)\big)$ is a homogeneous stationary solution) or $\lambda(x)$ is a sum of Dirac delta functions on $S$, i.e., $\lambda(x) = \sum_{k=1}^K \alpha_k \delta_k(x)$. Finally, we prove that the homogeneous stationary solution is always unstable by using the linearized principle.

These results then suggest that $\lambda(t,x)$ converges as $t \to \infty$ to a sum of Dirac delta functions on $S$. Meanwhile, these results show us very good agreements to the numerical computations illustrated in the next section.

**5.1 $\omega$-limit set.** So far, we treated (4.15) as an evolution equation in $L^1(S)$. Numerical results explained in the next section, however, suggest that the global solution to (4.15) converges as $t \to \infty$ to a measure on $S$. From these observations we are naturally led to introduce the space of measures on $S$ which is denoted by $\mathcal{M}(S)$ and consider the asymptotic behavior of solutions in the extended space $\mathcal{M}(S)$. As well known, $\mathcal{M}(S)$ includes the space $L^1_{\mathcal{M}}(S)$ as a closed subspace with isometric embedding and includes also all the Dirac delta functions.

As a matter of fact, it is possible to formulate (1.1) as an evolution equation in $\mathcal{M}(S)$ of the same form (4.15). That is, under similar assumptions, we can construct an $\mathcal{M}(S)$-valued solution for any initial value $\lambda_0 \in \mathcal{M}(S)$ such that $\lambda_0 \geq 0$ with $\|\lambda_0\|_{\mathcal{M}(S)} = 1$. Such a generalization is, however, not necessarily indispensable in the subsequent discussions.

Let $\lambda_0 \in L^1_{\mathcal{M}}(S)$ and let $\lambda(t;\lambda_0)$, $0 \leq t < \infty$, be the unique global solution of (4.15). We then define the weak* $\omega$-limit set of $\lambda_0$ by

$$\text{w}^*\text{-}\omega(\lambda_0) = \bigcap_{t \geq 0} \overline{\{\lambda(t';\lambda_0);\ t \leq t' < \infty\}} \quad \text{(closure in the weak* topology of } \mathcal{M}(S)).$$

**Theorem 5.1.** *For each $\lambda_0 \in L^1_{\mathcal{M}}(S)$, $w^*$-$\omega(\lambda_0)$ is a non-empty set of $\mathcal{M}(S)$.*

*Proof.* In general, it is known that, for a separable Banach space $\mathfrak{X}$, the closed unit ball of its dual space $\mathfrak{X}^*$ is sequentially weak* compact, i.e., if $\|f_n\|_{\mathfrak{X}^*} \leq 1$, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which is such that $f_{n_k} \to \overline{f}$ in the weak* topology of $\mathfrak{X}^*$. Furthermore, any closed bounded ball of $\mathfrak{X}^*$ is sequentially weak* compact.

Since $\|\lambda\|_{L_1} = 1$, we can apply this result with $\mathfrak{X} = \mathcal{C}(S)$ and $\mathfrak{X}^* = \mathcal{M}(S)$ (see [1]) to conclude that there exists an increasing sequence $t_n$ of time such that $\lambda(t_n;\lambda_0) \to \overline{\lambda}$ in the weak* topology of $\mathcal{M}(S)$, namely, $\overline{\lambda} \in \text{w}^*\text{-}\omega(\lambda_0) \neq \emptyset$. □

**5.2 Some Heuristic Arguments.** To know what is the weak* $\omega$-limit lying in w*-$\omega(\lambda_0)$ is, of course, a very important problem. As mentioned, our numerical results suggest that the limit function might be given by the form $\sum_{k=1}^K \alpha_k \delta_k(x)$, where $0 < \alpha_k < 1$ with $\sum_{k=1}^K \alpha_k = 1$ and $\delta_k(x)$ is a Dirac delta function on $S$ with center $x_k \in S$. But it is very hard to prove those evidences analytically. In this subsection, we assume that $\mu = 1$ and try to conclude that, if $\big(\lambda(x), w(x), G(x)\big)$ is a stationary solution to (1.1) such that $\lambda(x) \in \mathcal{M}(S)$ and $w(x)$, $G(x) \in \mathcal{C}(S)$, then either $\lambda(x)$ is in $L^1(S)$ even more is identically equal to $\frac{1}{2\pi}$ or $\lambda(x)$ is given by the form $\sum_{k=1}^K \alpha_k \delta_k(x)$. But the arguments will be rather heuristic.

Let us first apply the Lebesgue decomposition theorem (see [5]) to $\lambda(x)$, and assume that $\lambda(x) = \lambda_\infty(x) + \sum_{k=1}^K \alpha_k \delta_k(x)$, where $\lambda_\infty(x) \in L^1(S)$ and $\delta_k(x)$ is a Dirac delta function with center $x_k \in S$. Putting

$$\begin{aligned}\overline{\omega} &= \int_S \omega(y)\lambda(y)dy \\ &= \int_S \omega(y)\lambda_\infty(y)dy + \sum_{k=1}^{K} \alpha_k \omega(x_k),\end{aligned}$$

we decompose $S$ into three parts $S_- = \{x \in S;\ \omega(x) < \overline{\omega}\}$, $S_0 = \{x \in S;\ \omega(x) = \overline{\omega}\}$ and $S_+ = \{x \in S;\ \omega(x) > \overline{\omega}\}$. If $\lambda(x) > 0$ at some point $x \in S_-$, then $\lambda$ must be strictly decreasing in $t$ at this point, which contradicts the stationariness of $\lambda(x)$; thereby, $\lambda(x) = 0$ for every $x \in S_-$. Similarly, if $\lambda(x) > 0$ at some point $x \in S_+$, then $\lambda$ must be strictly increasing in $t$ at this point, which again contradicts the stationariness of $\lambda(x)$, i.e., $\lambda(x) = 0$ for every $x \in S_+$. Therefore, since $\lambda(x_k) = \infty$, all the centers $x_k$ belong to $S_0$.

Let now $x$ vary in $S_0$. Since $\omega(x) \equiv \overline{\omega}$ on $S_0$, $w(x) = \overline{\omega}G(x)$ for all $x \in S_0$ (due to $\mu = 1$). Moreover, we obtain from (1.1) that

$$\begin{aligned}G(x)^\sigma &= \overline{\omega}^{1-\sigma} \int_{S_0} \lambda(y)G(y)^\sigma e^{-\tau(\sigma-1)|x-y|}dy \\ &= \overline{\omega}^{1-\sigma}\left[\sum_{k=1}^{K} \alpha_k G(x_k)^\sigma e^{-\tau(\sigma-1)|x-x_k|} + \int_{S_0} \lambda_\infty(y)G(y)^\sigma e^{-\tau(\sigma-1)|x-y|}dy\right], \quad x \in S_0,\end{aligned}$$

and

$$\begin{aligned}G(x)^{1-\sigma} &= \overline{\omega}^{1-\sigma} \int_{S_0} \lambda(y)G(y)^{1-\sigma} e^{-\tau(\sigma-1)|x-y|}dy \\ &= \overline{\omega}^{1-\sigma}\left[\sum_{k=1}^{K} \alpha_k G(x_k)^{1-\sigma} e^{-\tau(\sigma-1)|x-x_k|} + \int_{S_0} \lambda_\infty(y)G(y)^{1-\sigma} e^{-\tau(\sigma-1)|x-y|}dy\right], \quad x \in S_0.\end{aligned}$$

This means that both $G(x)^\sigma$ and $G(x)^{1-\sigma}$ satisfy the same integral equation

$$u(x) = \overline{\omega}^{1-\sigma}\left[\sum_{k=1}^{K} \alpha_k u(x_k) e^{-\tau(\sigma-1)|x-x_k|} + \int_{S_0} \lambda_\infty(y)u(y) e^{-\tau(\sigma-1)|x-y|}dy\right], \quad x \in S_0.$$

We further assume that this integral equation has a unique non-trivial solution. Then, $G(x)^\sigma \equiv CG(x)^{1-\sigma}$ on $S_0$ with some constant $C > 0$, and hence $G(x) \equiv C^{1/(2\sigma-1)}$ must also be constant on $S_0$. Therefore,

$$\sum_{k=1}^{K} \alpha_k e^{-\tau(\sigma-1)|x-x_k|} + \int_{S_0} \lambda_\infty(y) e^{-\tau(\sigma-1)|x-y|}dy \equiv \overline{\omega}^{\sigma-1} \qquad \text{on } S_0.$$

Then, it may possibly hold that

$$\sum_{k=1}^{K} \alpha_k e^{-\tau(\sigma-1)|x-x_k|} \equiv \overline{\omega}_1 \quad \text{and} \quad \int_{S_0} \lambda_\infty(y) e^{-\tau(\sigma-1)|x-y|}dy \equiv \overline{\omega}_2 \qquad \text{on } S_0$$

with some constants $\overline{\omega}_1$ and $\overline{\omega}_2$ such that $\overline{\omega}_1 + \overline{\omega}_2 = \overline{\omega}^{\sigma-1}$.

In order that the first functional equivalence holds true, it is necessary that either $S_0 = \{x_1, x_2, \ldots, x_K\}$ (i.e. $\lambda_\infty(x) \equiv 0$), together with

$$\sum_{k=1}^{K} \alpha_k e^{-\tau(\sigma-1)|x_j-x_k|} \equiv \overline{\omega}_1 \qquad \text{for } j = 1, 2, \ldots, K,$$

or $\alpha_1 = \alpha_2 = \cdots = \alpha_K = 0$ (i.e., $\lambda(x) = \lambda_\infty(x)$). In the former case, it trivially holds that

$$\int_{S_0} \lambda_\infty(y) e^{-\tau(\sigma-1)|x-y|} dy \equiv 0$$

and $\overline{\omega}_1 = \overline{\omega}^\sigma - 1$. In the latter case, the measure of $S_0$ cannot vanish (otherwise $\lambda(x) = 0$ almost everywhere on $S$) and the functional equivalence

$$\int_{S_0} \lambda(y) e^{-\tau(\sigma-1)|x-y|} dy \equiv \overline{\omega}^{\sigma-1}$$

takes place on $S_0$. Therefore, $S_0$ must coincide with $S$ together with $\lambda(x) \equiv \frac{1}{2\pi}$. As noticed above, $S_0 = S$ implies that $w(x)$ and $G(x)$ are also constant on $S$, i.e., $\big(\lambda(x), w(x), G(x)\big)$ is a homogeneous stationary solution.

**5.3 Linearized Principle.** We prove that the homogeneous stationary solutions to (1.1) are all unstable. Throughout this subsection, $\phi(x)$ is assumed to be $\phi(x) \equiv \frac{1}{2\pi}$ on $S$.

Let us first notice that, when $0 < \mu < 1$, (1.1) has a unique homogeneous stationary solution given by

$$\begin{cases} \lambda(x) \equiv \overline{\lambda} = \dfrac{1}{2\pi} & \text{on } S, \\ w(x) \equiv \overline{w} = 1 & \text{on } S, \\ G(x) \equiv \overline{G} = \left\{ \dfrac{1 - e^{-\tau(\sigma-1)\pi}}{\tau(\sigma-1)\pi} \right\}^{\frac{1}{1-\sigma}} & \text{on } S, \end{cases}$$

On the other hand, when $\mu = 1$, (1.1) has an infinite number of homogeneous stationary solutions that are given by

$$\begin{cases} \lambda(x) \equiv \overline{\lambda} = \dfrac{1}{2\pi} & \text{on } S, \\ w(x) \equiv \overline{w} & \text{on } S, \\ G(x) \equiv \overline{G} = \overline{w} \left\{ \dfrac{1 - e^{-\tau(\sigma-1)\pi}}{\tau(\sigma-1)\pi} \right\}^{\frac{1}{1-\sigma}} & \text{on } S, \end{cases}$$

where $\overline{w} > 0$ is an arbitrary positive number.

Let $(\overline{\lambda}, \overline{w}, \overline{G})$ be a homogeneous stationary of (1.1). In a neighborhood of this solution, put the unknown functions $\Delta\lambda = \lambda - \overline{\lambda}$, $\Delta w = w - \overline{w}$ and $\Delta G = G - \overline{G}$, and rewrite (1.1). Using the notations introduced in Subsection 3.2, we have

$$\begin{cases} \overline{w} + \Delta w(t) = \Phi\big(\overline{w} + \Delta w(t), \overline{\lambda} + \Delta\lambda(t)\big), \\ \dfrac{d}{dt}\Delta\lambda(t) = \Psi\big(\overline{w} + \Delta w(t), \overline{\lambda} + \Delta\lambda(t)\big). \end{cases}$$

Since

$$\begin{cases} \Phi(\overline{w} + \Delta w, \overline{\lambda} + \Delta\lambda) = \Phi(\overline{w}, \overline{\lambda}) + \Phi_w(\overline{w}, \overline{\lambda})\Delta w + \Phi_\lambda(\overline{w}, \overline{\lambda})\Delta\lambda + o(\|\Delta w\| + \|\Delta\lambda\|), \\ \Psi(\overline{w} + \Delta w, \overline{\lambda} + \Delta\lambda) = \Psi(\overline{w}, \overline{\lambda}) + \Psi_w(\overline{w}, \overline{\lambda})\Delta w + \Psi_\lambda(\overline{w}, \overline{\lambda})\Delta\lambda + o(\|\Delta w\| + \|\Delta\lambda\|), \end{cases}$$

and since $\Phi(\overline{w}, \overline{\lambda}) = \overline{w}$ and $\Psi(\overline{w}, \overline{\lambda}) = 0$, we get the linearized equation at $(\overline{\lambda}, \overline{w}, \overline{G})$ which is given by

$$\begin{cases} [I - \Phi_w(\overline{w}, \overline{\lambda})]\Delta w(t) = \Phi_\lambda(\overline{w}, \overline{\lambda})\Delta\lambda(t), \\ \dfrac{d}{dt}\Delta\lambda(t) = \Psi_w(\overline{w}, \overline{\lambda})\Delta w(t) + \Psi_\lambda(\overline{w}, \overline{\lambda})\Delta\lambda(t). \end{cases}$$

Furthermore, we introduce the Fourier expansions

$$\Delta w(t,x) = \sum_{n=-\infty}^{\infty} W_n(t)e^{in\xi} \quad \text{and} \quad \Delta\lambda(t,x) = \sum_{n=-\infty}^{\infty} \varLambda_n(t)e^{in\xi}, \quad \xi \in [-\pi,\pi],$$

where $x = e^{i\xi}$. Then, thanks to Lemma 5.1 below, it follows that

$$\begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \left[1 - \frac{\mu}{\sigma}X_n - \left\{\frac{\mu(\sigma-1)}{\sigma} + \frac{(1-\mu)(\sigma-1)}{\sigma}\overline{w}^{-1}\right\} X_n^2\right] W_n(t)e^{inx} \\ \displaystyle\qquad = \sum_{n=-\infty}^{\infty} 2\pi \left[\frac{\mu}{\sigma}\overline{w}X_n - \left\{\frac{\mu}{\sigma}\overline{w} + \frac{1-\mu}{\sigma}\right\} X_n^2\right] \varLambda_n(t)e^{inx}, \\ \displaystyle\frac{d}{dt}\sum_{n=-\infty}^{\infty} \varLambda_n(t)e^{inx} = \sum_{n=-\infty}^{\infty} \overline{G}^{-\mu}\left\{\frac{1-\mu X_n}{2\pi}W_n(t) + \frac{\mu\overline{w}}{\sigma-1}X_n\varLambda_n(t)\right\} e^{inx} \end{cases}$$

where

$$X_n = \frac{\tau^2(\sigma-1)^2\left(1+(-1)^{|n|+1}e^{-\tau(\sigma-1)\pi}\right)}{\left(\tau^2(\sigma-1)^2+n^2\right)\left(1-e^{-\tau(\sigma-1)\pi}\right)}.$$

Eliminating $W_n(t)$, it is finally reduced to

$$\frac{d}{dt}\sum_{n=-\infty}^{\infty} \varLambda_n(t)e^{in\xi} = \sum_{n=-\infty}^{\infty} J_n\varLambda_n(t)e^{in\xi},$$

where the coefficients $J_n$ are calculated as

$$J_n = \overline{G}^{-\mu}\left\{\frac{\left(\frac{\mu}{\sigma}\overline{w} + \frac{1-\mu}{\sigma}\right)X_n^2 - \frac{\mu}{\sigma}\overline{w}X_n}{\left\{\frac{\mu(\sigma-1)}{\sigma} + \frac{(1-\mu)(\sigma-1)}{\sigma}\overline{w}^{-1}\right\}X_n^2 + \frac{\mu}{\sigma}X_n - 1}(1-\mu X_n) + \frac{\mu\overline{w}}{\sigma-1}X_n\right\} \quad (n \neq 0).$$

Note that $\int_S \Delta\lambda(t)dx = 0$ implies $\varLambda_0(t) = 0$. We are now in a position to state the theorem.

**Theorem 5.2.** *When $0 < \mu < 1$, there is an integer $N > 0$ such that $J_n > 0$ for any $|n| \geq N$. When $\mu = 1$, $J_n > 0$ for any $n \neq 0$. Therefore, $(\overline{\lambda}, \overline{w}, \overline{G})$ is unstable in both cases.*

*Proof.* It is easy to see that $J_n \lesseqgtr 0$ if and only if $X_n \gtreqless \frac{\mu(2\sigma-1)}{\sigma-1+\sigma\mu^2}$. Since $\lim_{|n|\to\infty} X_n = 0$, there is an $N > 0$ for which $X_n < \frac{\mu(2\sigma-1)}{\sigma-1+\sigma\mu^2}$ holds true for any $|n| \geq N$. In particular, when $\mu = 1$, $X_n < 1$ implies $J_n > 0$. Meanwhile, $X_n < 1$ is always the case. □

**Remark.** Theorem 5.2 has already been announced in [9] but without full proof. We here wrote a sketch of the whole proof.

**Lemma 5.1.** *Let $u \in L^1(S)$ and let $u(x) = \sum_{n=-\infty}^{\infty} u_n e^{in\xi}$ be its Fourier expansion, where $x = e^{i\xi}$ for $x \in S$ and $\xi \in [-\pi,\pi]$. The Fourier expansion of the function $\int_S u(y)e^{-\tau(\sigma-1)|x-y|}dy$ is given by*

$$\int_S u(y)e^{-\tau(\sigma-1)|x-y|}dy = 2\pi\sum_{n=-\infty}^{\infty} Y_n u_n e^{in\xi}, \tag{5.1}$$

*where*

$$Y_n = \frac{\tau(\sigma-1)\left(1+(-1)^{|n|+1}e^{-\tau(\sigma-1)\pi}\right)}{\pi\left(\tau^2(\sigma-1)^2+n^2\right)}. \tag{5.2}$$

*Proof.* The function $\int_S u(y)e^{-\tau(\sigma-1)|x-y|}dy$ is none other than the convolution of $u(z)$ and $e^{-\tau(\sigma-1)|z|}$ over $S$. Thereby, since Fourier coefficients for $e^{-\tau(\sigma-1)|z|}$, $z \in [-2\pi, 2\pi]$ is calculated as (5.2), the Fourier expansion of the convolution is given by (5.1). $\square$

**6 Numerical Results.** This section is devoted to illustrating some examples of numerical computations.

**6.1 Numerical Methods.** Identifying $S$ with the interval $[-\pi, \pi]$ with mod $2\pi$, let us discretize the variable $x \in S$ into $I$ nodal points $x_i = -\pi + (i-1)\Delta x$, $i = 1, 2, \dots, I$, where $\Delta x = \frac{2\pi}{I}$. Temporal variable $t \in [0, \infty)$ is discretized by $t_n = (n-1)\Delta t$, $n = 1, 2, 3, \dots$ with $\Delta t > 0$. Approximate values of $\lambda(t_n, x_i)$, $w(t_n, x_i)$, $G(t_n, x_i)$ and $\phi(x_i)$ are denoted by $\lambda_i^n$, $w_i^n$, $G_i^n$ and $\phi_i$, respectively. We use also the notations $\lambda^n = [\lambda_1^n, \lambda_2^n, \dots, \lambda_I^n]$, $w^n = [w_1^n, w_2^n, \dots, w_I^n]$, $G^n = [G_1^n, G_2^n, \dots, G_I^n]$ and $\phi = [\phi_1, \phi_2, \cdots, \phi_I]$.

We remember that there are two problems to be discretized. One is the fixed point problem (3.2), and the other is the evolution problem (4.15).

As for (3.2), we use the iteration method. Indeed, let $\lambda^n$ be given and define a mapping $\widehat{\Phi}_\sigma(\cdot, \lambda^n) : \mathbb{R}^I \to \mathbb{R}^I$ by

$$\left[\widehat{\Phi}_\sigma(W, \lambda^n)\right]_i = \sum_{j=1}^{I} \frac{\mu\lambda_j^n + (1-\mu)\phi_j}{\sum_{l=1}^{I} \lambda_l^n W_l^{1/\sigma - 1} e^{-(\sigma-1)\tau|x_j - y_l|}\Delta x} e^{-(\sigma-1)\tau|x_i - y_j|}\Delta x,\ i = 1, 2, \cdots, I,$$
$$W = [W_1, W_2, \dots, W_I] \in \mathbb{R}^I.$$

Then, the iteration is given by

$$\begin{cases} W^{(1)} = [w^{n-1}]^\sigma, \\ W^{(k+1)} = \widehat{\Phi}_\sigma(W^{(k)}, \lambda^n) \qquad (k = 1, 2, \dots) \end{cases} \tag{6.1}$$

This iteration will be stopped if $\|W^{(k+1)} - W^{(k)}\|_\infty \le \varepsilon_1$, where, $\varepsilon_1 > 0$ is a positive number that is a priori fixed. Then, we denote $W^{(k)}$ by $W^{(k)} = \widehat{\Phi}_{\sigma,\mathrm{f}}(\lambda^n)$, and $w^n$ will be set by $w_i^n = \left(\left[\widehat{\Phi}_{\sigma,\mathrm{f}}(\lambda^n)\right]_i\right)^{1/\sigma}$, $i = 1, 2, \cdots, I$. Furthermore, $G^n$ is determined from $w^n$ by the formula

$$G_i^n = \left[\sum_{j=1}^{N} \lambda_j^n \left(w_j^n\right)^{1-\sigma} e^{-(\sigma-1)\tau|x_i - y_j|}\Delta x\right]^{\frac{1}{1-\sigma}}, \quad i = 1, 2, \dots, I.$$

By using $w^n$, $G^n$ thus obtained, we introduce a mapping $\hat{\omega} : \mathbb{R}^I \to \mathbb{R}^I$ by

$$[\widehat{\omega}(\lambda^n)]_i = w_i^n \left(G_i^n\right)^{-\mu}, \quad i = 1, 2, \dots, I.$$

As for (4.15), we use the explicit Runge-Kutta method of order 4. Indeed, let $\lambda^n$ be given and let $\widehat{\omega}$ be determined by the method above. Define a mapping $\widehat{\psi} : \mathbb{R}^I \to \mathbb{R}^I$ by

$$\left[\widehat{\psi}(\lambda^n)\right]_i = \left[[\widehat{\omega}(\lambda^n)]_i - \sum_{j=1}^{I}[\widehat{\omega}(\lambda^n)]_j \lambda_j^n \Delta x\right]\lambda_i^n, \quad i = 1, 2, \cdots, I,$$

and put $k_1^n,\ k_2^n,\ k_3^n,\ k_4^n$ as

$$\begin{cases} k_1^n = \widehat{\psi}\left(\lambda^n\right), \\ k_2^n = \widehat{\psi}\left(\lambda^n + \dfrac{\Delta t}{2} k_1^n\right), \\ k_3^n = \widehat{\psi}\left(\lambda^n + \dfrac{\Delta t}{2} k_2^n\right), \\ k_4^n = \widehat{\psi}\left(\lambda^n + \Delta t k_3^n\right), \end{cases}$$

respectively, then

$$\begin{cases} \lambda^0 = [\lambda_0(x_1), \lambda_0(x_2), \dots, \lambda_0(x_I)], \\ \lambda^{n+1} = \lambda^n + \dfrac{\Delta t}{6}\Big[k_1^n + 2k_2^n + 2k_3^n + k_4^n\Big], \qquad (n = 0, 1, 2, \dots). \end{cases}$$

As before, this procedure will be stopped if $\|\lambda^{n+1} - \lambda^n\|_\infty \le \varepsilon_2$, where $\varepsilon_2 > 0$ is an a priori fixed positive number. We then take $\lambda^n$ as an approximation of the weak* $\omega$-limit of the global solution starting from the initial function $\lambda_0(x)$.

**6.2 Setting Up.** In our computations, we will set $\mu = 0.5$ and $\phi(x) \equiv \frac{1}{2\pi}$ on $S$. The initial function $\lambda_0(x)$ is a small perturbation of the homogeneous distribution $\frac{1}{2\pi}$. For discretization scheme, we set $I = 1024$, $\Delta t = 0.01$, $\varepsilon_1 = \varepsilon_2 = 10^{-10}$.

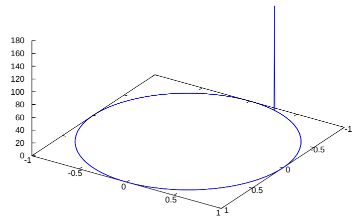
The exponents $\sigma$ and $\tau$ are treated as control parameters. We know that the conditions given by (4.12) guarantee convergence of the iteration scheme to solve the fixed point problem (3.2). Naturally, those may guarantee convergence of the approximated scheme (6.1). The pairs $(\sigma, \tau)$ we will pick up here do not necessarily satisfy those conditions. But the stopping criterion $\|W^{(k+1)} - W^{(k)}\|_\infty \le \varepsilon_1$ can actually be fulfilled by some suitable iteration times $k$.

**6.3 Numerical Examples.** The following Figures illustrate $\overline{\lambda}(x)$ and $\overline{\omega}(x) = \overline{w}(x)\overline{G}(x)^{-\mu}$ of stationary solutions $\big(\overline{\lambda}(x), \overline{w}(x), \overline{G}(x)\big)$ to (1.1) obtained numerically.

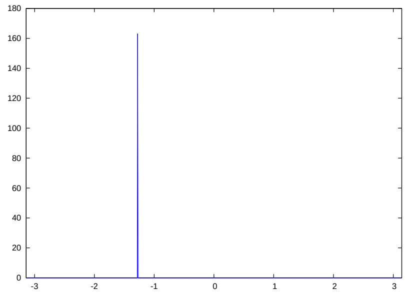
First, let $\sigma = 2.3$ and tune $\tau$ from 4 to 10. When $\tau = 4$, every solution tends to a stationary solution $\big(\overline{\lambda}(x), \overline{w}(x), \overline{G}(x)\big)$ in which $\overline{\lambda}(x)$ has only a single spike, see Figure 1. When $\tau = 4.5$, every solution still tends to a stationary solution $\big(\overline{\lambda}(x), \overline{w}(x), \overline{G}(x)\big)$ but $\overline{\lambda}(x)$ has at most double spikes, see Figure 2. When $\tau = 10$, the maximum number of spikes increases to 4, see Figure 3.

Next, let $\sigma = 3$ and tune $\tau$ from 1 to 8.5. As Figures 4-6 illustrate, when $\tau = 1$, $\overline{\lambda}(x)$ has only a single spike, but as $\tau$ is enhanced, the maximum number of spikes increases up to 10. The tendency that the maximum number of spikes of $\overline{\lambda}(x)$ increases as $\tau$ is enhanced is evidently the same as in the case of $\sigma = 2.3$. But the maximum number increases earlier in $\tau$ than before.

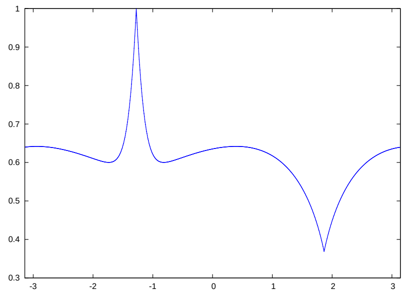
Finally, Figures 7-9 illustrate a graph of the function $\overline{\lambda}(x)$ for the cases where $\sigma = 4$ is fixed and $\tau = 0.6$, 1.5, 7. The maximum number of spikes of $\overline{\lambda}(x)$ arrives at 4 only by $\tau = 1.5$. It is quite earlier, since it was $\tau = 10$ when $\sigma = 2.3$ and $\tau = 3.5$ when $\sigma = 3$. The maximum number finally reaches 17 only by $\tau = 7$.




(a) Graph of $\overline{\lambda}(x)$ on $S$

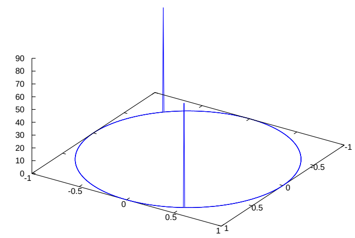



(b) Graph of $\overline{\lambda}(x)$ on $[-\pi, \pi]$

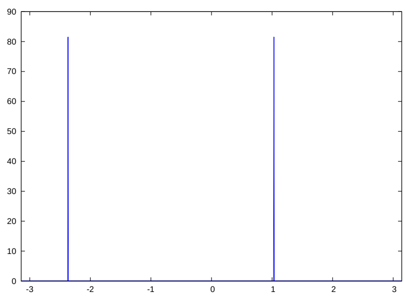



(c) Graph of $\overline{\omega}(x)$ on $[-\pi, \pi]$

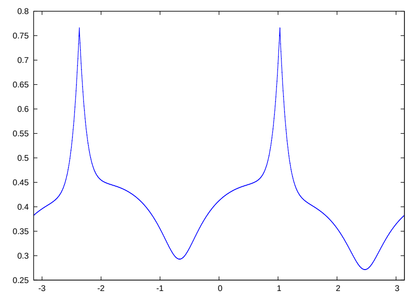
Fig. 1: Case where $\sigma = 2.3$, $\tau = 4$




(a) Graph of $\overline{\lambda}(x)$ on $S$

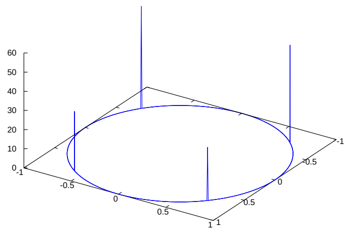



(b) Graph of $\overline{\lambda}(x)$ on $[-\pi, \pi]$

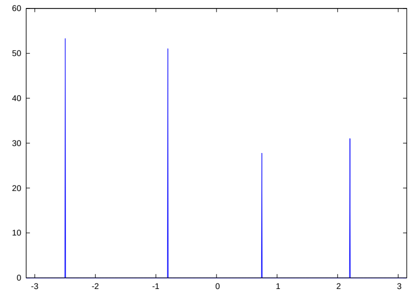



(c) Graph of $\overline{\omega}(x)$ on $[-\pi, \pi]$

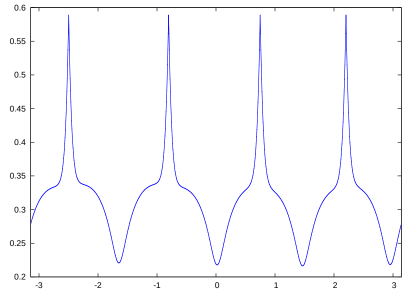
Fig. 2: Case where $\sigma = 2.3$, $\tau = 4.5$




(a) Graph of $\overline{\lambda}(x)$ on $S$

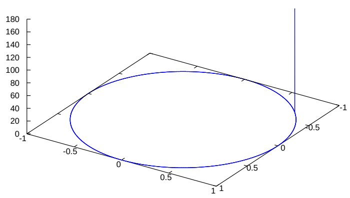



(b) Graph of $\overline{\lambda}(x)$ on $[-\pi, \pi]$

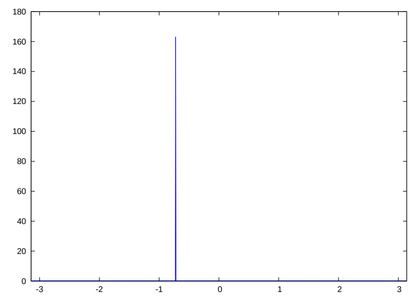



(c) Graph of $\overline{\omega}(x)$ on $[-\pi, \pi]$

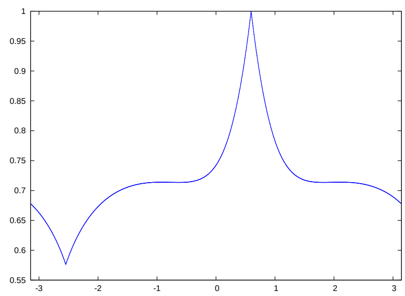
Fig. 3: Case where $\sigma = 2.3$, $\tau = 10$




(a) Graph of $\overline{\lambda}(x)$ on $S$

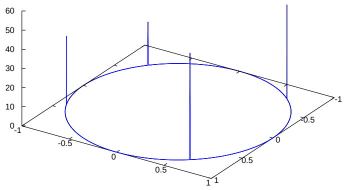



(b) Graph of $\overline{\lambda}(x)$ on $[-\pi, \pi]$

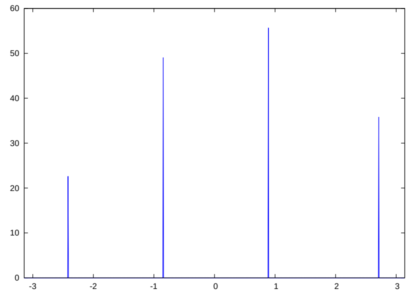



(c) Graph of $\overline{\omega}(x)$ on $[-\pi, \pi]$

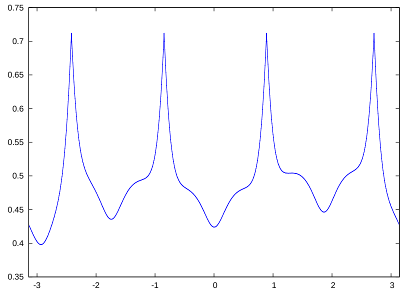
Fig. 4: Case where $\sigma = 3$, $\tau = 1$




(a) Graph of $\overline{\lambda}(x)$ on $S$

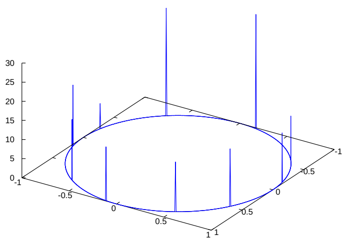



(b) Graph of $\overline{\lambda}(x)$ on $[-\pi, \pi]$

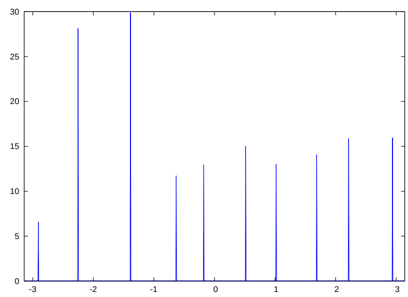



(c) Graph of $\overline{\omega}(x)$ on $[-\pi, \pi]$

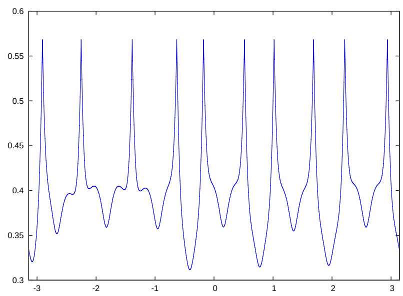
Fig. 5: Case where $\sigma = 3$, $\tau = 3.5$




(a) Graph of $\overline{\lambda}(x)$ on $S$

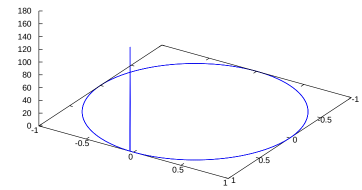



(b) Graph of $\overline{\lambda}(x)$ on $[-\pi, \pi]$

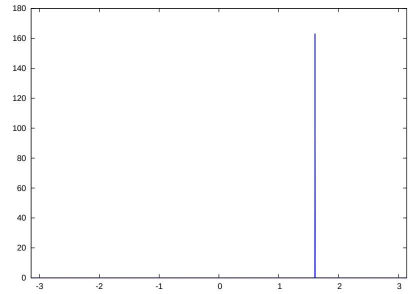



(c) Graph of $\overline{\omega}(x)$ on $[-\pi, \pi]$

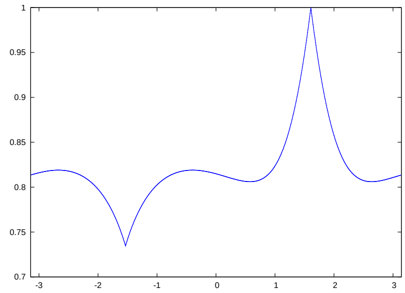
Fig. 6: Case where $\sigma = 3$, $\tau = 8.5$



(a) Graph of $\overline{\lambda}(x)$ on $S$

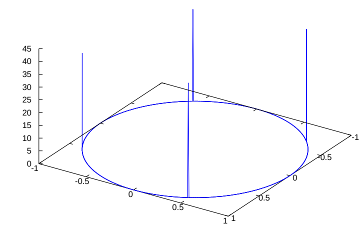



(b) Graph of $\overline{\lambda}(x)$ on $[-\pi, \pi]$

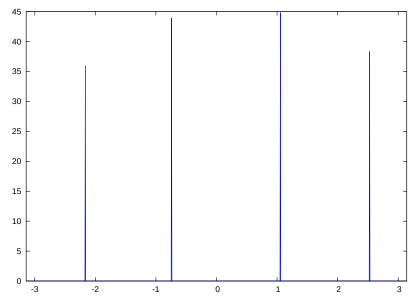



(c) Graph of $\overline{\omega}(x)$ on $[-\pi, \pi]$

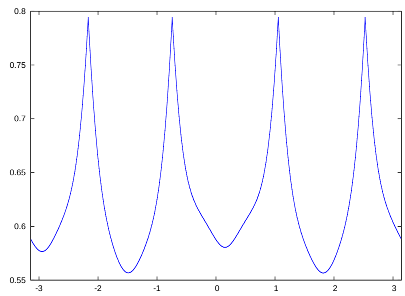
Fig. 7: Case where $\sigma = 4$, $\tau = 0.6$



(a) Graph of $\overline{\lambda}(x)$ on $S$

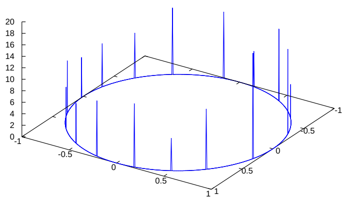



(b) Graph of $\overline{\lambda}(x)$ on $[-\pi, \pi]$

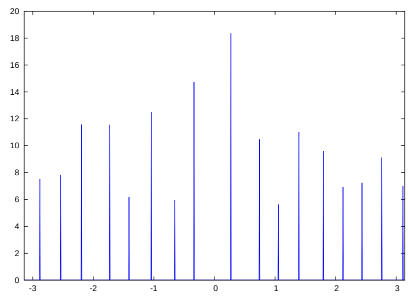



(c) Graph of $\overline{\omega}(x)$ on $[-\pi, \pi]$

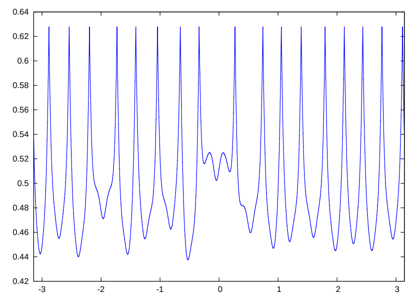
Fig. 8: Case where $\sigma = 4$, $\tau = 1.5$




(a) Graph of $\overline{\lambda}(x)$ on $S$




(b) Graph of $\overline{\lambda}(x)$ on $[-\pi, \pi]$




(c) Graph of $\overline{\omega}(x)$ on $[-\pi, \pi]$

Fig. 9: Case where $\sigma = 4$, $\tau = 7$

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