On generalized digital lines *

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Abstract. In the present paper, we introduce and study the concept of *generalized digital lines*, say $(\mathbb{Z}, \kappa(q,n))$, where q and n are positive integers with $2 \leq q < n$ and $n \not\equiv 0 \pmod q$; especially, for $q = 2$ and $n = 3$, $(\mathbb{Z}, \kappa(2,3))$ is identical with the digital line $(\mathbb{Z}, \kappa)$ (=the Khalimsky line due to E.D. Khalimsky).

1 Introduction and preliminaries *The Khalimsky line* or so called *the digital line* is the set $\mathbb{Z}$ of integers equipped with the topology κ having $\mathcal{G}_\kappa := \{\{2m-1, 2m, 2m+1\} \mid m \in \mathbb{Z}\}$ as a subbase ([25]: e.g. [26], [27, p.905, p.906], [28, Definition 2, p.175], [10, Example 4.6, p.23], [8, p.50], [13, p.164], [14, p.31], [44, p.601], [43, p.46], [18, p.926], [37, Example 2.4], [19, p.1034, p.1035], [36, Section 3(I)]). In 1970, the concept of the digital line was published by Khalimsky [25] above from Russia. In 1990, Khalimsky, Kopperman and Meyer [26] investigated the concepts of *connected ordered topological spaces*, *digital planes* and a proof of digital Jordan closed curve theorem using purely digital topological methods (cf. the references of [26], [27]). The digital line is denoted by $(\mathbb{Z}, \kappa)$. Roughly speaking, $(\mathbb{Z}, \kappa)$ has a covering $\mathcal{G}_\kappa$ by infinitely many open subsets which are three points subset $\{2m-1, 2m, 2m+1\}$, where $m \in \mathbb{Z}$, and two adjacent open sets $\{2m-1, 2m, 2m+1\}$ and $\{2m+1, 2m+2, 2m+3\}$ are connected with a singleton $\{2m+1\}$ as their intersection of two such open subsets. For any integer m, the singleton $\{2m+1\}$ is open in $(\mathbb{Z}, \kappa)$ and $\{2m\}$ is closed in $(\mathbb{Z}, \kappa)$. From a point of view in general topology approaches, the digital line $(\mathbb{Z}, \kappa)$ is a typical and geometrical example of a topological space which satisfies a $T_{1/2}$ separation axiom. In 1970, Levine [31] published, from Italy, the concept of $T_{1/2}$-spaces by introducing the concept of *generalized closed subsets* [31, Definition 2.1] of a topological space; a topological space is called $T_{1/2}$ [31, Definition 5.1] if every generalized closed set is closed. The class of $T_{1/2}$-spaces is properly placed between the classes of T_0- and T_1-spaces [31, Corollary 5.6]. In 1977, Dunham [11, Theorem 2.5] proved that a topological space (X, τ) is $T_{1/2}$ if and only if each singleton $\{x\}$ is open or closed in (X, τ), where $x \in X$. Therefore, we know that $(\mathbb{Z}, \kappa)$ is $T_{1/2}$ (cf. [26, p.7], [10, Example 4.6]). In 1996, Dontchev and Ganster [10] investigated the class of $T_{3/4}$-spaces which is properly placed between the classes of T_1- and $T_{1/2}$-spaces; and the authors proved that $(\mathbb{Z}, \kappa)$ is $T_{3/4}$ [10, Example 4.6].

The purpose of the present paper is to construct *generalized digital lines*, say $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2 below) and investigate its fundamental properties (cf. Theorem A below and related properties).

Throughout the present paper, (X, τ) represents a nonempty topological space on which no separation axioms are assumed unless otherwise mentioned and $P(X)$ denotes the power set of X.

Theorem A *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line in the sense of* Definition 2.2,*where the integers* q *and* n *satisfy the following conditions:* $2 \leq q < n$ *and* $n \not\equiv 0 \pmod q$, *say* $n \equiv r \pmod q$ $(1 \leq r \leq q-1)$. *Then, we have the following fundamental properties.*

(i) $\kappa(q,n) \neq P(\mathbb{Z})$ *holds*;

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(ii) (ii-1) *if* $2 \leq r$, *then* $(\mathbb{Z}, \kappa(q,n))$ *is pre-*T_2; (ii-2) *if* $r = 1$, *then* $(\mathbb{Z}, \kappa(q,n))$ *is semi-*T_2*; especially if* $q = 2$, *then* $(\mathbb{Z}, \kappa(q,n))$ *is* $T_{3/4}$;
(iii) $(\mathbb{Z}, \kappa(q,n))$ *is connected.*

The proof of Theorem A(i) (resp. (ii), (iii)) is shown in Section 5 (resp. Section 6, Section 7). When $q = 2$ and $n = 3$, then we see $(\mathbb{Z}, \kappa(2,3)) = (\mathbb{Z}, \kappa)$ (cf. Remark 2.3).

In the present paper, sometimes, we use the following notation:

Notation. For integers $a, b \in \mathbb{Z}$ with $a \leq b$, $[a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ (by [6], this set is called a *digital interval* if $a \lneq b$). For a set A, we denote by $|A|$ the cardinality of A (e.g. Lemma 2.8, Proof of Theorem 5.1(ii)).

2 Open sets and classifications of generalized digital lines

Definition 2.1 Let n and q be given two positive integers. Let $\mathcal{G}(q,n) := \{B_k(q,n) \mid k \in \mathbb{Z}\}$ be the family of subsets $B_k(q,n)$ of Z, where $k \in \mathbb{Z}$ and $B_k(q,n) := \{kq+i \in \mathbb{Z} \mid 1 \leq i \leq n\}$.

Definition 2.2 (the generalized digital line) Suppose that the following conditions: $2 \leq q < n$ and $n \equiv r \pmod q$ $(1 \leq r \leq q-1)$ hold for the integers q and n in Definition 2.1 above. Then, a *generalized digital line* is the set of the integers, $\mathbb{Z}$, equipped with the topology $\kappa(q,n)$ having $\mathcal{G}(q,n)$ as a subbase. It is denoted by $(\mathbb{Z}, \kappa(q,n))$.

Remark 2.3 In Definition 2.2 above, let $q = 2$ and $n = 3$. Then, for each $k \in \mathbb{Z}, B_k(2,3) = \{2(k+1)-1, 2(k+1), 2(k+1)+1\}$ and the space $(\mathbb{Z}, \kappa(2,3))$ coincides with the digital line $(\mathbb{Z}, \kappa)$ (cf. [26], e.g. [10], Section 1 above).

We investigate the smallest open set (resp. closed set) containing a point of $(\mathbb{Z}, \kappa(q,n))$.

Definition 2.4 For a subset A of a topological space (X, τ),
(i) $Ker(A) := \bigcap\{U \mid A \subset U, U \in \tau\}$, (e.g. in [35, Definition 2.1], $Ker(A)$ is denoted by A^{Λ});
(ii) $Cl(A) := \bigcap\{F \mid A \subset F, F$ is closed in $(X,\tau)\}$.

Definition 2.5 Let (X, τ) be a topological space, A and B subsets of (X, τ) and $x \in X$.
(i) A is called *the smallest open set containing* x if $x \in A, A \in \tau$ and $G = A$ holds for any open set G such that $x \in G$ and $G \subset A$. The uniqueness of the smallest open sets is assured by Remark 2.6(i) below.
(ii) B is called *the smallest closed set containing* x, if $x \in B, X \setminus B \in \tau$ and $F = B$ holds for any closed set F such that $x \in F$ and $F \subset B$.

Remark 2.6 (i) If subsets A and B are the smallest open subsets containing $x \in X$, then $A = B$.
(ii) For an open subset A of X and a point $x \in A$, the following properties are equivalent:
(1) A is the smallest open set containing x;
(2) for any open set U containing $x, A \subset U$ holds.

Lemma 2.7 *Let* (X, τ) *be a topological space and* $A \subset X, x \in X$.
(i) *If* A *is the smallest open set containing* x, *then* $Ker(\{x\}) = A$ *holds.*
(ii) *If* $Ker(\{x\}) = A$ *and* $A \in \tau$, *then* A *is the smallest open set containing* x.
(iii) A *is the smallest closed set containing* x *if and only if* $Cl(\{x\}) = A$ *holds.* □

Lemma 2.8 *Let* X *be a set and* $\mathcal{G} = \{V_i| \ i \in \mathcal{A}\}$ *be a collection of subsets of* X. *Let* (X, τ) *be a topological space, where* τ *is the topology having* $\mathcal{G}$ *as subbase. Suppose that, for each point* $w \in X$, *the collection* $\{V \mid V \in \mathcal{G}, w \in V\}$*:=*$\mathcal{G}_w$ *is a finite subcollection of* $\mathcal{G}$*,i.e.,* $|\mathcal{G}_w| < \infty$. *Then, for a point* $x \in X$ *and a subset* $A \subset X$, *the following properties on* $Ker(\{x\}), Cl(\{x\})$ *and* $Cl(A)$ *hold.*

(i) $Ker(\{x\}) = \bigcap\{V \mid V \in \mathcal{G}, x \in V\}(= \bigcap\{V \mid V \in \mathcal{G}_x\})$ *and it is the smallest open set containing* x.

(ii) *Moreover, suppose that* $Ker(\{x\}) \cap Ker(\{y\}) = \emptyset$ *or* $Ker(\{x\}) = Ker(\{y\})$ *hold for any distinct points* x, y *of* X.

Then, $Cl(\{x\}) = Ker(\{x\})$.

(iii) $Cl(A) = X \setminus U_A$, *where* $U_A = \{y \in X \mid Ker(\{y\}) \cap A = \emptyset\}$.

Proof. **(i)** We claim that $Ker(\{x\}) \supset \bigcap\{V \mid V \in \mathcal{G}_x\}$ holds. For each open set G containing x, we are able to set $G = \bigcup\{B_i \mid i \in I\}$, where the subset B_i is a finite intersection of some elements of $\mathcal{G}$ and I is an index set. For each open set G, there exists an element $i_0 \in I$ such that $x \in B_{i_0}$ and $B_{i_0} = \bigcap\{V_j \mid V_j \in \mathcal{G}_x, j \in J\}$ for some finite set $J \subset \mathcal{A}$. Then, we have $G \supset B_{i_0} \supset \bigcap\{V \mid V \in \mathcal{G}_x\} \ni x$ and so $Ker(\{x\}) \supset \bigcap\{V \mid V \in \mathcal{G}_x\}$. Conversely, the implication $Ker(\{x\}) \subset \bigcap\{V \mid V \in \mathcal{G}_x\}$ is easily proved. Thus we have that $Ker(\{x\}) = \bigcap\{V \mid V \in \mathcal{G}_x\}$ holds and it is open. By Lemma 2.7 (ii), the set $Ker(\{x\})$ is the smallest open set containing x.

(ii) For a given point $x \in X$, let $F := X \setminus U$, where $U := \bigcup\{Ker(\{y\}) \mid y \notin Ker(\{x\})\}$. Then, by the assumption in (ii), $F = Ker(\{x\})$ holds. Indeed, first we show that $U \subset X \setminus Ker(\{x\})$. Let $z \in U$. Then, there exists a point $y \in X$ such that $y \notin Ker(\{x\})$ and $z \in Ker(\{y\})$. It is shown that $Ker(\{y\}) \cap Ker(\{x\}) = \emptyset$ holds; and so $z \notin Ker(\{x\})$. Thus, we have the property that $U \subset X \setminus Ker(\{x\})$. Finally, we show that $U \supset X \setminus Ker(\{x\})$, because $U := \bigcup\{Ker(\{y\}) \mid y \notin Ker(\{x\})\} \supset \bigcup\{\{y\} \mid y \notin Ker(\{x\})\} = X \setminus Ker(\{x\})$. Therefore, $U = X \setminus Ker(\{x\})$ holds, i.e., $F = Ker(\{x\})$ holds. Since $Ker(\{y\})$ is open by (i), $F := X \setminus U$ is a closed subset containing x and so $Cl(\{x\}) \subset F = Ker(\{x\})$. Conversely, we claim that $Ker(\{x\}) \subset Cl(\{x\})$. Let y be a point such that $y \notin Cl(\{x\})$. Then, there exists an open subset V_y containing y such that $V_y \cap \{x\} = \emptyset$. Since $Ker(\{y\}) \subset V_y$, we have $Ker(\{y\}) \cap \{x\} = \emptyset$ and so $Ker(\{x\}) \neq Ker(\{y\})$. Using assumption we have $Ker(\{x\}) \cap Ker(\{y\}) = \emptyset$ and hence $y \notin Ker(\{x\})$ for any $y \notin Cl(\{x\})$. Thus we conclude that $Cl(x) = Ker(\{x\})$ holds.

(iii) It is shown that $Cl(A) \subset X \setminus U_A$. Indeed, let $a \notin X \setminus U_A$. Then, $Ker(\{a\}) \cap A = \emptyset$ and so $a \notin Cl(A)$ (cf. (i) above). Conversely, let $b \notin Cl(A)$. Then, there exists an open set V containing the point b such that $V \cap A = \emptyset$. Thus, we have that $Ker(\{b\}) \cap A = \emptyset$ and so $b \notin X \setminus U_A$. This shows that $X \setminus U_A \subset Cl(A)$ holds. □

Remark 2.9 (i) The following example shows that even if A is the smallest open set containing a point x there exists a proper open subset G such that $G \subset A$. Let $(\mathbb{Z}, \kappa)$ be the digital line, $x := 0$ and $A =: \{-1, 0, 1\}$ be the smallest open set containing x. Then, $Ker(\{x\}) = A$; however, subsets $G := \{1\}, G' := \{-1\}$ are open proper subsets of A. Note that $x \notin G$ and $x \notin G'$.

(ii) The following example shows that the converse of Lemma 2.7 (i) is not true in general. Let $(\mathbb{R}, \tau)$ be the Euclidian line. A subset $A := \{0\}$ is not open; $Ker(\{0\}) = \{0\}$ holds.

Lemma 2.10 *Assume that* $2 \leq q < n$ *and* $n = sq + r$, *where* $r, s \in \mathbb{N}$ *with* $1 \leq r \leq q - 1$. *Then, a subset* $\{y \in \mathbb{Z} \mid kq + 1 \leq y \leq (k + t)q + r\}$ *is open in* $(\mathbb{Z}, \kappa(q, n))$, *where* $k \in \mathbb{Z}$ *and* $t \in \mathbb{Z}$ *with* $1 \leq t \leq s$.

Proof. Using notation above (cf. the end of Section 1), we show that $[kq+1, kq+n]_{\mathbb{Z}} \cap [(k-(s-t))q+1, (k-(s-t))q+n]_{\mathbb{Z}} = [kq+1, (k+t)q+r]_{\mathbb{Z}}$ holds, because $kq-(s-t)q+1 \leq kq+1 \leq (k-(s-t))q+n \leq kq+n$. Since $[kq+1, kq+n] \in \mathcal{G}(q, n)$ and $[(k-(s-t))q+1, (k-(s-t))q+n]_{\mathbb{Z}} \in \mathcal{G}(q, n)$ (cf. Definition 2.1), we show that $[kq+1, (k+t)q+r]_{\mathbb{Z}} \in \kappa(q, n)$ (cf. Defintion 2.2). □

Lemma 2.11 *Suppose that* $2 \leq q < n$ *for the integers* q *and* n *of the sets* $B_k(q, n) \subset \mathbb{Z}(k \in \mathbb{Z})$ *and the family* $\mathcal{G}(q, n) \subset P(\mathbb{Z})$ *in* Definition 2.1. *Let* $n = sq + r(s, r \in \mathbb{Z}$ *with*

$0 \leq r \leq q-1$). *For a point $x \in \mathbb{Z}$ and $B_{k'}(q,n) \in \mathcal{G}(q,n)$, where $k' \in \mathbb{Z}$* (cf. Definition 2.1), *the following properties hold.*

(i) *Assume that $n \equiv 0 \pmod q$. For a point $x = kq+i$, where $k, i \in \mathbb{Z}$ with $1 \leq i \leq q, x \in B_{k'}(q,n)$ if and only if $k' \in \{y \in \mathbb{Z} \mid k-(s-1) \leq y \leq k\}$.*

(ii) *Assume that $n \equiv r \pmod q$, where $0 < r \leq q-1$.*

(b1) *For a point $x = kq+i$, where $k, i \in \mathbb{Z}$ with $1 \leq i \leq r, x \in B_{k'}(q,n)$ if and only if $k' \in \{y \in \mathbb{Z} \mid k-s \leq y \leq k\}$.*

(b2) *For a point $x = kq+j$, where $k, j \in \mathbb{Z}$ with $r+1 \leq j \leq q, x \in B_{k'}(q,n)$ if and only if $k' \in \{y \in \mathbb{Z} \mid k-s+1 \leq y \leq k\}$.*

Proof. First we recall that $B_{k'}(q,n) = [k'q+1, k'q+n]_{\mathbb{Z}}$ for $k' \in \mathbb{Z}$ (cf. Definition 2.1).

(i) Suppose that $x = kq+i \in B_{k'}(q,n)$ $(1 \leq i \leq q)$ and $n = sq$, where $s \in \mathbb{Z}$. Then, $k'q+1 \leq kq+i \leq k'q+sq$ and so $kq-sq < kq-sq+i \leq k'q \leq kq+i-1 \leq kq+q-1 < kq+q$. Thus we have $k-s < k' < k+1$,i.e., $k' \in [k-s+1, k]_{\mathbb{Z}}$. Conversely, if $k' \in [k-s+1, k]_{\mathbb{Z}}$, then $kq-sq+i \leq kq-sq+q \leq k'q \leq kq \leq kq+i-1$ and so $kq+i \leq k'q+sq$ and $k'q+1 \leq kq+i$. Thus, we have $x = kq+i \in [k'q+1, k'q+sq]_{\mathbb{Z}} = [k'q+1, k'q+n]_{\mathbb{Z}} = B_{k'}(q,n)$.

(ii)(b1) Suppose that $n = sq+r$ $(0 < r \leq q-1)$ and $x = kq+i \in B_{k'}(q,n)$ $(1 \leq i \leq r)$. Then, $k'q+1 \leq kq+i \leq k'q+sq+r$ and so $kq-sq+i-r \leq k'q \leq kq+i-1$. Then, we have $kq-sq+i-(q-1) \leq kq-sq+i-r \leq k'q \leq kq+i-1$ and so $kq-sq-q < kq-sq+1-(q-1) \leq kq-sq+i-(q-1) \leq k'q \leq kq+r-1 \leq kq+(q-2) < kq+q$. Thus, we have $k' \in [k-s, k]_{\mathbb{Z}}$. Conversely, if $k' \in [k-s, k]_{\mathbb{Z}}$, then $kq-sq \leq k'q \leq kq$ and so $kq-sq+i-r \leq k'q \leq kq+i-1$. Thus, we show that $k'q+1 \leq kq+i \leq k'q+sq+r = k'q+n$ and so $x \in [k'q+1, k'q+n]_{\mathbb{Z}} = B_{k'}(q,n)$.

(b2) Suppose that $n = sq+r$ $(0 < r \leq q-1)$ and $x = kq+j \in B_{k'}(q,n)$ $(r+1 \leq j \leq q)$. Then, $k'q+1 \leq kq+j \leq k'q+sq+r$ and so $kq-sq+j-r \leq k'q \leq kq+j-1$. Thus we have $kq-sq < kq-sq+j-r \leq k'q \leq kq+j-1$ and so $kq-sq < k'q < kq+q$. Namely, we have $k' \in [k-s+1, k]_{\mathbb{Z}}$. Conversely, if $k' \in [k-s+1, k]_{\mathbb{Z}}$, then $kq-sq+q \leq k'q \leq kq$ and so $kq-sq-r+j < kq-sq+j \leq kq-sq+q \leq k'q < kq+j-1$. Thus, we show that $k'q+1 < kq+j < k'q+sq+r = k'q+n$ and so $x \in [k'q+2, k'q+n-1]_{\mathbb{Z}} \subset [k'q+1, k'q+n]_{\mathbb{Z}} = B_{k'}(q,n)$. ☐

Remark 2.12 For the generalized digital line $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2), its topology $\kappa(q,n)$ satisfies the assumptions in Lemma 2.8. Indeed, for each point $x \in \mathbb{Z}$, by Lemma 2.11, it is shown that $\mathcal{G}_x = \{B_{k'}(q,n) | \ x \in B_{k'}(q,n)\}$ is a finite subcollection of $\mathcal{G}(q,n)$. Namely, $\{k' | \ x \in B_{k'}(q,n)\}$ is a finite set for each point $x \in \mathbb{Z}$. Thus, for each point $x \in \mathbb{Z}$, we can get $Ker(\{x\}) = \bigcap\{B_{k'}(q,n) | \ x \in B_{k'}(q,n)\}$. We note that $Ker(\{x\})$ is the smallest open set containng x in $(\mathbb{Z}, \kappa(q,n))$.

We are able to determine the structure of $Ker(\{x\})$ for a point x in $(\mathbb{Z}, \kappa(q,n))$, where $q < n$, using Lemma 2.8 (i) and Remark 2.12 and also $Cl(\{x\})$ using Lemma 2.8 (iii), cf. Theorem 2.13 below.

Theorem 2.13 *Let $(\mathbb{Z}, \kappa(q,n))$ be a generalized digital line* (cf. Definition 2.2). *Assume that $n \equiv r \pmod q$, where $1 \leq r \leq q-1$. The following properties hold:*

(b1) *For a point $x = kq+i$, where $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq r$, $Ker(\{x\}) = \{y \in \mathbb{Z} \mid kq+1 \leq y \leq kq+r\}$ and it is the smallest open set containing x.*

(b2) *For a point $x = kq+j$, where $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ with $r+1 \leq j \leq q, Ker(\{x\}) = \{y \in Z \mid kq+1 \leq y \leq (k+1)q+r\}$ and it is the smallest open set containing x.*

(b1)$'$ *For a point $x = kq+i$, where $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq r$, $Cl(\{x\}) = \{y \in \mathbb{Z} \mid (k-1)q+r+1 \leq y \leq kq+q\}$ holds;*

(b2)$'$ *For a point $x = kq+j$, where $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ with $r+1 \leq j \leq q$, $Cl(\{x\}) = \{y \in \mathbb{Z} \mid kq+r+1 \leq y \leq kq+q\}$ holds.*

Proof. We recall that $2 \leq q < n, n = sq + r$ ($s, r \in \mathbb{Z}$ with $1 \leq r \leq q-1$) and the family $\mathcal{G}(q,n) := \{B_{k'}(q,n) |\ k' \in \mathbb{Z}\}$ generates the topology $\kappa(q,n)$ on $\mathbb{Z}$ and $B_{k'}(q,n) = \{y \in \mathbb{Z} \mid k'q+1 \leq y \leq k'q+n\}$ is open in $(\mathbb{Z}, \kappa(q,n))$, where $k' \in \mathbb{Z}$.

(b1) Let $x = kq + i \in \mathbb{Z}$ be a point with $1 \leq i \leq r$. We have the following property (cf. Lemma 2.11 (ii) (b1)):
$(*2)$ $x = kq+i \in [k'q+1, k'q+sq+r]_{\mathbb{Z}}$ $(1 \leq i \leq r)$ if and only if $k' \in [k-s, k]_{\mathbb{Z}}$.
Using $(*2)$ and Lemma 2.8 (i) (cf. Remark 2.12), we show that $Ker(\{x\}) = \bigcap\{B_{k'}(n,q) \mid k' \in [k-s,k]_{\mathbb{Z}}\} = \bigcap\{[(k-a)q+1, (k-a)q+sq+r]_{\mathbb{Z}} \mid a \in [0,s]_{\mathbb{Z}}\} = [kq+1, kq+r]_{\mathbb{Z}}$ and $Ker(\{x\})$ is the smallest open set containing x.

(b2) Let $x = kq + j \in \mathbb{Z}$ be a point with $r+1 \leq j \leq q$. We have the followng property (cf. Lemma 2.11 (ii)(b2)):

$(*3)$ $x = kq+j \in [k'q+1, k'q+sq+r]_{\mathbb{Z}}$ $(r+1 \leq j \leq q)$ if and only if $k' \in [k-s+1, k]_{\mathbb{Z}}$.

Using $(*3)$ and Lemma 2.8 (i) (cf. Remark 2.12), we show that $Ker(\{x\})=\bigcap\{B_{k'}(q,n) \mid k' \in [k-s+1,k]_{\mathbb{Z}}\} =\bigcap\{[(k-a)q+1, (k-a)q+sq+r]_{\mathbb{Z}} \mid a \in [0,s-1]_{\mathbb{Z}}\} =[kq+1, (k+1)q+r]_{\mathbb{Z}}$ and $Ker(\{x\})$ is the smallest open set containing x.

$(\mathbf{b1})'$ We prove $(\text{b1})'$ using Lemma 2.8 (iii). Let $U_{\{x\}} := \{y \in \mathbb{Z} \mid Ker(\{y\}) \cap \{x\} = \emptyset\}$ for given point x. For $x = kq + i$ with $1 \leq i \leq r$, we claim that

$(*)$ $U_{\{x\}} = [(k+1)q+1, +\infty)_{\mathbb{Z}} \cup (-\infty, (k-1)q+r]_{\mathbb{Z}}$, where $[d, +\infty)_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid d \leq z\}$ and $(-\infty, e]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid z \leq e\}$ for some integers $d, e \in \mathbb{Z}$.

First we show that

$(*)^1$ $[(k+1)q+1, +\infty)_{\mathbb{Z}} \cup (-\infty, (k-1)q+r]_{\mathbb{Z}} \subset U_{\{x\}}$ holds.

Let $y \in [(k+1)q+1, +\infty)_{\mathbb{Z}} \cup (-\infty, (k-1)q+r]_{\mathbb{Z}}$.

Case 1. $y \in [(k+1)q+1, +\infty)_{\mathbb{Z}}$: if $y = tq+i$ $(1 \leq i \leq r$ and $t \in \mathbb{Z}$ with $k+1 \leq t)$, then $Ker(\{y\}) = [tq+1, tq+r]_{\mathbb{Z}}$; it is shown by replacing the point y for the point x in the result of (b1) above. If $y = tq+j$ $(r+1 \leq j \leq q$ and $t \in \mathbb{Z}$ with $k+1 \leq t)$, then $Ker(\{y\}) = [tq+1, (t+1)q+r]_{\mathbb{Z}}$; it is obtained by replacing the point y for x in the result of (b2) above. Thus, we show that $x = kq+i \notin Ker(\{y\})$ $(1 \leq i \leq r)$ for this case and so $y \in U_{\{x\}}$.

Case 2. $y \in (-\infty, (k-1)q+r]_{\mathbb{Z}}$: if $y = tq+i$ $(1 \leq i \leq r$ and $t \in \mathbb{Z}$ with $t \leq k-1)$, then $Ker(\{y\}) = [tq+1, tq+r]_{\mathbb{Z}}$ (cf. the result of (b1) above). If $y = tq+j$ $(r+1 \leq j \leq q$ and $t \in \mathbb{Z}$ with $t \leq k-2)$, then $Ker(\{y\}) = [tq+1, (t+1)q+r]_{\mathbb{Z}}$ (cf. the result of (b2) above). For this case, we have $x = kq+i \notin Ker(\{y\})$ $(1 \leq i \leq r)$ and so $y \in U_{\{x\}}$.

Finally, we show the converse implication:

$(*)^2$ $U_{\{x\}} \subset [(k+1)q+1, +\infty)_{\mathbb{Z}} \cup (-\infty, (k-1)q+r]_{\mathbb{Z}}$.

Let $y \in [(k-1)q+r+1, (k+1)q]_{\mathbb{Z}}$ be any point. By the result of (b2) above, it is shown that $Ker(\{y\}) = [(k-1)q+1, kq+r]_{\mathbb{Z}}$ if $y \in [(k-1)q+r+1, kq]_{\mathbb{Z}}$. By the result of (b1) above, it is shown that $Ker(\{y\}) = [kq+1, kq+r]_{\mathbb{Z}}$ if $y \in [kq+1, kq+r]_{\mathbb{Z}}$. Moreover, if $y \in [kq+r+1, kq+q]_{\mathbb{Z}}$, we have that $Ker(\{y\}) = [kq+1, (k+1)q+r]_{\mathbb{Z}}$ holds (cf. the result of (b2) above). Thus, we show that, for these points y above, $x = kq+i \in Ker(\{y\})$ and so $y \notin U_{\{x\}}$, where $1 \leq i \leq r$. This concludes that $(*)^2$ above holds.

Using $(*)^1$ and $(*)^2$ above, we have done the proof of the claim $(*)$ above. Therefore, by Lemma 2.8 (iii) (cf. Remark 2.12), it is obtained that $Cl(\{x\}) = X \setminus U_{\{x\}} = [(k-1)q+r+1, (k+1)q]_{\mathbb{Z}}$.

$(\mathbf{b2})'$ We claim that, for a given point $x = kq + j$ $(r+1 \leq j \leq q)$,

$(**)$ $U_{\{x\}} = [(k+1)q+1, +\infty)_{\mathbb{Z}} \cup (-\infty, kq+r]_{\mathbb{Z}}$ holds, where $U_{\{x\}}$ is defined in the top of the proof of $(\text{b1})'$ above. The property $(**)$ is proved by argument similar to that in the proof of $(*)$ in $(\text{b1})'$ above. By Lemma 2.8 (iii) (cf. Remark 2.12), it is obtained that $Cl(\{x\}) = X \setminus U_{\{x\}} = [kq+r+1, (k+1)q]_{\mathbb{Z}}$. □

In the end of the present section, the following Corollary 2.14 shows the classification of families of topologies: • $\{\kappa(q,n) | n \in \mathbb{Z}$ with $2 \leq q < n$ and $n \not\equiv 0 \pmod q\}$, for a given positive integer $q \in \mathbb{Z}$ with $2 \leq q$. Throughout the proof of Corollary 2.14, the kernel of a singleton $\{x\}$ in a topological space (X, τ) also denoted by τ-$Ker(\{x\})$.

Corollary 2.14 *Let n, n' and q be positive integers such that $2 \leq q < n$, $2 \leq q < n'$, $n \not\equiv 0$ (mod q) and $n' \not\equiv 0$ (mod q). Then, $\kappa(q, n) = \kappa(q, n')$ if and only if $n \equiv n'$ (mod q).*

Proof. We denote shortly the kernel of a singleton $\{x\}$ in $(\mathbb{Z}, \kappa(q, n))$ (resp. $(\mathbb{Z}, \kappa(q, n'))$) by κ-$Ker(\{x\})$ (resp. κ'-$Ker(\{x\})$).

(Necessity) It follows from assumption that κ-$Ker(\{x\}) = \kappa'$-$Ker(\{x\})$ holds for each point $x \in \mathbb{Z}$. Let $n \equiv r$ (mod q) and $n' \equiv r'$ (mod q) for some integer r and r' with $1 \leq r \leq q-1$ and $1 \leq r' \leq q-1$. We shall show $r = r'$. First we suppose $r \leq r'$. Take a point $x := kq+i$, where $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq r$; then we have κ-$Ker(\{x\}) = [kq+1, kq+r]_{\mathbb{Z}}$ (cf. Theorem 2.13 (b1)). Since $x = kq+i$ $(1 \leq i \leq r')$, by Theorem 2.13 (b1) for the singleton $\{x\}$ in $(\mathbb{Z}, \kappa(q, n'))$ it is shown κ'-$Ker(\{x\}) = [kq+1, kq+r']_{\mathbb{Z}}$. Thus we have $r = r'$ for this first case, because κ-$Ker(\{x\}) = \kappa'$-$Ker(\{x\})$. Finally, we suppose $r' \leq r$. By the similar fashion to above first case, it is obtained that $r' = r$ for this case. Therefore, we show $r = r'$; and so we conclude that $n \equiv n'$ (mod q).

(Sufficiency) In oder to prove the sufficiency, we claim the following properties (1) and (2) of topological spaces; (2) is proved by (1).
Claim: Let (X, τ) and (X, τ') be two topological spaces.

(1) If U is an open set in (X, τ), then $U = \bigcup\{\tau\text{-}Ker(\{x\}) |\ x \in U\}$ holds.

(2) If τ-$Ker(\{x\}) \in \tau$, τ'-$Ker(\{x\}) \in \tau'$ and τ-$Ker(\{x\}) = \tau'$-$Ker(\{x\})$ hold for each point $x \in X$, then $\tau = \tau'$ and so $(X, \tau) = (X, \tau')$.

We prove the sufficiency of the present Corollary 2.14. Let $(\mathbb{Z}, \kappa(q, n))$ and $(\mathbb{Z}, \kappa(q, n'))$ be two generalized digital lines. We suppose $n \equiv r$ (mod q) and $n' \equiv r$ (mod q) for an integer r with $1 \leq r \leq q-1$. Let $x \in \mathbb{Z}$ and $x = kq+i$ for some $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq q-1$. We consider the following Case 1 and Case 2 on the point x.
Case 1. $x = kq+i$, where $1 \leq i \leq r$: by Theorem 2.13 (b1) for the point $x = kq+i$ in $(\mathbb{Z}, \kappa(q, n))$, it is obtained that κ-$Ker(\{x\}) = [kq+1, kq+r]_{\mathbb{Z}}$; and by Theorem 2.13(b1) for the point $x = kq+i$ in $(\mathbb{Z}, \kappa(q, n'))$, it is obtained that κ'-$Ker(\{x\}) = [kq+1, kq+r]_{\mathbb{Z}}$. Thus, for the point $x = kq+i$ $(1 \leq i \leq r)$, κ-$Ker(\{x\}) = \kappa'$-$Ker(\{x\})$ holds.
Case 2. $x = kq+j$, where $r+1 \leq j \leq r$: by Theorem 2.13 (b2) for the point $x = kq+j$ in $(\mathbb{Z}, \kappa(q, n))$, it is obtained that κ-$Ker(\{x\}) = [kq+1, kq+q+r]_{\mathbb{Z}}$; and, by Theorem 2.13 (b2) for the point $x = kq+i$ in $(\mathbb{Z}, \kappa(q, n'))$, it is obtained that κ'-$Ker(\{x\}) = [kq+1, kq+q+r]_{\mathbb{Z}}$. Thus, for the point $x = kq+j$ $(r+1 \leq j \leq q)$, κ-$Ker(\{x\}) = \kappa'$-$Ker(\{x\})$ holds.

Therefore, for both cases above we see κ-$Ker(\{x\}) = \kappa'$-$Ker(\{x\})$ for any point x. By using Theorem 2.13 (b1), (b2) and the claim (2) above, we have $\kappa(q, n) = \kappa(q, n')$. □

Remark 2.15 Kojima [29] investigated the classification of a family $\{\tau(3, m) | m \in \mathbb{Z}\}$ of the natural fuzzy topologies on $\mathbb{Z}$.

3 Semi-open sets in generalized digital lines In the first of the present section, we recall some notation with definitions and some properties (3.1) - (3.11) on familes of generalized open sets of a topological space (X, τ) (i.e., semi-open sets, preopen sets, α-open sets, β-open sets, semi-preopen sets, b-open sets):

(3.1) $SO(X, \tau) := \{A \mid A$ is semi-open in $(X, \tau)\} = \{A | A \subset Cl(Int(A))\} = \{A \mid$ there exists a subset $U \in \tau$ such that $U \subset A \subset Cl(U)\}$ [30],

(3.2) $PO(X, \tau) := \{A \mid A$ is preopen in $(X, \tau)\} = \{A \mid A \subset Int(Cl((A))\} = \{A|$ there exists a subset $V \in \tau$ such that $A \subset V \subset Cl(A)\}$ [34],

(3.3) $\tau^\alpha := \{A \mid A$ is α-open in $(X, \tau)\} = \{A \mid A \subset Int(Cl(Int(A)))\}$ [38].

(3.4) For every topological space (X, τ), $PO(X, \tau) \cap SO(X, \tau) = \tau^\alpha$ holds [42] and τ^α is a topology on X [38] (e.g., [40]);

(3.5) $\beta O(X, \tau) := \{A \mid A$ is β-open in $(X, \tau)\} = \{A \mid A \subset Cl(Int(Cl(A)))\}$ [1], (3.6) $SPO(X,\tau) := \{A \mid A$ is semi-preopen in $(X,\tau)\} = \{A \mid$ there exists a preopen set U such that $U \subset A \subset Cl(U)\}$ [4].

(3.7) For every topological space (X,τ),$SPO(X,\tau) = \beta O(X,\tau)$ holds [4, Theorem 2.4].

(3.8) $BO(X,\tau) := \{A \mid A$ is b-open in $(X,\tau)\} = \{A \mid A \subset Int(Cl(A)) \cup Cl(Int(A))\}$ [5].

(3.9) For every topological space (X,τ),

$\tau \subset PO(X,\tau) \cap SO(X,\tau) \subset PO(X,\tau) \cup SO(X,\tau) \subset BO(X,\tau) \subset \beta O(X,\tau) = SPO(X,\tau)$ hold [4, Theorem 2.2], [5, p.60] (e.g., [17, Proposition 1.1]).

(3.10) The following properties are well known and important ones:

if $V_i \in SO(X,\tau)$ (resp. $PO(X,\tau), SPO(X,\tau), BO(X,\tau)), i \in \Gamma$, then $\bigcup\{V_i \mid i \in \Gamma\} \in SO(X,\tau)$ (resp. $PO(X,\tau), SPO(X,\tau), BO(X,\tau)$), where the index set Γ is not necessarily finite.

(3.11) The complement of a semi-open set (resp. preopen set, α-open set, β-open set, pre-semi-open set, b-open set) is called a semi-close set (resp. preclosed set, α-closed set, β-closed set, pre-semi-closed set, b-closed set).

In the present section, we investigate mainly the semi-closure and the semi-kernel of a singleton of $(\mathbb{Z}, \kappa(q,n))$ (cf. Theorem 3.2). We note that [39, Lemma 2] if A is a nonempty semi-open set of (X,τ), then $Int(A) \neq \emptyset$.

Lemma 3.1 *Let $(\mathbb{Z}, \kappa(q,n))$ be a generalized digital line* (cf. Definition 2.2) *and $A \in SO(\mathbb{Z}, \kappa(n,q))$ with a point $x \in A$. Assume that $n \equiv r \pmod{q}$, where $r \in \mathbb{Z}$ with $1 \leq r \leq q-1$.*

(b1) *If $x = kq + i \in A$, where $k \in \mathbb{Z}$, and $i \in Z$ with $1 \leq i \leq r$, then there exists a subset $U_1(x) \in \kappa(q,n)$ such that $x \in U_1(x) \subset A$ and $U_1(x)$ is the smallest open set containing x, where $U_1(x) := \{y \in \mathbb{Z} \mid kq+1 \leq y \leq kq+r\}$.*

(b2) *If $x = kq + j \in A$, where $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ with $r+1 \leq j \leq q$, then there exist a point $kq+h$ $(1 \leq h \leq q+r)$ such that $kq+h \in Int(A)$ and an open set V such that $V \subset A$, where V is defined as follows:*

$V := \{y \in \mathbb{Z} \mid kq+1 \leq y \leq kq+r\}$ if $1 \leq h \leq r$; $V := \{y \in \mathbb{Z} \mid kq+1 \leq y \leq (k+1)q+r\}$ if $r+1 \leq h \leq q$; $V := \{y \in \mathbb{Z} \mid (k+1)q+1 \leq y \leq (k+1)q+r\}$ if $q+1 \leq h \leq q+r$.

Proof. **(b1)** Suppose that $x = kq+i$ $(1 \leq i \leq r), x \in A$ and $A \in SO(\mathbb{Z}, \kappa(q,n))$. Since $x \in Cl(Int(A))$ holds, by using Theorem 2.13 (b1) for the point x, there exists the smallest open set $Ker(\{x\}) = [kq+1, kq+r]_{\mathbb{Z}}$ containing x, say $U_1(x)$, such that $U_1(x) \cap Int(A) \neq \emptyset$. Take a point $y_x \in \mathbb{Z}$ such that $y_x \in U_1(x) \cap Int(A)$, say $y_x = kq+h$ $(1 \leq h \leq r)$. Then, using Theorem 2.13 (b1) for the point $y_x = kq+h$ $(1 \leq h \leq r)$, the set $Ker(\{y_x\}) = [kq+1, kq+r]_{\mathbb{Z}}$ is the smallest open set containing y_x and so $y_x \in [kq+1, kq+r]_{\mathbb{Z}} \subset Int(A) \subset A$. Thus, it is obtained that $U_1(x) = [kq+1, kq+r]_{\mathbb{Z}}$ is the smallest open set containing x such that $U_1(x) \subset A$.

(b2) By using Theorem 2.13 (b2) for the point x, there exists the smallest open set $Ker(\{x\}) = [kq+1, (k+1)q+r]_{\mathbb{Z}}$ containing x. Since $x \in A$ and $A \subset Cl(Int(A))$ hold, we have $[kq+1, (k+1)q+r]_{\mathbb{Z}} \cap Int(A) \neq \emptyset$ and so there exists a point $kq+h \in Int(A)$ with $1 \leq h \leq q+r$. Thus we investigate the following Case 1, Case 2 and Case 3.

Case 1. $kq+h \in Int(A)$, where $1 \leq h \leq r$; Case 2. $kq+h \in Int(A)$, where $r+1 \leq h \leq q$; Case 3. $kq+h \in Int(A)$, where $q+1 \leq h \leq q+r$.

For Case 1, by using Theorem 2.13 (b1) for the point $kq+h$ and the definition of V, it is shown that $Ker(\{kq+h\}) = [kq+1, kq+r]_{\mathbb{Z}} \subset Int(A) \subset A$ hold and so $V \subset A$. We note $x \notin V$ for this case. For Case 2, by using Theorem 2.13 (b2) for the point $kq+h$ and the definition of V, it is shown that $Ker(\{kq+h\}) = [kq+1, (k+1)q+r]_{\mathbb{Z}} \subset Int(A) \subset A$ hold and so $V \subset A$. We note $x \in V$ for this case. For Case 3, by using Theorem 2.13 (b1) for the point $kq+h = (h+1)q+h'$, where $h' \in \mathbb{Z}$ with $1 \leq h' \leq r$, and the definition of V, it is shown that $Ker(\{kq+h\}) = [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}} \subset Int(A) \subset A$ hold and so $V \subset A$. We note $x \notin V$ for this case. □

For the digital line $(\mathbb{Z}, \kappa), \kappa(2,3) = \kappa$, i.e., $q = 2, n = 3$ and so $r = 1$, it is known that $SO(\mathbb{Z}, \kappa(2,3)) \neq \kappa(2,3)$ and $\kappa(2,3) \subsetneq SO(\mathbb{Z}, \kappa(2,3))$. For example, a subset $\{q+r, q+q\} = \{3,4\}$ is a semi-open set, where $q = 2$ and $r = 1$; it is not open in $(\mathbb{Z}, \kappa(2,3))$.

We recall the following definitions: for a subset B of a topological space (X, τ),
$sKer(B) = \bigcap\{U \mid U \in SO(X,\tau), B \subset U\}$; $sCl(B) = \bigcap\{F \mid X \setminus F \in SO(X,\tau), B \subset F\}$. It is well nown that [4, Theorem 2.1 (a)] $sCl(A) = A \cup Int(Cl(A))$ holds for any subset A of (X, τ).

Theorem 3.2 *Let $(\mathbb{Z}, \kappa(q,n))$ be a generalized digital line* (cf. Definition 2.2) *and a point $x \in \mathbb{Z}$. Assume that $n \equiv r \pmod q$, where $r \in \mathbb{Z}$ with $1 \le r \le q-1$. The following properties hold:*
(b1) *Let $x = kq + i$, where $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \le i \le r$. Then,*
(b1-1) *there exists a subset $U_1(x) \in SO(\mathbb{Z}, \kappa(q,n))$ such that $x \in U_1(x)$, where $U_1(x) := \{y \in \mathbb{Z} \mid kq+1 \le y \le kq+r\}$;*
(b1-2) *if there exists a semi-open set A_1 containing the point x such that $A_1 \subset U_1(x)$, then $A_1 = U_1(x)$ and $x \in U_1(x)$ hold, where $U_1(x)$ is defined in* (b1-1) *above;*
(b1-3) *$sKer(\{x\}) = \{y \in \mathbb{Z} \mid kq+1 \le y \le kq+r\} \in SO(\mathbb{Z}, \kappa(q,n))$ and $sKer(\{x\})$ is semi-open in $(\mathbb{Z}, \kappa(q,n))$.*
(b2) *Let $x = kq + j \in \mathbb{Z}$, where $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ with $r+1 \le j \le q$. Then,*
(b2-1) *there exist two subsets $V_i(x) \in SO(\mathbb{Z}, \kappa(q,n)), i \in \{1,2\}$, such that $\{x\} = V_1(x) \cap V_2(x)$, where $V_1(x) := \{x\} \cup \{y \in \mathbb{Z} \mid kq+1 \le y \le kq+r\}$ and $V_2(x) := \{x\} \cup \{y \in \mathbb{Z} \mid (k+1)q+1 \le y \le (k+1)q+r\}$;*
(b2-2) *$sKer(\{x\}) = \{x\}$ and $\{x\}$ is not semi-open in $(\mathbb{Z}, \kappa(q,n))$;*
(b2-3) *if there exists a semi-open set G_1* (resp. *a semi-open set G_2*) *such that $x \in G_1 \subset V_1(x)$* (resp. $x \in G_2 \subset V_2(x)$), *then $G_1 = V_1(x)$* (resp. $G_2 = V_2(x)$), *where $V_1(x)$ and $V_2(x)$ are defined in* (b2-1) *above.*
(b1)$'$ *For a point $x = kq + i$, where $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \le i \le r$,*
$sCl(\{x\}) = \{y \in \mathbb{Z} \mid kq+1 \le y \le kq+r\} = sKer(\{x\})$ hold.
(b2)$'$ *For a point $x = kq + j$, where $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ with $r+1 \le j \le q$,*
$sCl(\{x\}) = \{x\} = sKer(\{x\})$ hold.

Proof. **(b1) (b1-1)** Let $x = kq + i$ $(1 \le i \le r)$. By using Lemma 3.1 (b1) for the semi-open set $\mathbb{Z}$ of $(\mathbb{Z}, \kappa(q,n))$ and the point $x \in \mathbb{Z}$ and a fact that $\kappa(q,n) \subset SO(\mathbb{Z}, \kappa(q,n))$, there exists a subset $U_1(x) \in SO(\mathbb{Z}, \kappa(q,n))$ such that $x \in U_1(x)$, where $U_1(x) = [kq+1, kq+r]_{\mathbb{Z}}$.
(b1-2) Suppose that there exists a semi-open set A_1 such that $x \in A_1 \subset U_1(x)$. Then, by Lemma 3.1 for A_1 and x, it is shown that $x \in U_1(x) \subset A_1$ and so $A_1 = U_1(x)$.
(b1-3) By (b1-2) above, it is obtained that $sKer(\{x\}) = U_1(x)$ holds and $sKer(\{x\})$ is semi-open in $(\mathbb{Z}, \kappa(q,n))$.

(b2) Throughout (b2) we recall that $x = kq + j$ $(r+1 \le j \le q)$.
(b2-1) First we claim that $V_1(x) := \{x\} \cup [kq+1, kq+r]_{\mathbb{Z}}$ is a semi-open set containing x. Put $V_1 := [kq+1, kq+r]_{\mathbb{Z}}$. Using Theorem 2.13 (b1) for a point $y \in V_1$, $Ker(\{y\}) = V_1$ is the smallest open set containing y. It is shown that $V_1(x) \subset Cl(V_1)$. Indeed, by Theorem 2.13 (b1)$'$, $Cl(V_1) = \bigcup\{Cl(\{kq+h\}) \mid h \in [1,r]_{\mathbb{Z}}\} = [(k-1)q+r+1, (k+1)q]_{\mathbb{Z}}$ and so $V_1(x) \subset Cl(V_1)$. Thus, there exists an open set V_1 such that $V_1 \subset V_1(x) \subset Cl(V_1)$. Namely, $V_1(x)$ is a semi-open set containing x. Finally, we can prove that $V_2(x) := \{x\} \cup [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}}$ is a semi-open set containing x. Put $V_2 := [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}}$. Using Theorem 2.13 (b2) for a point $z \in V_2$, $Ker(\{z\}) = V_2$ is the smallest open set containing z. By Theorem 2.13 (b1)$'$, $Cl(V_2) = \bigcup\{Cl(\{(k+1)q+h\}) \mid h \in [1,r]_{\mathbb{Z}}\} = [kq+r+1, (k+1)q+q]_{\mathbb{Z}}$ and $x \in Cl(V_2)$. Thus, there exists an open set V_2 such that $V_2 \subset V_2(x) \subset Cl(V_2)$. Namely, $V_2(x)$ is a semi-open set containing x. Obviously, we have $\{x\} = V_1(x) \cap V_2(x)$.
(b2-2) It follows from (b2-1) above that $\{x\} \subset sKer(\{x\}) \subset V_1(x) \cap V_2(x) = \{x\}$ and so $sKer(\{x\}) = \{x\}$. By Theorem 2.13 (b2), it is obtained that $Int(\{x\}) = \emptyset$ and so

$\{x\} \not\subset Cl(Int((\{x\})) = \emptyset$, i.e., {x} is not semi-open in $(\mathbb{Z}, \kappa(q,n))$.

(b2-3) Let $\xi := \{[kq+1, kq+r]_{\mathbb{Z}}, [kq+1, (k+1)q+r]_{\mathbb{Z}}, [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}}\}$ throughout the present proof. First, we claim that $V_1(x) = G_1$. Indeed, using Lemma 3.1 (b2) for G_1 and the point x, there exists an open set V such that $V \subset G_1$; by Lemma 3.1 (b2), it is shown explicitly that $V \in \xi$. Because of $V \subset G_1 \subset V_1(x) = \{kq+j\} \cup [kq+1, kq+r]_{\mathbb{Z}}$, where $r+1 \leq j \leq q$, we have $V = [kq+1, kq+r]_{\mathbb{Z}}$. Thus, $V_1(x) = \{x\} \cup V \subset \{x\} \cup G_1 = G_1 \subset V_1(x)$ and hence $V_1(x) = G_1$. Finally, we prove that $V_2(x) = G_2$. Using Lemma 3.1 (b2) for the semi-open set G_2 and the point x, there exists an open set V such that $V \subset G_2$; explicitly that $V \in \xi$. Because of $V \subset G_2 \subset V_2(x) = \{kq+j\} \cup [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}}$, where $r+1 \leq j \leq q$, we conclude that $V = [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}}$. Thus, we obtain $V_2(x) = \{x\} \cup V \subset \{x\} \cup G_2 = G_2 \subset V_2(x)$ and hence $V_2(x) = G_2$.

(b1)′ By Theorem 2.13 (b1)′, (b1) and (b2), for a point $x = kq+i$ $(1 \leq i \leq r)$, it is shown that $Int(Cl(\{x\})) = Int([(k-1)q+r+1, kq+q]_{\mathbb{Z}}) = [kq+1, kq+r]_{\mathbb{Z}}$. Then, $sCl(\{x\}) = \{x\} \cup Int(Cl(\{x\})) = [kq+1, kq+r]_{\mathbb{Z}}$ hold. We have $sCl(\{x\}) = sKer(\{x\})$ (cf. (b1) above).

(b2)′ Let $x = kq+j$ $(r+1 \leq j \leq q)$. By Theorem 2.13 (b2′), $Cl(\{x\}) = [kq+r+1, kq+q]_{\mathbb{Z}}$. By Theorem 2.13 (b2), it is obtained that $Int(Cl(\{x\})) = Int([kq+r+1, kq+q]_{\mathbb{Z}}) = \emptyset$ and so $sCl(\{x\}) = \{x\}$. It is noted that $sCl(\{x\}) = sKer(\{x\})$ (cf. (b2-2) above). □

Remark 3.3 It is shown that $sKer(\{x\})$ is not necessarily semi-open (cf. Theorem 3.2 (b2-2)).

4 Preopen sets of generalized digital lines

In the present section, we investigate prekernels and preclosures of singletons in $(\mathbb{Z}, \kappa(q,n))$. We recall the following definitions: for a subset A of a topological space (X, τ), $pKer(A) := \bigcap\{U \mid A \subset U, U \in PO(X,\tau)\}$ [21]; $pCl(A) := \bigcap\{F \mid A \subset F, X \setminus F \in PO(X,\tau)\}$ [12]. It is well known that [4, Theorem 1.5 (e)] $pCl(A) = A \cup Cl(Int(A))$ holds for any subset A of (X, τ).

Lemma 4.1 *Let $(\mathbb{Z}, \kappa(q,n))$ be a generalized digital line* (cf. Definition 2.2)*. Assume that $n \equiv r \pmod q$, where $r \in \mathbb{Z}$ with $1 \leq r \leq q-1$. Let $x = kq+j \in Z$, where $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ with $r+1 \leq j \leq q$. If $A \in PO(\mathbb{Z}, \kappa(q,n))$ and $x \in A$, then there exist two points $kq+a$ and $kq+q+b$ such that $\{kq+a, kq+q+b\} \subset A$ for some integers a and b with $1 \leq a \leq r$ and $1 \leq b \leq r$.*

Proof. There exists a subset $W \in \kappa(q,n)$ such that $x \in W \subset Cl(A)$, because $x \in A \subset Int(Cl(A))$. Since $Ker(\{x\}) \subset W$, by Theorem 2.13 (b2), $[kq+1, kq+q+r]_{\mathbb{Z}} \subset Cl(A)$ holds. Thus, we have $kq+1 \in Cl(A)$ and $kq+q+r \in Cl(A)$. By using Theorem 2.13 (b1) for the above two points, it is obtained that $[kq+1, kq+r]_{\mathbb{Z}} \cap A \neq \emptyset$ and $[kq+q+1, kq+q+r]_{\mathbb{Z}} \cap A \neq \emptyset$, respectively. Then there exist two points $kq+a \in A$ and $kq+q+b \in A$ for some integers a, b with $1 \leq a \leq r$ and $1 \leq b \leq r$. □

Theorem 4.2 *Let $(\mathbb{Z}, \kappa(q,n))$ be a generalized digital line* (cf. Definition 2.2)*. Assume that $n \equiv r \pmod q$, where $r \in \mathbb{Z}$ with $1 \leq r \leq q-1$.*

(b1) *For a point $x = kq+i \in Z$, where $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq r$, the following properties hold.*

(b1-1) *$pKer(\{x\}) = \{x\}$ and $\{x\}$ is preopen.*

(b1-1)′ *If $r \geq 2$, then $pCl(\{x\}) = \{x\}$, i.e., $\{x\}$ is preclosed.*

If $r = 1$, then $x = kq+1$ and $pCl(\{x\}) = \{y \in \mathbb{Z} \mid (k-1)q+2 \leq y \leq kq+q\}$.

(b2) *For a point $x = kq+j \in \mathbb{Z}$, where $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ with $r+1 \leq j \leq q$, the following properties* (b2-1) - (b2-4) *and* (b2-3)′ *hold. Let $V_{h,h'}(x) := \{kq+h, x, kq+q+h'\}$, where $h, h' \in \mathbb{Z}$ with $1 \leq h \leq r$ and $1 \leq h' \leq r$.*

(b2-1) *$V_{h,h'}(x) \in PO(\mathbb{Z}, \kappa(q,n))$ and $pKer(\{x\}) \subset V_{h,h'}(x)$ for each integers h and h' with $1 \leq h \leq r, 1 \leq h' \leq r$.*

(b2-2) *Suppose that* $r = 1$. *If there exists a preopen set* G *containing the point* x, *then* $x \in V_{1,1}(x) \subset G$.

(b2-3) $pKer(\{x\}) = V_{1,1}(x)$ *if* $r = 1$; $pKer(\{x\}) = \{x\}$ *if* $r \geq 2$; *for the singleton* $\{x\}, \{x\} \notin PO(\mathbb{Z}, \kappa(q,n))$.

(b2-4) *If there exists a subset* $G \in PO(\mathbb{Z}, \kappa(q,n))$ *such that* $x \in G \subset V_{h,h'}(x)$, *then* $G = V_{h,h'}(x)$.

(b2-3)′ $pCl(\{x\}) = \{x\}$, *i.e.,* $\{x\}$ *is preclosed.*

Proof. **(b1) (b1-1)** For the point $x = kq+i$ $(1 \leq i \leq r)$, by using Theorem 2.13 (b1)′, (b1) and (b2), it is shown that $Int(Cl(\{x\})) = Int([(k-1)q+r+1, kq+q]_{\mathbb{Z}}) = [kq+1, kq+r]_{\mathbb{Z}} \supset \{x\}$ and so $\{x\} \in PO(\mathbb{Z}, \kappa(q,n))$. This implies $pKer(\{x\}) = \{x\}$.
(b1-1)′ By Theorem 2.13 (b1), it is shown that, for the case where $r \geq 2$, $Int(\{x\}) = \emptyset$ and so $pCl(\{x\}) = \{x\} \cup Cl(Int(\{x\})) = \{x\}$. For the case where $r = 1$, $x = kq+1$ holds. And, by Theorem 2.13 (b1) and (b1)′, it is shown that $Cl(Int(\{x\})) = Cl(\{x\}) = [(k-1)q+2, kq+q]_{\mathbb{Z}}$ and so $pCl(\{kq+1\}) = [(k-1)q+2, kq+q]_{\mathbb{Z}}$.

(b2) (b2-1) Put $V_{h,h'}(x) := \{x, kq+h, kq+q+h'\}$ for a point $x = kq+j$ $(r+1 \leq j \leq q)$ and each integers h and h' with $1 \leq h \leq r$ and $1 \leq h' \leq r$. Then, by Theorem 2.13, it is shown that $Int(Cl(V_{h,h'}(x)))$=$Int([kq+r+1, kq+q]_{\mathbb{Z}} \cup [(k-1)q+r+1, kq+q]_{\mathbb{Z}} \cup [kq+r+1, (k+1)q+q]_{\mathbb{Z}})$=$Int([(k-1)q+r+1, (k+1)q+q]_{\mathbb{Z}})$=$[kq+1, (k+1)q+r]_{\mathbb{Z}} \supset V_{h,h'}(x)$ and so $V_{h,h'}(x) \in PO(\mathbb{Z}, \kappa(q,n))$. Thus, we show that $pKer(\{x\}) \subset V_{h,h'}(x)$ for each integers h and h' with $1 \leq h \leq r$ and $1 \leq h' \leq r$.
(b2-2) If $r = 1$, then $V_{1,1}(x) = \{kq+1, x, kq+q+1\} \subset G$ for any preopen set G containing x (cf. Lemma 4.1).
(b2-3) Using (b2-1) and (b2-2) above, we have that $pKer(\{x\}) = V_{1,1}(x)$ if $r = 1$. If $r \geq 2$, then there exist two preopen sets $V_{1,1}(x)$ and $V_{2,2}(x)$ such that $V_{1,1}(x) \cap V_{2,2}(x) = \{x\}$. Thus we have that $pKer(\{x\}) = \{x\}$ if $r \geq 2$. By Theorem 2.13 (b2)′ and (b2), it is shown that $\{x\} \not\subset Int(Cl(\{x\})) = \emptyset$ and so $\{x\} \notin PO(\mathbb{Z}, \kappa(q,n))$.
(b2-4) Let $G \in PO(\mathbb{Z}, \kappa(q,n))$ such that $G \subset V_{h,h'}(x)$ and $x \in G$. We claim that $G = V_{h,h'}(x)$ holds. Indeed, by Lemma 4.1, $\{kq+a, kq+q+b\} \subset G \subset V_{h,h'}(x)$, for some $a, b \in \mathbb{Z}$ with $1 \leq a \leq r$ and $1 \leq b \leq r$. Thus, we have $a = h, b = h'$ and so $G = V_{h,h'}(x)$, because $x \in G$.
(b2-3)′ By Theorem 2.13 (b2), $pCl(\{x\}) = \{x\} \cup Cl(Int(\{x\})) = \{x\} \cup Cl(\emptyset) = \{x\}$. Thus $\{x\}$ is preclosed. □

5 Proof of Theorem A(i) and related properties

In the present section, the proof of Theorem A(i) (cf. Section 1) shall be given (cf. Theorem 5.1 (i) or (ii) below); moreover we investigate some related properties on structures of $SO(\mathbb{Z}, \kappa(q,n))$ and $PO(\mathbb{Z}, \kappa(q,n))$ (cf. Theorems 5.1 and 5.2 below).

For a topological space (X, τ), we recall that (X, τ) is said to be *extremally disconnected* if the closure of every open set is open; by [23, Proposition 4.1], [22], it is well known that a topological space (X, τ) is extremally disconnected if and only if $SO(X, \tau) \subset PO(X, \tau)$ holds. A topological space (X, τ) is said to be a *PS-space* [2] if $PO(X, \tau) \subset SO(X, \tau)$ holds. It is well known that the following properties are equivalent to each others: (X, τ) is a PS-space; $SO(X, \tau) = SPO(X, \tau); \tau^{\alpha} = PO(X, \tau)$; (X, τ^{α}) is submaximal; (X, τ) is quasi-submaximal (cf. [15, Theorem 4], [16, Proposition 8]; [2, Theorem 2.1]; [3, Theorem 3.4], e.g. [43, Theorem 3.4]).

Theorem 5.1 *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2). *Assume that* $n \equiv r \pmod q$, *where* $r \in \mathbb{Z}$ *with* $1 \leq r \leq q-1$. *Then, the following properties hold.*

(i) *A singleton* $\{kq+j\}$ *is not preopen in* $(\mathbb{Z}, \kappa(q,n))$, *where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r+1 \leq j \leq q$. *Namely,* $PO(\mathbb{Z}, \kappa(q,n)) \neq P(\mathbb{Z})$ *holds.*

(ii) *A singleton* $\{kq+j\}$ *is not semi-open in* $(\mathbb{Z}, \kappa(q,n))$*, where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r+1 \le j \le q$*. Namely,* $SO(\mathbb{Z}, \kappa(q,n)) \ne P(\mathbb{Z})$ *holds.*

(iii) *Especially, assume that* $2 \le r$*. For a singleton* $\{kq+i\}$*, where* $k \in \mathbb{Z}$ *and* $i \in \mathbb{Z}$ *with* $1 \le i \le r$*, we have* $\{kq+i\} \in PO(\mathbb{Z}, \kappa(q,n))$ *and* $\{kq+i\} \notin SO(\mathbb{Z}, \kappa(q,n))$*.*

(iv) *There exists a subset* V *such that* $V \notin PO(\mathbb{Z}, \kappa(q,n))$ *and* $V \in SO(\mathbb{Z}, \kappa(q,n))$*.*

(v) (e.g., [13, Theorem 2.1 (i)(b)]) *Especially, if* $q = 2, n = 3$ *and* $r = 1$*, then* $PO(\mathbb{Z}, \kappa(2,3)) \subset SO(\mathbb{Z}, \kappa(2,3))$ *and* $\kappa(2,3)^{\alpha} = \kappa(2,3)$ *hold.*

Proof. **(i)** By using Theorem 4.2 (b2)(b2-3) for the point $x := kq+j$ $(r+1 \le j \le q)$, it is obtained that $\{kq+j\} \notin PO(\mathbb{Z}, \kappa(q,n))$ and so $PO(\mathbb{Z}, \kappa(q,n)) \subsetneq P(\mathbb{Z})$.

(ii) We claim that the singleton $\{kq+j\}$ is not semi-open in $(\mathbb{Z}, \kappa(q,n))$, where $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ with $r+1 \le j \le q$. Suppose that $\{kq+j\}$ is semi-open in $(\mathbb{Z}, \kappa(q,n))$. By Theorem 3.2 (b2)(b2-1), there exists a semi-open set $V_1(kq+j) = \{kq+j\} \cup [kq+1, kq+r]_{\mathbb{Z}}$. Then, by using Theorem 3.2 (b2)(b2-3) for the point $x := kq+j$ and the semi-open set $G_1 := \{kq+j\}$, it is shown that $\{kq+j\} = V_1(kq+j)$ holds. Thus, we have $|\{kq+j\}| = 1 = |V_1(kq+j)| = r+1$ and so $r = 0$; thus this contradicts to the assumption. Thus, $\{kq+j\} \notin SO(\mathbb{Z}, \kappa(q,n))$ and so $SO(\mathbb{Z}, \kappa(q,n)) \subsetneq P(\mathbb{Z})$.

(iii) By using Theorem 4.2 (b1)(b1-1) for the point $x := kq+i$ $(1 \le i \le r)$, the singleton $\{kq+i\}$ is preopen in $(\mathbb{Z}, \kappa(q,n))$. Since $2 \le r$, the singleton $\{kq+i\}$ is not semi-open in $(\mathbb{Z}, \kappa(q,n))$, because $sKer(\{kq+i\}) = [kq+1, kq+r]_{\mathbb{Z}} \supsetneq \{kq+i\}$ and $sKer(\{kq+i\})$ is the intersection of all semi-open sets containing the point $kq+i$ (cf. Theorem 3.2 (b1)(b1-3)).

(iv) By using Theorem 3.2 (b2)(b2-1) for the point $x := kq+j$ $(r+1 \le j \le q)$, there exists a semi-open set $V_1(kq+j) := \{kq+j\} \cup [kq+1, kq+r]_{\mathbb{Z}}$. We put $V := V_1(kq+j)$ and so $V \in SO(\mathbb{Z}, \kappa(q,n))$. We claim that $V \not\subset Int(Cl(V))$. Indeed, by using Theorem 2.13 (b2)$'$ and (b1$'$) for the point $kq+j$ and points $kq+i$ $(1 \le i \le r)$, respectively, it is shown that $Cl(V) = Cl(\{kq+j\}) \cup (\bigcup_{i=1}^{r} Cl(\{kq+i\})) = [(k-1)q+r+1, kq+q]_{\mathbb{Z}}$. Using Theorem 2.13 (b1) and (b2), we have $Int(Cl(V)) = [kq+1, kq+r]_{\mathbb{Z}}$ and hence $V := V_1(kq+j) = \{kq+j\} \cup [kq+1, kq+r]_{\mathbb{Z}} \not\subset [kq+1, kq+r]_{\mathbb{Z}} = Int(Cl(V))$. Therefore, we have $V \notin PO(\mathbb{Z}, \kappa(q,n))$ and $V \in SO(\mathbb{Z}, \kappa(q,n))$. □

Proof of Theorem A(i) The proof is shown by using Theorem 5.1 (i) or (ii) above, because $\kappa(q,n) \subset PO(\mathbb{Z}, \kappa(q,n))$ or $\kappa(q,n) \subset SO(\mathbb{Z}, \kappa(q,n))$ hold in general. □

Theorem 5.1 (iii) and (v) (resp. (iv)) suggest the property of Theorem 5.2 (i) (resp. (ii)) below.

Theorem 5.2 *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2)*. Assume that* $n \equiv r \pmod q$*, where* $r \in \mathbb{Z}$ *with* $1 \le r \le q-1$*.*

(i) $PO(\mathbb{Z}, \kappa(q,n)) \subset SO(\mathbb{Z}, \kappa(q,n))$ *holds if and only if* $n \equiv 1 \pmod q$*.*

(ii) *A non-implicaton* $SO(\mathbb{Z}, \kappa(q,n)) \not\subset PO(\mathbb{Z}, \kappa(q,n))$ *holds.*

(iii) *The topology* $\kappa(q,n)$ *is a proper subfamily of* $SO(\mathbb{Z}, \kappa(q,n))$*. And, if* $q+r > 3$ *then* $\kappa(q,n)$ *is a proper subfamily of* $PO(\mathbb{Z}, \kappa(q,n))$*.*

Proof. **(i) (Necessity)** By Theorem 4.2 (b1)(b1-1) for a point $x := kq+i$ $(1 \le i \le r)$, it is shown that $\{kq+i\} = pKer(\{kq+i\}) \in PO(\mathbb{Z}, \kappa(q,n))$. It follows our assumption that $\{kq+i\} \in SO(\mathbb{Z}, \kappa(q,n))$; by definition, $sKer(\{kq+i\}) = \{kq+i\}$ holds. Using Theorem 3.2 (b1)(b1-3) for the point $kq+i$, we have $sKer(\{kq+i\}) = [kq+1, kq+r]_{\mathbb{Z}}$ and so $|[kq+1, kq+r]_{\mathbb{Z}}| = 1$; therefore $r = 1$.

(**Sufficiency**) Suppose that $r = 1$. Let $V \in PO(\mathbb{Z}, \kappa(q,n))$. The set V has a decomposition $V = A_V \cup B_V$, where $A_V := \bigcup\{V \cap \{kq+1\} |\ k \in \mathbb{Z}\}$ and $B_V := \bigcup\{V \cap [kq+2, kq+q]_{\mathbb{Z}} |\ k \in \mathbb{Z}\}$.

First, we show that: $(*1)$ $A_V \in SO(\mathbb{Z}, \kappa(q,n))$. Indeed, we have that $V \cap \{kq+1\} = \{kq+1\}$ or $\emptyset$ and $sKer(\{kq+1\}) = [kq+1, kq+r]_{\mathbb{Z}} = \{q+1\}$ hold and $\{kq+1\} \in SO(\mathbb{Z}, \kappa(q,n))$ by Theorem 3.2 (b1)(b1-3); thus $A_V \in SO(\mathbb{Z}, \kappa(q,n))$.

Secondly, we show that: $(*2)$ for a point $x \in B_V$, there exist a preopen set $V_{1,1}(x) := \{kq+1, x, kq+q+1\}$ such that $x \in V_{1,1}(x)$ and $V_{1,1}(x) \subset V$. Indeed, the point $x \in B_V$, there exist integers k and j with $r+1 = 2 \leq j \leq q$ such that $x = kq+j$. Since $x \in [kq+2, kq+q]_{\mathbb{Z}}$, $x \in V$ and $V \in PO(\mathbb{Z}, \kappa(q,n))$, we use Theorem 4.2 (b2)(b2-1) and (b2-2) for the point $x = kq+j$, the preopen set V, $r = 1$ and $h = h' = 1$. Then, there exist a preopen set $V_{1,1}(x)$ such that $x \in V_{1,1}(x)$ and $V_{1,1}(x) \subset V$, where $V_{1,1}(x) := \{kq+1, x, kq+q+1\} \subset V$.

Thus, by using $(*2)$, it is obtained that: $(*2')$ $B_V \subset \bigcup\{V_{1,1}(x) | \ x \in B_V\} \subset V$ hold.

Thirdly, we show that: $(*3)$ $V_{1,1}(x) \in SO(\mathbb{Z}, \kappa(q,n))$ for the point $x = kq+j \in B_V$. Indeed, using Theorem 3.2 (b2)(b2-1) for the point $x = kq+j$ and $r = 1$, fortunately, we have two semi-open sets $V_1(x) = \{x\} \cup [kq+1, kq+r]_{\mathbb{Z}} = \{x, kq+1\}$ and $V_2(x) = \{x\} \cup [(k+1)q+1, (k+1)q+r]_{\mathbb{Z}} = \{x, kq+q+1\}$. Since $V_1(x) \cup V_2(x) = \{kq+1, x, kq+q+1\}$ and $V_i(x) \in SO(\mathbb{Z}, \kappa(q,n))$ for each $i \in \{1,2\}$, we have $V_1(x) \cup V_2(x) = V_{1,1}(x)$ and $V_{1,1}(x) \in SO(\mathbb{Z}, \kappa(q,n))$ for the point $x = kq+j \in B_V$.

Finally, by the properties $(*1)$, $(*2')$ and $(*3)$ above, it is shown that $V = A_V \cup B_V \subset A_V \cup (\bigcup\{V_{1,1}(x) | \ x \in B_V\}) \subset V$ and so $V = A_V \cup (\bigcup\{V_{1,1}(x) | \ x \in B_V\})$ and hence $V \in SO(\mathbb{Z}, \kappa(q,n))$ (cf. (3.10) in Section 3). Therefore, $PO(\mathbb{Z}, \kappa(q,n)) \subset SO(\mathbb{Z}, \kappa(q,n))$ holds if $q < n$ and $n \equiv 1 \pmod q$.

(ii) By Theorem 5.1 (iv), there exists a semi-open set, say V, such that $V \notin PO(\mathbb{Z}, \kappa(q,n))$; this shows $SO(\mathbb{Z}, \kappa(q,n)) \not\subset PO(\mathbb{Z}, \kappa(q,n))$.

(iii) First, let $V_1(x) := \{x\} \cup [kq+1, kq+r]_{\mathbb{Z}}$ be the semi-open set in Theorem 3.2 (b2) (b2-1), where $x := kq+j$ $(r+1 \leq j \leq q, k \in \mathbb{Z})$. The semi-open set $V_1(x)$ is not open because $V_1(x) \subsetneq Ker(\{x\})$ and $Ker(\{x\})$ is the smallest open set containing x (cf. Theorem 2.13 (b2), $Ker(\{x\}) = [kq+1, kq+q+r]_{\mathbb{Z}}$). Thus, we have that $V_1(x) \in SO(\mathbb{Z}, \kappa(q,n))$ and $V_1(x) \notin \kappa(q,n)$ (i.e., $\kappa(q,n)$ is a proper subfamily of $SO(\mathbb{Z}, \kappa(q,n))$, because $\kappa(q,n) \subset SO(\mathbb{Z}, \kappa(q,n))$ holds in general). Finally, let $V_{h,h'}(x) := \{kq+h, x, kq+q+h'\}$ be the preopen set containing x in Theorem 4.2 (b2), where $x := kq+j (r+1 \leq j \leq q, \ k \in \mathbb{Z})$ and $h, h' \in [1, r]_{\mathbb{Z}}$ (cf. (b2-1)). However, the preopen set $V_{h,h'}(x)$ is not open in $(\mathbb{Z}, \kappa(q,n))$ if $q+r > 3$. Indeed, $Ker(\{x\}) = [kq+1, (k+1)q+r]_{\mathbb{Z}}$ is the smallest open set containing the point $x := kq+j$ (cf. Theorem 2.13 (b2)), $|Ker(\{x\})| = q+r$ and $|V_{h,h'}(x)| = 3$ hold; and so the point x is not an interior point of $V_{h,h'}(x)$. Thus, we have that $V_{h,h'}(x) \in PO(\mathbb{Z}, \kappa(q,n))$ and if $q+r > 3$ then $V_{h,h'}(x) \notin \kappa(q,n)$ (i.e., $\kappa(q,n)$ is a proper subfamily of $PO(\mathbb{Z}, \kappa(q,n))$, because $\kappa(q,n) \subset PO(\mathbb{Z}, \kappa(q,n))$ holds in general). □

6 Some separation axioms of generalized digital lines and proof of Theorem A(ii)

The purpose of the present section is to investigate some separation axioms of generalized digital lines (cf. Theorem A(ii) in Section 1; and Theorem 6.2, Tables 1 and 2 below). The proof of Theorem A(ii) shall be given by quoting some results in Theorem 6.2 below.

We first recall the following properties (6.1) - (6.6) for a topological space (X, τ).

(6.1) (X, τ) is $T_{1/2}$ if and only if every singleton $\{x\}, x \in X$, is open or closed in (X, τ) ([11, Theorem 2.5]).

(6.2) (X, τ) is $T_{3/4}$ if and only if every singleton $\{x\}$ of (X, τ) is δ-open or closed (equivalently, regular open or closed) in (X, τ) ([10, Theorem 4.3, Example 4.6]).

(6.3) (X, τ) is semi-pre-$T_{1/2}$ if and only if every singleton $\{x\}$ of (X, τ) is semi-preopen or closed (=preopen or closed) in (X, τ) ([9, Theorem 4.1]).

(6.4) For each integer $i \in \{2, 1, 0\}$, the semi-T_i axiom [32] (resp. pre-T_i axiom [24], β-T_i axiom [33]) is defined by using as ordinary T_i axiom except each open set replaced by semi-open set (resp. preopen sets, β-open set(=semi-preopen sets)).

(6.5) (X, τ) is semi-T_1 (resp. pre-T_1, β-T_1) if and only if every singleton $\{x\}, x \in X$, is semi-closed (resp. preclosed, β-closed) in (X, τ).

(6.6) The following implications of separation axioms above are well known:

$\cdot$ $T_2 \Rightarrow T_1 \Rightarrow T_{3/4} \Rightarrow T_{1/2} \Rightarrow T_0$,

· $T_2 \Rightarrow$ semi-$T_2 \Rightarrow$ semi-$T_1 \Rightarrow$ semi-$T_{1/2} \Rightarrow$ semi-T_0,
· $T_2 \Rightarrow$ pre-$T_2 \Rightarrow$ pre-$T_1 \Rightarrow$ pre-$T_{1/2} \Rightarrow$ pre-T_0,
· $T_2 \Rightarrow \beta$-$T_2 \Rightarrow \beta$-$T_1 \Rightarrow \beta$-$T_{1/2} \Rightarrow \beta$-$T_0$,
· for each $i \in \{2, 1, 1/2, 0\}$, $T_i \Rightarrow$ semi-$T_i \Rightarrow \beta$-T_i,
· for each $i \in \{2, 1, 1/2, 0\}$, $T_i \Rightarrow$ pre-$T_i \Rightarrow \beta$-T_i.

In order to investigate some separation axioms of the generalized digital line, we need the following theorem on topological properties of singletons $\{x\}$ of $(\mathbb{Z}, \kappa(q,n))$ (cf. Defintion 2.2).

Theorem 6.1 *For a generalized digital line* $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2) *and a point* $x \in \mathbb{Z}$*, the following properties hold. Assume that* $n \equiv r \pmod q$*, where* $r \in \mathbb{Z}$ *with* $1 \le r \le q-1$.

(b1) *For a point* $x := kq+i$*, where* $k \in \mathbb{Z}$ *and* $i \in \mathbb{Z}$ *with* $1 \le i \le r, \{x\}$ *is semi-preopen (=*β*-open). Especially, if* $2 \le r$*, then* $\{x\}$ *is semi-preclosed (=*β*-closed).*

(b2) *For a point* $x := kq+j$*, where* $k \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *with* $r+1 \le j \le q, \{x\}$ *is semi-closed and so semi-preclosed (=*β*-closed).*

Proof. **(b1)** By using Theorem 2.13 for the point $x = kq+i$ ($k \in \mathbb{Z}, 1 \le i \le r$), it is obtained that $Cl(Int(Cl(\{kq+i\}))) = Cl(Int([(k-1)q+r+1, kq+q]_{\mathbb{Z}})) = Cl([kq+1, kq+r]_{\mathbb{Z}}) = [(k-1)q+r+1, kq+q]_{\mathbb{Z}} \supset \{kq+i\}$; so $\{x\}$ is semi-preopen (cf. (3.7), (3.5) in Section 3). We shall show that if $2 \le r$ then the singleton $\{kq+i\}$ is semi-preclosed, where $1 \le i \le r$. Since $Ker(\{kq+i\}) = [kq+1, kq+r]_{\mathbb{Z}}$ (cf. Theorem 2.13 (b1)), we have that if $2 \le r$ then $Int(\{kq+i\}) = \emptyset$ and so $Int(Cl(Int(\{kq+i\}))) = \emptyset \subset \{kq+i\}$; therefore, $\{x\}$ is semi-preclosed (cf. (3.11) in Section 3).

(b2) Using Theorem 2.13 (b2)$'$ for the point $x = kq+j$ ($k \in \mathbb{Z}, r+1 \le j \le q$), we have $Cl(\{kq+j\}) = [kq+r+1, kq+q]_{\mathbb{Z}}$. Moreover, by using Theorem 2.13 (b2), it is shown that $Int([kq+r+1, kq+q]_{\mathbb{Z}}) = \emptyset$ and hence $Int(Cl(\{x\})) = \emptyset \subset \{x\}$. Namely, the singleton $\{x\}$ is semi-closed; it is semi-peclosed (cf. (3.7), (3.5) and (3.11) in Section 3). ☐

Theorem 6.2 *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2)*. Assume that* $n \equiv r \pmod q$ *and* $1 \le r \le q-1$.

(1) (T_i*-axioms, where* $i \in \{2, 1, 3/4, 1/2, 0\}$; cf. (6.1),(6.2)).
(1-1) *If* $2 \le r \le q-1$*, then* $(\mathbb{Z}, \kappa(q,n))$ *is not a* T_0*-space.*
(1-2) *If* $r = 1$ *and* $q = 2$*, then* $(\mathbb{Z}, \kappa(q,n))$ *is a* $T_{3/4}$*-space and so it is a* $T_{1/2}$*-space; it is not a* T_1*-space* (cf. [10, Definition 4, Example 4.6]).
(1-3) *If* $r = 1$ *and* $3 \le q$*, then* $(\mathbb{Z}, \kappa(q,n))$ *is not a* T_0*-space.*

(2) (*Semi-*T_i*-separation axioms, where* $i \in \{2, 1, 1/2, 0\}$; cf. (6.4), (6.5).
(2-1) *If* $r = 1$ *and* $2 \le q$*, then* $(\mathbb{Z}, \kappa(q,n))$ *is a semi-*T_2*-space.*
(2-2) *If* $2 \le r \le q-1$*, then* $(\mathbb{Z}, \kappa(q,n))$ *is not a semi-*T_0*-space.*

(3) (*Pre-*T_i*-separation axioms, where* $i \in \{2, 1\}$; cf. (6.4), (6.5)).
(3-1) *If* $r = 1$ *and* $2 \le q$*, then* $(\mathbb{Z}, \kappa(q,n))$ *is not a pre-*T_1*-space.*
(3-2) *If* $2 \le r \le q-1$*, then* $(\mathbb{Z}, \kappa(q,n))$ *is a pre-*T_2*-space.*

(4) (β*-*T_i*-separation axioms, where* $i \in \{2, 1, 1/2\}$; cf. (6.4), (6.5)).
$(\mathbb{Z}, \kappa(q,n))$ *is a* β*-*T_2*-space.*

(5) (*Semi-pre-*$T_{1/2}$*-space*; cf. (6.3))
(5-1) *If* $1 \le r \le q-2$*, then* $(\mathbb{Z}, \kappa(q,n))$ *is not semi-pre-*$T_{1/2}$.
(5-2) *If* $1 \le r = q-1$*, then* $(\mathbb{Z}, \kappa(q,n))$ *is semi-pre-*$T_{1/2}$.

Proof. **(1) (1-1)** Assume that $n \equiv r \pmod q$, where $2 \le r$ and $r \le q-1$. Let $x := kq+1 \in \mathbb{Z}$ and $y := kq+r \in \mathbb{Z}$ for some integer k. We have $x \ne y$ because of $r \ne 1$. By Theorem 2.13 (b1) for the point x (resp. y), $Ker(\{x\})$ (resp. $Ker(\{y\})$) is the smallest open set containing x (resp. y). And, since $Ker(\{x\}) = [kq+1, kq+r]_{\mathbb{Z}} = Ker(\{y\})$ hold, $y \in Ker(\{x\})$

and $x \in Ker(\{y\})$; and hence $(\mathbb{Z}, \kappa(q,n))$ is not a T_0 space, where $n \equiv r \pmod q$ and $2 \leq r \leq q-1$.

(1-2) We assume that $q = 2$; and we claim that $(\mathbb{Z}, \kappa(2,n))$ is a $T_{3/4}$-space and it is not T_1, where $q = 2 < n$ and $n \equiv 1 \pmod 2$. First, by using Corollary 2.14 for $q = 2, 2 < n$ and $n' = 3$, it is shown that $\kappa(2,n) = \kappa(2,3)$ holds, since $n \equiv 3 \pmod 2$, $q = 2 < 3$ and $q = 2 < n$. Thus, $(\mathbb{Z}, \kappa(2,n))$ is $T_{3/4}$ and it is not T_1, since it is well known that the digital line $(\mathbb{Z}, \kappa) = (\mathbb{Z}, \kappa(2,3))$ is $T_{3/4}$ (cf. [10, Example 4.6]) and it is not T_1. Finally, we note that an alternative proof is given by using Theorem 2.13; we can claim that every singleton $\{x\}$ is closed or regular open (cf. (6.2) above, [10, Theorem 4.3]) and some singleton is not closed. Indeed, by Theorem 2.13 (b2)$'$ for $j = 2 = r+1$ and assumptions that $q = 2 = r+1$, it is shown that a singleton $\{k2+2\}$ is closed, where $k \in \mathbb{Z}$. For a singleton $\{k2+1\}$, it is regular open, where $k \in \mathbb{Z}$; its proof is as follows. By using Theorem 2.13 (b1) (b2) (resp. (b1)$'$) and assumption that $q = 2 = r+1$, it is shown that $Ker(\{k2\}) = [(k-1)2+1, k2+1]_{\mathbb{Z}}, Ker(\{k2+1\}) = \{k2+1\}$ and $Ker(\{k2+2\}) = [k2+1, k2+3]_{\mathbb{Z}}$ (resp. $Cl(\{k2+1\}) = [k2, k2+2]_{\mathbb{Z}}$) hold; and so $Int([k2, k2+2]_{\mathbb{Z}}) = \{k2+1\}$. Thus, we have that $Int(Cl(\{k2+1\})) = \{k2+1\}$; and hence the singleton $\{k2+1\}$ is regular open. And, the above singleton $\{k2+1\}$ is not closed.

(1-3) We assume that $3 \leq q$ and $r = 1$. Let $x := kq+j \in \mathbb{Z}$, where $2 \leq j \leq q$ and $y := kq+j' \in \mathbb{Z}$, where $2 \leq j' \leq q$ and $j \neq j'$ for some integer k. We have $x \neq y$, because of $3 \leq q$ and $j \neq j'$. By Theorem 2.13 (b2) for $r = 1$, $Ker(\{x\}) = Ker(\{y\}) = [kq+1, (k+1)q+r]_{\mathbb{Z}}$ is the smallest open set containing x and also it is the smallest open set containing y. Thus, $(\mathbb{Z}, \kappa(q,n))$ is not a T_0-space, where $n \equiv 1 \pmod q$, $q < n$ and $3 \leq q$.

(2) (2-1) We first use Theorem 3.2 (b1) and (b2) for $r = 1$. For each ordered pair (x,y) of distinct points x and y, we take disjoint semi-open sets U_x and U_y containing x and y, respectively, as follows: let k, k', j and j' be integers such that $2 \leq j \leq q$ and $2 \leq j' \leq q$.

Case 1. $x = kq+1, y = kq+j$, where $2 \leq j \leq q$: $U_x := \{x\}, U_y := V_2(y) = \{y\} \cup \{(k+1)q+1\}$ (cf. Theorem 3.2 (b1), (b2)(b2-1)).

Case 2. $x = kq+1, y = k'q+1$, where $k \neq k'$: $U_x := \{x\}, U_y := \{y\}$ (cf. Theorem 3.2 (b1)).

Case 3. $x = kq+1, y = k'q+j$, where $2 \leq j \leq q, k \neq k'$: $U_x := \{x\}, U_y := V_1(y) = \{y\} \cup \{k'q+1\}$ (cf. Theorem 3.2 (b1), (b2)(b2-1)).

Case 4. $x = kq+j, y = kq+j'$, where $2 \leq j \leq q, 2 \leq j' \leq q$ and $j \neq j'$: $U_x := V_1(x) = \{x\} \cup \{kq+1\}, U_y := V_2(y) = \{y\} \cup \{(k+1)q+1\}$ (cf. Theorem 3.2 (b2)(b2-1)). Notice: for $q = 2, x = y$; Case 4 above is removed from the proof for $q = 2$.

Case 5. $x = kq+j, y = k'q+j'$, where $2 \leq j \leq q, 2 \leq j' \leq q$ and $k \neq k'$: $U_x := V_1(x) = \{x\} \cup \{kq+1\}, U_y := V_1(y) = \{y\} \cup \{k'q+1\}$ (cf. Theorem 3.2 (b2)(b2-1)).

These properties above conclude that $(\mathbb{Z}, \kappa(q,n))$ is a semi-T_2-space, where $q < n, n \equiv 1$ (mod q) and $q \geq 2$.

(2-2) Under assumption that $2 \leq r \leq q-1$, we can take two singletons $\{x\} := \{kq+1\}$ and $\{y\} := \{kq+r\}$, where $k \in \mathbb{Z}$, such that $x, y \in sKer(\{kq+i\}) = [kq+1, kq+r]_{\mathbb{Z}} \in SO(\mathbb{Z}, \kappa(q,n))$, where $i \in \mathbb{Z}$ with $1 \leq i \leq r$ (cf. Theorem 3.2 (b1)). Then, for every semi-open sets U_x and U_y containing x and y respectively, we have that $x \in [kq+1, kq+r]_{\mathbb{Z}} = sKer(\{y\}) \subset U_y$ and $y \in U_x$ hold. Thus, $(\mathbb{Z}, \kappa(q,n))$ is not semi-T_0.

(3) (3-1) We show that $(\mathbb{Z}, \kappa(q,n))$ is not a pre-T_1-space if $r = 1$ and $2 \leq q$. We use Theorem 4.2 (b1-1)$'$ for $r = 1$; $pCl(\{kq+1\}) = [(k-1)q+2, kq+q]_{\mathbb{Z}}$ holds and so there exists a point $kq+1$ such that $\{kq+1\}$ is not preclosed. Namely, $(\mathbb{Z}, \kappa(q,n))$ is not pre-T_1, where $q < n$ and $n \equiv 1 \pmod q$ (cf. (6.5)).

(3-2) We shall prove that $(\mathbb{Z}, \kappa(q,n))$ is pre-T_2 if $2 \leq r \leq q-1$. We recall that for a point $kq+j \in \mathbb{Z}$, $V_{h,h'}(kq+j) := \{kq+j\} \cup \{kq+h, kq+q+h'\}$ is a preopen set containing the point $kq+j$, where $k \in \mathbb{Z}, r+1 \leq j \leq q, 1 \leq h \leq r$ and $1 \leq h' \leq r'$ (cf. Theorem 4.2 (b2)(b2-1)); moreover, for a point $kq+i \in \mathbb{Z}, \{kq+i\}$ is a preopen set, where $1 \leq i \leq r$ (cf. Theorem 4.2 (b1)(b1-1)). Under the assumption that $2 \leq r \leq q-1$, we have that

$kq + 1 \neq kq + r$ and

($*$) $V_{1,1}(kq + j) \cap V_{r,r}(k'q + j') = \emptyset$ for two distinct points $kq + j$ and $k'q + j'$ with $r + 1 \leq j \leq q$ and $r + 1 \leq j' \leq q$ (we assume $j \neq j'$ if $k = k'$).

We claim that any two distinct points, say x and y, are separated by preopen sets containing the points respectively.

Case 1. $x = kq + j$ and $y = k'q + j'$, where $j, j' \in [r + 1, q]_{\mathbb{Z}}$ and $j \neq j'$ if $k = k'$: for these points x and y, we put $U_x := V_{1,1}(kq + j)$ and $U_y := V_{r,r}(k'q + j')$. Then, by ($*$) above, it is shown that $U_x \cap U_y = \emptyset$.

Case 2. $x = kq + i$ and $y = k'q + j'$, where $i \in [1, r]_{\mathbb{Z}}$ and $j' \in [r + 1, q]_{\mathbb{Z}}$: for these points x and y, we put $U_x := \{kq + i\} \in PO(\mathbb{Z}, \kappa(q, n))$ (cf. Theorem 4.2 (b1)(b1-1)) and $U_y := V_{r,r}(k'q + j')$ if $i = 1$ and $U_y := V_{1,1}(k'q + j')$ if $i \neq 1$. Then, it is directly shown that $kq + i \notin U_y$ and so $U_x \cap U_y = \emptyset$.

Case 3. $x = kq + i$ and $y = k'q + i'$, where $i, i' \in [1, r]_{\mathbb{Z}}$ and $i \neq i'$ if $k = k'$: for these points x and y, we put $U_x := \{kq+i\} \in PO(\mathbb{Z}, \kappa(q, n))$ and $U_y := \{k'q+i'\} \in PO(\mathbb{Z}, \kappa(q, n))$ (cf. Theorem 4.2 (b1)(b1-1)). Then, it is obvious that $U_x \cap U_y = \emptyset$.

Therefore, for each case it is shown that $x \in U_x, y \in U_y, U_x \cap U_y = \emptyset$ and U_x and U_y are preopen in $(\mathbb{Z}, \kappa(q, n))$ and so $(\mathbb{Z}, \kappa(q, n))$ is pre-T_2.

(4) By (2)(2-1) above, $(\mathbb{Z}, \kappa(q, n))$ is semi-T_2 if $r = 1$ and $2 \leq q$; and so it is β-T_2 (cf. (6.6)). By (3)(3-2) above, $(\mathbb{Z}, \kappa(q, n))$ is pre-T_2 if $2 \leq r \leq q - 1$; and so it is β-T_2 (cf. (6.6)).

(5)(5-1) Under assumption that $1 \leq r \leq q - 2$, a singleton $\{kq + j\}$ is not closed, where $r + 1 \leq j \leq q$. Indeed, $Cl(\{kq + j\}) = [kq + r + 1, kq + q]_{\mathbb{Z}} \neq \{kq + j\}$, because $r + 1 < q$ (cf. Theorem 2.13 (b2)$'$). And, the singleton $\{kq + j\}$ is not preopen, where $r + 1 \leq j \leq q$ (cf. Theorem 5.1 (i)). Thus, there exists a singleton which is neither closed nor preopen and so this generalized digital line $(\mathbb{Z}, \kappa(q, n))$ is not semi-pre-$T_{1/2}$ (cf. (6.3), i.e. [9, Theorem 4.1]).

(5-2) Let x be a point of $\mathbb{Z}$. If $x = kq+j$, where $r+1 = j = q$, then $Cl(\{kq+j\}) = \{kq+j\}$ (cf. Theorem 2.13 (b2)$'$); if $x = kq + i$, where $1 \leq i \leq r = q - 1$, then $\{x\}$ is preopen (cf. Theorem 4.2 (b1)(b1-1)). Thus, this generalized digital line $(\mathbb{Z}, \kappa(q, n))$ is semi-pre-$T_{1/2}$ (cf. (6.3), i.e., [9, Theorem 4.1]). □

Proof of Theorem A(ii) The result (ii-1) is obtained by Theorem 6.2 (3)(3-2) above; the result (ii-2) is obtained by Theorem 6.2 (2)(2-1) and (1)(1-2) above. □

Let us present the tables of separation axioms of $(\mathbb{Z}, \kappa(q, n))$(cf. Definition 2.2).

Table 1. Separation axioms of $(\mathbb{Z}, \kappa(q, n))$ for the case where $q < n$ and $n \equiv r \pmod q$ $(1 \leq r \leq q - 1)$

r, q	T_i-axioms	semi-T_i-axioms/pre-T_i-axioms	β-T_i-axioms
$r = 1, q = 2$	$T_{3/4}$, Non T_1	semi-T_2 / Non pre-T_1	β-T_2
$r = 1, q \geq 3$	Non T_0	semi-T_2 / Non pre-T_1	β-T_2
$2 \leq r \leq q - 1$	Non T_0	Non semi-T_0 / pre-T_2	β-T_2

Table 2. Semi-pre-$T_{1/2}$ separation axioms of $(\mathbb{Z}, \kappa(q, n))$ for the case where $q < n$ and $n \equiv r \pmod q$ $(1 \leq r \leq q - 1)$

r, q	semi-pre-$T_{1/2}$-axiom
$r = 1, q = 2$	semi-pre-$T_{1/2}$
$r = 1, q \geq 3$	Non semi-pre-$T_{1/2}$
$2 \leq r \leq q - 2$	Non semi-pre-$T_{1/2}$
$2 \leq r = q - 1$	semi-pre-$T_{1/2}$

7 The connectedness of generalized digital lines and Proof of Theorem A(iii)

We recall the following: a topological space (X, τ) is said to be *semi-connected* ([7]) (resp.

preconnected ([41])), if it cannot be represented as the disjoint union of two nonempty semi-open (resp. preopen) subsets. The class of *semi-connected* (resp. *preconnected*) topological spaces was introduced by Phullenda Das [7] (resp. Popa [41]) in 1974 (resp. 1987).

Theorem 7.1 *Let* $(\mathbb{Z}, \kappa(q,n))$ *be a generalized digital line* (cf. Definition 2.2). *Suppose that* $n \equiv r \pmod q$, *where* $1 \le r \le q-1$. *Then,*
(i) $(\mathbb{Z}, \kappa(q,n))$ *is connected;*
(ii) $(\mathbb{Z}, \kappa(q,n))$ *is not semi-connected;*
(iii) *if* $2 \le r$, *then* $(\mathbb{Z}, \kappa(q,n))$ *is not preconnected;*
(iv) *if* $r = 1$, *then* $(\mathbb{Z}, \kappa(q,n))$ *is preconnected.*

Proof. **(i)** Suppose that $(\mathbb{Z}, \kappa(q,n))$ is not connected; i.e., there exists a nonempty open and closed subset U such that $U \ne \mathbb{Z}$. We shall show a contradiction (cf. $(*5)$, $(*6)$ below). Since $U \ne \emptyset$, we pick a point x of $\mathbb{Z}$ such that
$\cdot(*1)$ $x \in U$; let $x := kq + s$, where $k \in \mathbb{Z}$ and $s \in \mathbb{Z}$ with $1 \le s \le q$.

First, using above integer "k" of $x := kq + s$ $(1 \le s \le q)$, we construct the following sequences of points, $\{x_a\}_{a \in \mathbb{N}}$ and $\{x_a^-\}_{a \in \mathbb{N}}$ defined by:
$\cdot(*2)$ $x_a := (k+a)q$ and $x_a^- := (k-a+1)q$ for each $a \in \mathbb{N}$. Then, it is easily shown that: for each $a \in \mathbb{N}$,
$\cdot(*3)$ $x_a < x_{a+1},\ x_{a+1}^- < x_a^-$ and $x < x_a$ (if $a \ge 2$)$, x \le x_1,\ x_a^- < x$.

Secondly, we claim that: for each $a \in \mathbb{N}$,
$\cdot(*4)^a$ $[x, x_a]_{\mathbb{Z}} \subset U$ and $\cdot(**4)^a$ $[x_a^-, x]_{\mathbb{Z}} \subset U$.

Proof of $(*4)^a$. The proof is done by induction on $a \in \mathbb{N}$. For $a = 1$, we show $(*4)^1$. Indeed, by Theorem 2.13 (b1)$'$ (resp. (b2)$'$), it is shown that if the point x has a form $x = kq + i$ $(1 \le i \le r)$ (resp. $x = kq + j$ $(r+1 \le j \le q)$) then $[x, x_1]_{\mathbb{Z}} \subset [(k-1)q + r + 1, kq + q]_{\mathbb{Z}} = Cl(\{kq+i\}) \subset U$ (resp. $[x, x_1]_{\mathbb{Z}} \subset [kq + r + 1, kq + q]_{\mathbb{Z}} = Cl(\{kq+j\}) \subset U$) hold, because $x \in U$ and U is closed.

We suppose that $(*4)^t$ is true for an integer $t \in \mathbb{N}$ with $t \ge 2$, i.e., $[x, x_t]_{\mathbb{Z}} \subset U$, where $x_t = (k+t)q$ (cf. $(*2)$ above) and $t \ge 2$. We use Theorem 2.13 (b2) for the point $x_t = (k+t-1)q + j$, where $j = q$, and the assumption of induction, we have $Ker(\{x_t\}) = [(k+t-1)q+1, (k+t)q+r]_{\mathbb{Z}} \subset U$ because $x_t \in U$ and U is open; and so $(k+t)q + r \in U$. By using Theorem 2.13 (b1)$'$ for the above point $(k+t)q + r \in U$, it is shown that $Cl(\{(k+t)q+r\}) = [(k+t-1)q + r + 1, (k+t)q + q]_{\mathbb{Z}} \subset U$, because U is a closed subset such that $(k+t)q + r \in U$. Thus, we prove that $(k+t+1)q \in U$ (i.e., $x_{t+1} \in U$) and $[x_t, x_{t+1}]_{\mathbb{Z}} \subset [(k+t-1)q + r + 1, (k+t+1)q]_{\mathbb{Z}} = Cl(\{(k+t)q + r\}) \subset U$. Since $[x, x_{t+1}]_{\mathbb{Z}} = [x, x_t]_{\mathbb{Z}} \cup [x_t, x_{t+1}]_{\mathbb{Z}}$, we have that $[x, x_{t+1}]_{\mathbb{Z}} \subset U$ holds. Namely, we have the required property $(*4)^a$ for $a = t+1$. Thus, for any integer $a \in \mathbb{N}$, we have $(*4)^a$. $\diamond$

Proof of $(**4)^a$. The proof is also done by induction on $a \in \mathbb{N}$ as follows. For $a = 1$, the property $(**4)^1$ is true. Indeed, if $x = kq + i$ $(1 \le i \le r)$, then $[x_1^-, x]_{\mathbb{Z}} \subset [(k-1)q + r + 1, kq+q]_{\mathbb{Z}} = Cl(\{kq+i\}) = Cl(\{x\}) \subset U$ hold (cf. Theorem 2.13 (b1)$'$); and so $[x_1^-, x]_{\mathbb{Z}} \subset U$. If $x = kq + j$ $(r+1 \le j \le q)$, then $Ker(\{x\}) = [kq+1, (k+1)q + r]_{\mathbb{Z}} \subset U$ (cf. Theorem 2.13 (b2)); and so $kq + 1 \in U$. By using Theorem 2.13 (b1)$'$ for the point $kq+1$ above, it is shown that $x_1^- = kq \in [x_1^-, x]_{\mathbb{Z}} \subset [(k-1)q + r + 1, kq + q]_{\mathbb{Z}} = Cl(\{kq+1\}) \subset U$; and so $[x_1^-, x]_{\mathbb{Z}} \subset U$ hold.

We suppose that $(**4)^t$ is true for an integer $t \in \mathbb{N}$ with $t \ge 2$, i.e., $[x_t^-, x]_{\mathbb{Z}} \subset U$, where $x_t^- = (k-t+1)q$ (cf. $(*2)$ above) and $t \ge 2$. We see $(k-t)q + 1 \in U$. Indeed, using Theorem 2.13(b2) for the point $x_t^- = (k-t)q + j'$ with $j' = q$ and the assumption of induction, we have $(k-t)q+1 \in [(k-t)q+1, (k-t+1)q+r]_{\mathbb{Z}} = Ker(\{(k-t)q+q) = Ker(\{x_t^-\}) \subset U$ and so $(k-t)q + 1 \in U$. Now, by using Theorem 2.13 (b1)$'$ for the above point $(k-t)q+1$, it is shown that $Cl(\{(k-t)q+1\}) = [(k-t-1)q + r + 1, (k-t)q + q]_{\mathbb{Z}} \subset U$. Thus, for the point $x_{t+1}^- := (k-t)q$, we prove that $[x_{t+1}^-, x_t^-]_{\mathbb{Z}} \subset [(k-t-1)q + r + 1, (k-t+1)q]_{\mathbb{Z}} \subset U$ hold. Since $[x_{t+1}^-, x]_{\mathbb{Z}} = [x_{t+1}^-, x_t^-]_{\mathbb{Z}} \cup [x_t^-, x]_{\mathbb{Z}}$, we have that $[x_{t+1}^-, x]_{\mathbb{Z}} \subset U$ holds. Namely, we have the required property $(**4)^a$ for $a = t+1$. Thus, for any integer $a \in \mathbb{N}$, we have that $(**4)^a$ is true. $\diamond$

Finally, we proceed the proof as follows: take a point $y \in \mathbb{Z}$ such that
$\cdot(*5)$ $y \notin U$, because $U \neq \mathbb{Z}$; and let $y = s_0q + i_0$, where $s_0 \in \mathbb{Z}$ and $i_0 \in \mathbb{Z}$ with $1 \leq i_o \leq q$. Then, we consider the following two cases.

Case 1. $x < y$: for this case, using the sequence of points $\{x_a\}_{a\in\mathbb{N}}$ investigated by $(*2), (*3)$ and $(*4)$, we can pick a point $x_{t(0)}$ with $t(0) \in \mathbb{N}$ such that $y \leq x_{t(0)}$. Indeed, we take the integer $t(0)$ as $t(0) := s_0 - k + 1$ (cf. the integer k is given in $(*1)$ above); then $t(0) \geq 1$ and $y = s_0q + i_0 \leq (s_0+1)q = (t(0)+k-1+1)q = (k+t(0))q = x_{t(0)}$ (cf. $(*2)$ above); and so $x < y < x_{t(0)}$. By $(*4)^a$ above, it is shown that $y \in [x, x_{t(0)}]_\mathbb{Z} \subset U$; and so $y \in U$.

Case 2. $y < x$: for this case, using the sequence of points $\{x_a^-\}_{a\in\mathbb{N}}$ investigated by $(*2), (*3)$ and $(*4)$, we can pick a point $x^-_{t(1)}$ with $t(1) \in \mathbb{N}$, such that $x^-_{t(1)} \leq y$. Indeed, we take the integer $t(1)$ as $t(1) := k - s_0 + 1$; then $t(1) \geq 1$ and $y = s_0q + i_0 > s_0q = (k - t(1) + 1)q = x^-_{t(1)}$ (cf. $(*2)$ above); and so $x^-_{t(1)} < y < x$. By $(**4)^a$ above, it is shown that $y \in [x^-_{t(1)}, x]_\mathbb{Z} \subset U$; and so $y \in U$.

By both cases above, it is obtained that:
$\cdot(*6)$ $y \in U$ holds for the point $y \notin U$ (cf. $(*5)$ above).
This shows a contradiction; therefore, $(\mathbb{Z}, \kappa(q,n))$ is a connected topological space, where $n \equiv r \pmod q$ with $1 \leq r \leq q-1$.

(ii) For $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2) and a point $x := kq + i$, where $k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq r$, it is known that $sKer(\{x\}) = sCl(\{x\}) = [kq+1, kq+r]_\mathbb{Z}$ and $sKer(\{x\})$ is a nonempty semi-open proper subset of $(\mathbb{Z}, \kappa(q,n))$ and $sCl(\{x\})$ is semi-closed in $(\mathbb{Z}, \kappa(q,n))$ (cf. Theorem 3.2 (b1) and (b1)$'$). Therefore, $(\mathbb{Z}, \kappa(q,n))$ is not semi-connected.

(iii) For $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2) and a point $x := kq + i$ ($k \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $1 \leq i \leq r$), $pKer(\{x\}) = \{x\}$ holds and it is preopen (cf. Theorem 4.2 (b1)(b1-1)); if $2 \leq r$, then $\{x\}$ is preclosed (cf. Theorem 4.2 (b1)(b1-1)$'$). Thus, the singleton $\{x\}$ is a preopen and preclosed in $(\mathbb{Z}, \kappa(q,n))$ if $2 \leq r$; and so $(\mathbb{Z}, \kappa(q,n))$ is not preconnected if $2 \leq r$.

(iv) We assume that $n \equiv r \pmod q$ and $r = 1$. In order to prove that $(\mathbb{Z}, \kappa(q,n))$ is preconnected, we suppose that there exists a preopen and preclosed subset V such that $V \neq \emptyset$ and $V \neq \mathbb{Z}$. Since $V \neq \emptyset$, we pick a point $x \in \mathbb{Z}$ such that
$\cdot(*7)$ $x \in V$; let $x := kq + s$, where $k \in \mathbb{Z}$ and $s \in \mathbb{Z}$ with $1 \leq s \leq q$.
Using the above integer "k" of $x := kq + s$ $(1 \leq s \leq q)$, let $\{x_a\}_{a\in\mathbb{N}}$ and $\{x_a^-\}_{a\in\mathbb{N}}$ be the similar sequences of points (cf. $(*2)$ in the proof of (i) above) defined by:
$\cdot(*8)$ $x_a := (k+a)q$ and $x_a^- := (k-a+1)q$ for each $a \in \mathbb{N}$. And, they have the following same properties:
$\cdot(*9)$ $x_a < x_{a+1},\ x^-_{a+1} < x^-_a$ and $x < x_a$ (if $a \geq 2$), $x \leq x_1,\ x^-_a < x$ hold.

We first claim that: under the assumption that $x := kq+s \in V$ for some s with $1 \leq s \leq q$,
$\cdot(*10)$ $kq + 1 \in V$ holds; and
$\cdot(*11)$ $[x, x_1]_\mathbb{Z} \subset V$ and $[x_1^-, x]_\mathbb{Z} \subset V$ hold.

Proof of $(*10)$. If $x = kq+s$, where $s = 1$, then $kq+1 \in V$ (cf. $(*7)$ above). If $x = kq+s \in V$, where $2 \leq s \leq q$, we use Theorem 4.2 (b2)(b2-3) for the point $kq + j$, where $j = s$ and $2 \leq j \leq q$; and so we have $pKer(\{kq+s\}) = V_{1,1}(kq+s) = \{kq+1, kq+s, (k+1)q+1\} \subset V$, because V is preopen and $x := kq + s \in V$; thus $kq + 1 \in V$. $\diamond$

Proof of $(*11)$. Using Theorem 4.2 (b1)(b1-1)$'$ for the point $kq + 1$, we have $[x, x_1]_\mathbb{Z} \subset [(k-1)q+2, (k+1)q]_\mathbb{Z} = pCl(\{kq+1\}) \subset V$, because V is preclosed and $kq + 1 \in V$ (cf. $(*10)$ above). For the points $x_1^- = kq$ and $x = kq + s$ $(1 \leq s \leq q)$, we see that $[x_1^-, x]_\mathbb{Z} \subset pCl(\{kq+1\}) \subset V$. $\diamond$

Secondly, we claim that: for each $a \in \mathbb{N}$,
$\cdot(*12)^a$ $[x, x_a]_\mathbb{Z} \subset V$ and $\cdot(**12)^a$ $[x_a^-, x]_\mathbb{Z} \subset V$ hold.

Proof of $(*12)^a$. We shall use induction on a. The former part of $(*11)$ above shows that the case where $a = 1$ is true. We suppose the statement $(*12)^a$ for the case where $a = t > 1$ is true; then $[x, x_t]_{\mathbb{Z}} \subset V$. By Theorem 4.2 (b2)(b2-1) and (b2-3) for the point $x_t = (k+t-1)q+j \in V$, where $j = q$, it is shown that $pKer(\{x_t\}) = V_{1,1}((k+t-1)q+q) = \{(k+t-1)q+1, x_t, (k+t-1)q+q+1\}$; and so $(k+t)q+1 \in V$ holds, because $pKer(\{x_t\}) \subset V$. For the point $(k+t)q+1 \in V$, we use Theorem 4.2 (b1)(b1-1)$'$; then, we have $[x_t, x_{t+1}]_{\mathbb{Z}} = [(k+t)q, (k+t+1)q]_{\mathbb{Z}} \subset [(k+t-1)q+2, (k+t+1)q]_{\mathbb{Z}} = pCl(\{(k+t)q+1\}) \subset V$; and so $[x_t, x_{t+1}]_{\mathbb{Z}} \subset V$ hold. Since $[x, x_{t+1}]_{\mathbb{Z}} = [x, x_t]_{\mathbb{Z}} \cup [x_t, x_{t+1}]_{\mathbb{Z}}$, we show that $[x, x_{t+1}]_{\mathbb{Z}} \subset V$ holds. Therefore, by induction on a, the statement $(*12)^a$ is proved. ◇

The property $(**12)^a$ is proved by argument similar to that in the proof of $(*12)^a$ above; and so it is omitted. ◇

Finally, we shall find the following contradiction (cf. $(*14)$ bellow). There exists a point $y \in \mathbb{Z}$ such that:

$\cdot(*13)$ $y \notin V$, because $V \neq \mathbb{Z}$; and let $y = s_0q + i_0$, where $s_0 \in \mathbb{Z}$ and $i_0 \in \mathbb{Z}$ with $1 \leq i_0 \leq q$. Since $x \neq y$, we have the following two cases:

Case 1. $x < y$: for this case, we pick the following point x_b such that $x_b \geq y$, where $b := s_0 - k + 1$. Indeed, we have that $b \geq 1$ and $x_b = (k+b)q = s_0q + q \geq y$ hold. By $(*12)^a$ for $a = b$, it is shown that $y \in [x, x_b]_{\mathbb{Z}} \subset V$; and so $y \in V$.

Case 2. $y < x$: for this case, we pick the following point x_d^- such that $x_d^- < y$, where $d := k - s_0 + 1$. Indeed, we have that $d \geq 1$ and $x_d^- = (k-d+1)q = s_0q < y$ hold, because $1 \leq i_0 \leq q$. By $(**12)^a$ for $a = d$, it is shown that $y \in [x_d^-, x]_{\mathbb{Z}} \subset V$; and so $y \in V$.

By the both cases above, it is obtained that:

$\cdot(*14)$ $y \in V$ holds for the point $y \notin V$(cf. $(*13)$ above). This $(*14)$ shows a contradiction; therefore, $(\mathbb{Z}, \kappa(q,n))$ is preconnected, where $n \equiv 1 \pmod q$ (i.e. $r = 1$). □

Proof of Theorem A(iii) The proof is shown by Theorem 7.1 (i) above. □

We present the table of connectedness of $(\mathbb{Z}, \kappa(q,n))$ from Theorem 7.1.

Table. The connectedness of $(\mathbb{Z}, \kappa(q,n))$ (cf. Definition 2.2)

n, q		connectedness; semi-connectedness; preconnectedness
$n \equiv r \pmod q (1 \leq r \leq q-1)$	$\Rightarrow$	connected; non semi-connected
$n \equiv r \pmod q$ $(2 \leq r \leq q-1)$	$\Rightarrow$	connected; non preconnected
$n \equiv 1 \pmod q$	$\Rightarrow$	preconnected

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