Scientiae Mathematicae Japonicae

STABILITY OF INHOMOGENEOUS STATIONARY SOLUTIONS TO RACETRACK MODEL IN SPATIAL ECONOMY

Kensuke Ohtake, Atsushi Yagi

Received January 26, 2017 ; revised March 28, 2017

Abstract. We continue our study on the racetrack model. In the previous paper, we have shown that the global solution has an ω-limit which is a stationary solution. In this paper, we introduce a simplified racetrack model and study stability and instability of stationary solutions by using the linearization principle.

1 Introduction. We continue our study on the racetrack model which has been presented in [9] by M. Fujita, P. Krugman, A. Venables in order to describe the dynamics of a tutorial economic system on a circumference driven by economic incentives. The model is written by

$$(1.1)\quad \begin{cases} w(t,x) = \left[\int_S \{\mu\lambda(t,y)w(y,t) + (1-\mu)\phi(y)\} G(t,y)^{\sigma-1} e^{-(\sigma-1)\tau|x-y|} dy\right]^{\frac{1}{\sigma}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[0,\infty)\times S, \\ G(t,x) = \left[\int_S \lambda(t,y)w(t,y)^{1-\sigma} e^{-(\sigma-1)\tau|x-y|} dy\right]^{\frac{1}{1-\sigma}} \qquad (t,x)\in[0,\infty)\times S, \\ \omega(t,x) = w(t,x)G(t,x)^{-\mu} \qquad (t,x)\in[0,\infty)\times S, \\ \dfrac{\partial\lambda}{\partial t}(t,x) = \gamma\left[\omega(t,x) - \int_S \omega(t,y)\lambda(t,y)dy\right]\lambda(t,x) \qquad (t,x)\in[0,\infty)\times S, \\ \lambda(0,x) = \lambda_0(x) \qquad x\in S. \end{cases}$$

Here, S is a circumference on which economic regions exist continuously and x is a spatial variable varying on S. The unknown function $\lambda(t,x)$ is a function such that $\mu\lambda(t,x)$ denotes population density of manufacturing workers at time $t\in[0,\infty)$ at a position $x\in S$. The other unknown function $w(t,x)$ denotes nominal wage at $(t,x)\in[0,\infty)\times S$. The function $G(t,x)$ and $\omega(t,x)$ denote respectively, price index and real wage at $(t,x)\in[0,\infty)\times S$. The function ϕ is a given function such that $(1-\mu)\phi(x)$ denotes the density of agricultural workers on S. It is assumed that $0\le\phi\in L^1(S)$ and $\int_S\phi(x)dx=1$. The function $|x-y|$ denotes a symmetric distance between $x,\ y\in S$ along S. The exponent $0<\mu\le 1$ denotes a ratio of the manufacturing workers on S to the total number of (manufacturing and agricultural) workers. Meanwhile $\sigma>1$ stands for an index of preference for manufacturing goods, and $\tau>0$ stands for a parameter concerning the transportation cost.

In the previous paper [11], we have studied (1.1) mathematically and numerically. In fact, we have shown, after discussing the global existence, that the global solution has an ω-limit which is a stationary solution of (1.1) and that any stationary solution to (1.1) is either the homogeneous solution on S or an inhomogeneous solution whose manufacturing density is a sum of Dirac delta functions.

We are then interested in investigating stability of stationary solutions to (1.1). As mentioned in [9] (and indeed reviewed in [11]), the homogeneous stationary solution is always unstable. So, in this paper, our interest is addressed to considering inhomogeneous stationary solutions. Meanwhile, our numerical computations suggest that there are no continuous shifts of manufacturing regions along S but there is growth or decline in manufacturing population at each manufacturing region. In order to simplify the situations, we are here led to introduce a modified racetrack model of a finite number of manufacturing regions. We assume that the whole manufacturing population agglomerates only in a finite number of fixed positions $x_1, \cdots, x_M \in S$ which will be called manufacturing regions of the model. Then, (1.1) reduces to

$$(1.2)\quad \begin{cases} w_i(t)^\sigma = \mu \sum_{j=1}^{M} \lambda_j(t) w_j G_j(t)^{\sigma-1} e^{-(\sigma-1)\tau|x_i-x_j|} \\ \qquad\qquad + (1-\mu)\int_S \phi(y) G(t,y)^{\sigma-1} e^{-(\sigma-1)\tau|x_i-y|} dy, \\ \qquad\qquad\qquad\qquad\qquad\qquad t\in[0,\infty),\ i=1,2,\cdots,M, \\ G(t,x)^{1-\sigma} = \sum_{j=1}^{M} \lambda_j(t) w_j(t)^{1-\sigma} e^{-(\sigma-1)\tau|x_i-y|}, \quad t\in[0,\infty),\ x\in S, \\ \omega_i = w_i(t) G_i(t)^{-\mu}, \qquad t\in[0,\infty),\ i=1,2,\cdots,M, \\ \frac{d}{dt}\lambda_i(t) = \left\{ \omega_i(t) - \sum_{j=1}^{M} \lambda_j(t)\omega_j(t) \right\} \lambda_i(t), \qquad t\in[0,\infty),\ i=1,2,\cdots,M, \\ \lambda_i(0) = \lambda_{i,0}, \qquad i=1,2,\cdots,M. \end{cases}$$

Here, $w_i(t)$, $\lambda_i(t)$, and $\omega_i(t)$ are, respectively, nominal wage, manufacturing population size, and real wage, at time $t \in [0,\infty)$ and at manufacturing region $x_i \in S$.

We begin with deriving (1.2) from (1.1) by putting $\lambda(t,x) = \sum_{k=1}^{M} \lambda_k(t)\delta_k(x)$, $\delta_i(x)$ being the Dirac delta function with center x_i. Then, after formulating (1.2) mathematically in a suitable way, we construct a global solution as well its numerical computations. The second part of the paper is devoted to studying stability of stationary solutions to (1.2). As will been seen, (1.2) is finally reduced to an M dimensional ordinary differential equation but including complex implicit functions. Anyway, we can use the usual linearization principle for studying the stability. The linearization matrices at stationary solutions are calculated. In the case of $M = 2$ or 3, the signs of their eigenvalues are known, provided that the exponent $\alpha := \tau(\sigma - 1)$ is sufficiently small or sufficiently large. More refined results are obtained by numerical computations.

Generally speaking, increase of α provides existence of more stable stationary solutions under the no black hole condition

$$\frac{\sigma - 1}{\sigma} > \mu.$$

This condition suppresses too strong centripetal forces in spatial economy, and thus prevents the manufacturing workers from agglomerating to only single economic region (see [9, p59]). On the other hand, as α decreases, stationary solutions lose their stability and only stationary solutions with single manufacturing region remain stable (i.e., $M = 1$).

When M is fixed, say $M = 3$, the configuration of $x_1,\ x_2,\ x_3$ on S also influences the stability of its stationary solution. As will be seen, the symmetric configuration solution has the highest degree of stability, that is, can remain stable as α decreases than any other non-symmetric stationary solution.

Let us finally refer to some papers related to our paper. Actually, the racetrack model (1.1) is one of the many models which may be divided into discrete space models and continuous space models. The discrete space models have already been studied in many papers.

Especially, the bifurcation property of stationary solutions is well studied. We want to quote Castro-Correia da Silva-Mossay [5], Ikeda-Akamatsu-Kono [10], Akamatsu-Takayama [3], Akamatsu-Takayama-Ikeda [1], Akamatsu-Mori-Takayama [2]. Tabuchi-Thisse [13] consider the racetrack model in which the agricultural sector is distributed continuously, and the manufacturing sector is distributed discretely. This setting is similar to our model (1.2), however due to their assumption on a utility function of consumers, their model is quite different from (1.2). Most of the papers on stationary solutions to the racetrack model treat only the symmetric stationary solutions except a few paper Fabinger [7]. Using a discrete space model, Barbero-Zofío [4] discussed the relation between stability and a configuration (they call it space topology) of economic regions.

2 Modeling. In this section, we will sketch the derivation of (1.2) from (1.1). In what follows, α stands for $\alpha = \tau(\sigma-1)$. The manufacturing regions $x_1, \cdots, x_M \in S$ are positions at which all the manufacturing workers accumulate, and $\lambda_1(t), \cdots, \lambda_M(t)$ denote the manufacturing population size at time $t \in [0,\infty)$ at each manufacturing region. Then, the manufacturing population density $\lambda(t,x)$ on S is written in the form

$$\lambda(t,x) = \sum_{k=1}^{M} \lambda_k(t)\delta_k(x), \quad t \in [0,\infty), \quad x \in S, \tag{2.1}$$

where $\delta_k(x)$ is the Dirac delta function with center x_k. From $\int_S \lambda(t,x)dx = 1$, it holds that $\sum_{k=1}^{M} \lambda_k(t) = 1$ for any time t. By (2.1), the first equation of (1.1) becomes

$$\begin{aligned} w(t,x)^{\sigma} &= \mu \sum_{j=1}^{M} \lambda_j(t) w_j G(t,x_j)^{\sigma-1} e^{-\alpha|x-x_j|} \\ &\quad + (1-\mu)\int_S \phi(y) G(t,y)^{\sigma-1} e^{-\alpha|x-y|} dy, \quad t \in [0,\infty), \quad x \in S. \end{aligned} \tag{2.2}$$

So, the first equation of (1.2) is verified.

Let us write $w(t,x_i) = w_i(t)$ for $i = 1,\cdots,M$. By (2.1), the second equation of (1.1) becomes

$$G(t,x)^{1-\sigma} = \sum_{j=1}^{M} \lambda_j(t) w_j(t)^{1-\sigma} e^{-\alpha|x_i - x|}, \quad t \in [0,\infty), \quad x \in S, \tag{2.3}$$

hence the second equation of (1.2).

Let us write $\omega(t,x_i) = \omega_i(t)$ and $G(t,x_i) = G_i(t)$ for $i = 1,\cdots,M$. Then, the real wage at each manufacturing region is given by

$$\omega_i(t) = w_i(t) G_i(t)^{-\mu}, \quad t \in [0,\infty), \quad i = 1,\cdots,M. \tag{2.4}$$

Finally, the fourth equation of (1.1) reduces to

$$\frac{d}{dt}\lambda_i(t) = \left[\omega_i(t) - \sum_{k=1}^{M} \omega_k(t)\lambda_k(t)\right]\lambda_i(t), \quad i = 1,\cdots,M. \tag{2.5}$$

This is the fourth equation of (1.2).

3 Mathematical Formulation In this section, let us make mathematical formulation for (1.2).

3.1 Norms on $\mathbb{R}^M$. As seen, the unknown functions $w(t)$ and $\lambda(t)$ of (1.2) take both their values in $\mathbb{R}^M$. It is however convenient to use different norms of $\mathbb{R}^M$ for $w(t)$ and $\lambda(t)$.

We denote the space $\mathbb{R}^M$ equipped with the maximum norm $\|\cdot\|_\infty$ as E^∞, i.e.,

$$E^\infty = \left(\mathbb{R}^M, \|w\|_\infty = \max\{|w_1|, \cdots, |w_M|\}\right).$$

We further denote a positive subset of E^∞ as

$$E^\infty_+ = \{w \in E^\infty | w_i > 0,\ i = 1, \cdots, M\}.$$

It is reasonable to expect that $w(t) \in E^\infty_+$ for any $t > 0$.

On the other hand, we denote the space $\mathbb{R}^M$ equipped with the summation norm $\|\cdot\|_1$ as E^1, i.e.,

$$E^1 = \left(\mathbb{R}^M, \|\lambda\|_1 = |\lambda_1| + \cdots + |\lambda_M|\right).$$

We further consider a subset of E^1 such that

$$\mathcal{M} = \left\{\lambda \in E^1 | \lambda_i \geq 0,\ i = 1, \cdots, M, \quad \|\lambda\|_1 = 1\right\}.$$

It is reasonable to expect that $\lambda(t) \in \mathcal{M}$ for any $t > 0$.

3.2 Formation. We begin with formulating the first equation of (1.2) as a fixed point problem in E^∞. To do so, let us introduce the operator G which maps $E^\infty_+ \times \mathcal{M}$ into the space of continuous functions $\mathcal{C}(S)$ defined by

$$[G(f, \lambda)](x) = \left[\sum_j \lambda_j f_j^{\frac{1}{\sigma}-1} e^{-\alpha|x-x_j|}\right]^{\frac{1}{1-\sigma}}, \quad x \in S. \tag{3.1}$$

Put $[G(f,\lambda)](x_i) = G(f,\lambda)_i$ for $i = 1, \cdots, M$. We also introduce the operator $\Phi : E^\infty_+ \times \mathcal{M} \to E^\infty_+$ by

$$\Phi(f,\lambda)_i = \sum_{j=1}^M \frac{\mu\lambda_j f_j^{\frac{1}{\sigma}} e^{-\alpha|x_i-x_j|}}{G(f,\lambda)_j^{1-\sigma}} + (1-\mu)\int_S \frac{\phi(y)e^{-\alpha|x_i-y|}}{[G(f,\lambda)](y)^{1-\sigma}} dy, \qquad i = 1, \cdots, M. \tag{3.2}$$

Then, by putting $f = w^\sigma$, it is observed that at each t the first and second equations of (1.2) are confined into

$$f = \Phi(f, \lambda), \qquad f \in E^\infty_+, \quad \lambda \in \mathcal{M}. \tag{3.3}$$

Next, we formulate the fourth equation of (1.2) as an ordinary differential equation in E^1. To do so, let us introduce the operator $\omega : E^\infty_+ \times \mathcal{M} \to E^\infty_+$ given by

$$\omega(f,\lambda)_i = f_i^{\frac{1}{\sigma}} [G(f,\lambda)_i]^{-\mu}, \qquad i = 1, \cdots, M, \tag{3.4}$$

and the operator $\Psi : E^\infty_+ \times \mathcal{M} \to E^1$ given by

$$\Psi(f,\lambda)_i = \left[\omega(f,\lambda)_i - \sum_{k=1}^M \omega(f,\lambda)_k \lambda_k\right] \lambda_i, \quad i = 1, \cdots, M. \tag{3.5}$$

Then, the fourth equation of (1.2) becomes

$$\frac{d\lambda}{dt}(t) = \Psi(w(t), \lambda(t)).$$

In this way, putting $f(t) = w(t)^\sigma$, the problem (1.2) has been formulated as stationary and evolution equations:

$$\begin{cases} f(t) = \Phi(f(t), \lambda(t)), & 0 \le t < \infty, \\ \dfrac{d\lambda}{dt}(t) = \Psi(f(t), \lambda(t)), & 0 \le t < \infty, \\ \lambda(0) = \lambda_0 \end{cases} \tag{3.6}$$

in the product space

$$E^\infty \times E^1 = \left\{ (f, \lambda) \middle| f \in E^\infty, \ \lambda \in E^1 \right\}.$$

The initial value λ_0 is taken in $\mathcal{M}$.

4 Global solution. In this section, we construct a global solution for (3.6). This section consists of two subsections. In Subsection 4.1, the fixed point problem (3.3) is handled for each fixed $\lambda \in \mathcal{M}$. Based on the results, a local solution is constructed in Subsection 4.2 and is extended to global one in Subsection 4.3.

4.1 Fixed Point Problem (3.3)**.** For real numbers $0 < r_1 < r_2$, we set a bounded closed subset $E^\infty_{r_1,r_2}$ of E^∞ by

$$E^\infty_{r_1,r_2} := \{u \in E^\infty | r_1 \le u_i \le r_2, \ i = 1, \cdots, M\}.$$

In addition, denote the maximal value of the distance between the manufacturing regions as

$$\overline{d} = \max_{i,j} |x_i - x_j|.$$

Theorem 4.1. *Assume that $\sigma > 1$ and $\tau > 0$ are sufficiently small so that*

$$e^{\alpha\pi} < 1/\mu. \tag{4.1}$$

And, put numbers a and b as

$$\begin{aligned} a &= \left[\frac{(1-\mu)e^{-\alpha\pi}}{1-\mu e^{-\alpha\overline{d}}} \right]^\sigma, \\ b &= \left[\frac{(1-\mu)e^{\alpha\pi}}{1-\mu e^{\alpha\overline{d}}} \right]^\sigma, \end{aligned} \tag{4.2}$$

respectively. Then, for any $\lambda \in \mathcal{M}$, (3.3) has at least one solution f in $E^\infty_{a,b}$.

Proof. The proof is based on the Brouwer fixed point theorem.

The bounded closed subset $E^\infty_{a,b}$ is convex. In fact, for any $u, v \in E^\infty_{a,b}$ and $\theta \in (0,1)$, we have

$$\begin{aligned} [\theta u + (1-\theta)v]_i &= \theta u_i + (1-\theta)v_i \\ &\le \theta b + (1-\theta)b = b, \quad i = 1, \cdots, M. \end{aligned}$$

Similarly,

$$[\theta u + (1-\theta)v]_i \ge \theta a + (1-\theta)a = a, \quad i = 1, \cdots, M.$$

The operator $\Phi(\cdot, \lambda)$ defined by (3.2) maps $E^\infty_{a,b}$ into itself. In fact, for $i = 1, \cdots, M$,

$$\begin{aligned}\Phi(f,\lambda)_i &\le \mu b \sum_j \frac{\lambda_j e^{-\alpha|x_i-x_j|}}{\sum_k \lambda_k e^{-\alpha|x_j-x_k|}} \\ &\quad + (1-\mu) b^{1-\frac{1}{\sigma}} \int_S \frac{\phi(y) e^{-\alpha|x_i-y|}}{\sum_k \lambda_k e^{-\alpha|y-x_k|}} dy \\ &\le \mu b e^{\alpha \overline{d}} + (1-\mu) b^{1-\frac{1}{\sigma}} e^{\alpha\pi} \\ &= b\end{aligned}$$

due to the definition of b. Similarly, for $i = 1, \cdots, M$,

$$\begin{aligned}\Phi(f,\lambda)_i &\ge \mu a \sum_j \frac{\lambda_j e^{-\alpha|x_i-x_j|}}{\sum_k \lambda_k e^{-\alpha|x_j-x_k|}} \\ &\quad + (1-\mu) a^{1-\frac{1}{\sigma}} \int_S \frac{\phi(y) e^{-\alpha|x_i-y|}}{\sum_k \lambda_k e^{-\alpha|y-x_k|}} dy \\ &\ge \mu a e^{-\alpha\pi} + (1-\mu) a^{1-\frac{1}{\sigma}} e^{-\alpha\pi} \\ &= a.\end{aligned}$$

The operator $\Phi(\cdot, \lambda)$ is continuous in $E^\infty_{a,b}$. More strongly, it is actually Lipschitz continuous. Indeed, for any $f, g \in E^\infty_{a,b}$,

$$\begin{aligned}&|\Phi(f,\lambda)_i - \Phi(g,\lambda)_i| \\ &\le \mu \left| \sum_{j=1}^M \frac{\lambda_j f_j^{\frac{1}{\sigma}} e^{-\alpha|x_i-x_j|}}{\sum_{k=1}^M \lambda_k f_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_j-x_k|}} - \sum_{j=1}^M \frac{\lambda_j g_j^{\frac{1}{\sigma}} e^{-\alpha|x_i-x_j|}}{\sum_{k=1}^M \lambda_k g_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_j-x_k|}} \right| \\ &\quad + (1-\mu) \left| \int_S \frac{\phi(y) e^{-\alpha|x_i-y|}}{\sum_{k=1}^M \lambda_k f_k^{\frac{1}{\sigma}-1} e^{-\alpha|y-x_k|}} dy - \int_S \frac{\phi(y) e^{-\alpha|x_i-y|}}{\sum_{k=1}^M \lambda_k g_k^{\frac{1}{\sigma}-1} e^{-\alpha|y-x_k|}} dy \right| \\ &\le \mu \sum_j \frac{\lambda_j f_j^{\frac{1}{\sigma}} \sum_k \lambda_k \left| g_k^{\frac{1}{\sigma}-1} - f_k^{\frac{1}{\sigma}-1} \right| e^{-\alpha|x_j-x_k|}}{\left(\sum_k \lambda_k f_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_j-x_k|}\right)\left(\sum_k \lambda_k g_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_j-x_k|}\right)} e^{-\alpha|x_i-x_j|} \\ &\quad + \mu \sum_s \frac{\sum_k \lambda_k f_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_j-x_k|} \cdot \lambda_s \left| f_s^{\frac{1}{\sigma}} - g_j^{\frac{1}{\sigma}} \right|}{\left(\sum_k \lambda_k f_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_s-x_k|}\right)\left(\sum_k \lambda_k g_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_j-x_k|}\right)} e^{-\alpha|x_i-x_j|} \\ &\quad + (1-\mu) \int_S \frac{\phi(y) \sum_k \lambda_k \left| g_k^{\frac{1}{\sigma}-1} - f_k^{\frac{1}{\sigma}-1} \right| e^{-\alpha|y-x_k|}}{\left(\sum_k \lambda_k f_k^{\frac{1}{\sigma}-1} e^{-\alpha|y-x_k|}\right)\left(\sum_k \lambda_k g_k^{\frac{1}{\sigma}-1} e^{-\alpha|y-x_k|}\right)} dy \\ &\le \left\{ \frac{\mu}{\sigma} \left(\frac{a}{b}\right)^{2(\frac{1}{\sigma}-1)} e^{\alpha\overline{d}} + \frac{\mu(\sigma-1)}{\sigma} \left(\frac{a}{b}\right)^{\frac{1}{\sigma}-2} e^{\alpha\overline{d}} \right. \\ &\qquad\qquad \left. + \frac{(1-\mu)(\sigma-1)}{\sigma} \frac{a^{\frac{1}{\sigma}-2}}{b^{2(\frac{1}{\sigma}-1)}} e^{\alpha\pi} \right\} \|f-g\|_\infty.\end{aligned}$$

Therefore, $\Phi(\cdot,\lambda)$ is Lipschitz continuous with the Lipschitz constant

$$L = \frac{\mu}{\sigma}\left(\frac{a}{b}\right)^{2(\frac{1}{\sigma}-1)} e^{\alpha\bar{d}} + \frac{\mu(\sigma-1)}{\sigma}\left(\frac{a}{b}\right)^{\frac{1}{\sigma}-2} e^{\alpha\bar{d}} + \frac{(1-\mu)(\sigma-1)}{\sigma}\frac{a^{\frac{1}{\sigma}-2}}{b^{2(\frac{1}{\sigma}-1)}}e^{\alpha\pi}. \tag{4.3}$$

As shown, $\Phi(\cdot,\lambda)$ is a Lipschitz continuous operator from the bounded closed convex subset $E^\infty_{a,b}$ into itself. Then, by the Brouwer fixed point theorem, (3.3) has at least one solution $f \in E^\infty_{a,b}$. □

Uniqueness of the solution is obtained by the following theorem.

Theorem 4.2. *In addition to* (4.1)*, assume that*

$$\frac{\mu}{\sigma}\left(\frac{a}{b}\right)^{2(1/\sigma-1)} e^{\alpha\bar{d}} + \frac{\mu(\sigma-1)}{\sigma}\left(\frac{a}{b}\right)^{1/\sigma-2} e^{\alpha\bar{d}} + \frac{(1-\mu)(\sigma-1)}{\sigma}\frac{a^{1/\sigma-2}}{b^{2(1/\sigma-1)}}e^{\alpha\pi} < 1. \tag{4.4}$$

Then, for any $\lambda \in \mathcal{M}$, the solution $f \in E^\infty_{a,b}$ to (3.3) *is unique.*

Proof. Since (4.4) means that $L < 1$, (4.4) implies that $\Phi(\cdot,\lambda)$ is a contraction on $E^\infty_{a,b}$. □

Because of the following theorem, the solution f constructed in Theorem 4.2 is unique in the whole space E^∞_+.

Theorem 4.3. *Under* (4.1)*, any solution to* (3.3) *in E^∞_+ actually lies in $E^\infty_{a,b}$.*

Proof. Let $f \in E^\infty_+$ be a solution to (3.3). Then, an upper estimate such as

$$\begin{aligned} f_i &= \Phi(f,\lambda)_i \\ &\le \mu \max_i |f_i| e^{\alpha\bar{d}} + (1-\mu)\left(\max_i |f_i|\right)^{1-\frac{1}{\sigma}} e^{\alpha\pi} \end{aligned}$$

holds. By solving this inequality for $\max_i |f_i|$, we see that $\max_i |f_i| \le b$.

On the other hand, a lower estimate such as

$$\begin{aligned} f_i &= \Phi(f,\lambda)_i \\ &\ge \mu \min_i |f_i| e^{-\alpha\bar{d}} + (1-\mu)\left(\min_i |f_i|\right)^{1-\frac{1}{\sigma}} e^{-\alpha\pi} \end{aligned}$$

holds, too. By solving this inequality for $\min_i |f_i|$, we see that $\min_i |f_i| \ge a$. □

The following proposition gives upper and lower bounds for $G(f,\lambda)$ and $\omega(f,\lambda)$ when (f,λ) varies in $E^\infty_{a,b} \times \mathcal{M}$.

Proposition 4.1. *For $i = 1, \cdots, M$, we have the estimates*

$$a^{\frac{1}{\sigma}} \le G(f,\lambda)_i \le b^{\frac{1}{\sigma}} e^{\tau\bar{d}}, \quad (f,\lambda) \in E^\infty_{a,b} \times \mathcal{M}, \tag{4.5}$$

$$a^{\frac{1}{\sigma}} b^{-\frac{\mu}{\sigma}} e^{-\mu\tau\bar{d}} \le \omega(f,\lambda)_i \le b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}}, \quad (f,\lambda) \in E^\infty_{a,b} \times \mathcal{M}. \tag{4.6}$$

Proof. These estimates are verified by direct calculations in view of the definitions of $G(f,\lambda)$ and $\omega(f,\lambda)$ and the range condition $a \le f_i \le b$, $i = 1, \cdots, M$. For example, the upper bound for $G(f,\lambda)$ is verified by

$$\begin{aligned} G(f,\lambda)_i &= \left[\sum_{j=1}^{M} \lambda_j f_j^{\frac{1}{\sigma}-1} e^{-\alpha|x_i-x_j|}\right]^{\frac{1}{1-\sigma}} \\ &\le \left[b^{\frac{1}{\sigma}-1} e^{-\alpha\overline{d}}\right]^{\frac{1}{1-\sigma}} \\ &= b^{\frac{1}{\sigma}} e^{\tau\overline{d}}. \end{aligned}$$

□

In the case when the fixed point problem (3.3) admits a unique solution $f \in E_+^\infty$ for $\lambda \in \mathcal{M}$, we denote it by $f = \Phi_{\rm f}(\lambda)$. Then, (3.6) ultimately reduces to the Cauchy problem

$$\begin{cases} \dfrac{d\lambda}{dt}(t) = \Psi(\Phi_{\rm f}(\lambda(t)), \lambda(t)), \qquad 0 \le t < \infty, \\ \lambda(0) = \lambda_0 \end{cases} \tag{4.7}$$

in E^1 with an initial value $\lambda_0 \in \mathcal{M}$.

4.2 Local Solution. We construct a local solution to (4.7) using the Banach fixed point theorem. The following proposition plays an important role.

Proposition 4.2. *Under* (4.1) *and* (4.4), *the estimates*

$$\|\Phi_{\rm f}(\lambda) - \Phi_{\rm f}(\kappa)\|_\infty \le \beta_1 \|\lambda - \kappa\|_1, \quad \lambda,\ \kappa \in \mathcal{M}(S), \tag{4.8}$$

$$\|G(\Phi_{\rm f}(\lambda),\lambda) - G(\Phi_{\rm f}(\kappa),\kappa)\|_\infty \le \beta_2 \|\lambda - \kappa\|_1, \quad \lambda,\ \kappa \in \mathcal{M}(S), \tag{4.9}$$

$$\|\omega(\Phi_{\rm f}(\lambda),\lambda) - \omega(\Phi_{\rm f}(\kappa),\kappa)\|_\infty \le \beta_3 \|\lambda - \kappa\|_1, \quad \lambda,\ \kappa \in \mathcal{M}(S) \tag{4.10}$$

hold true with some constants $\beta_1,\ \beta_2,\ \beta_3 > 0$.

Proof. It suffices to prove (4.8), because (4.9) and (4.10) are easily verified from (4.8).

For $\lambda,\ \kappa \in \mathcal{M}$, we write $f = \Phi_{\rm f}(\lambda),\ g = \Phi_{\rm f}(\kappa)$, and we use the following notations

$$\begin{aligned} A_j &= \sum_k \kappa_k g_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_j-x_k|}, \quad j = 1, \cdots, M, \\ B_j &= \sum_k \lambda_k f_k^{\frac{1}{\sigma}-1} e^{-\alpha|x_j-x_k|}, \quad j = 1, \cdots, M, \\ A(y) &= \sum_k \kappa_k g_k^{\frac{1}{\sigma}-1} e^{-\alpha|y-x_k|}, \quad y \in S, \\ B(y) &= \sum_k \lambda_k f_k^{\frac{1}{\sigma}-1} e^{-\alpha|y-x_k|}, \quad y \in S. \end{aligned}$$

Then,

$$\begin{aligned} |f_i - g_i| \le{}& \mu \sum_j \frac{\left|\lambda_j f_j^{\frac{1}{\sigma}} A_j - \kappa_j g_j^{\frac{1}{\sigma}} B_j\right|}{A_j B_j} e^{-\alpha|x_i-x_j|} \\ &+ (1-\mu) \int_S \frac{|A(y) - B(y)|}{A(y)B(y)} \phi(y) e^{-\alpha|x_i-y|} dy. \end{aligned} \tag{4.11}$$

Furthermore, it follows that

$$
\begin{aligned}
&\left|\lambda_j f_j^{\frac{1}{\sigma}} A_s - \kappa_j g_j^{\frac{1}{\sigma}} B_s\right| \\
&\le \lambda_j f_j^{\frac{1}{\sigma}} \sum_k \left|\kappa_k g_k^{\frac{1}{\sigma}-1} - \lambda_k f_k^{\frac{1}{\sigma}-1}\right| e^{-\alpha|x_j-x_k|} + B_s \left|\lambda_j f_j^{\frac{1}{\sigma}} - \kappa_j g_j^{\frac{1}{\sigma}}\right| \\
&\le \lambda_j f_j^{\frac{1}{\sigma}} \sum_k \left\{\kappa_k |g_k^{\frac{1}{\sigma}-1} - f_k^{\frac{1}{\sigma}-1}| + f_k^{\frac{1}{\sigma}-1}|\kappa_k - \lambda_k|\right\} e^{-\alpha|x_j-x_k|} \\
&\qquad + B_j \left\{\lambda_j |f_j^{\frac{1}{\sigma}} - g_j^{\frac{1}{\sigma}}| + g_j^{\frac{1}{\sigma}} |\lambda_j - \kappa_j|\right\} \\
&\le \left(\frac{\sigma-1}{\sigma}\right) a^{\frac{1}{\sigma}-2} b^{\frac{1}{\sigma}} \lambda_j \|f-g\|_\infty \sum_k \kappa_k e^{-\alpha|x_j-x_k|} \\
&\qquad + a^{\frac{1}{\sigma}-1} b^{\frac{1}{\sigma}} \lambda_s \|\kappa - \lambda\|_1 \\
&\qquad + \frac{1}{\sigma} a^{2\left(\frac{1}{\sigma}-1\right)} \lambda_j \|f-g\|_\infty \sum_k \lambda_k e^{-\alpha|x_j-x_k|} \\
&\qquad + a^{\frac{1}{\sigma}-1} b^{\frac{1}{\sigma}} |\lambda_j - \kappa_j| \sum_k \lambda_k e^{-\alpha|x_j-x_k|}.
\end{aligned} \tag{4.12}
$$

It is also verified by the similar calculations that

$$
\begin{aligned}
&|A(y) - B(y)| \le \\
&\left(\frac{\sigma-1}{\sigma}\right) a^{\frac{1}{\sigma}-2} \|f-g\|_\infty \sum_k \kappa_k e^{-\alpha|y-x_k|} + a^{\frac{1}{\sigma}-1} \|\lambda - \kappa\|_1.
\end{aligned} \tag{4.13}
$$

In addition, the estimates

$$
\begin{aligned}
A_j B_j &\ge b^{2\left(\frac{1}{\sigma}-1\right)} \left(\sum_k \kappa_k e^{-\alpha|x_j-x_k|}\right) \left(\sum_k \lambda_k e^{-\alpha|x_j-x_k|}\right), \\
A(y) B(y) &\ge b^{2\left(\frac{1}{\sigma}-1\right)} \left(\sum_k \kappa_k e^{-\alpha|y-x_k|}\right) \left(\sum_k \lambda_k e^{-\alpha|y-x_k|}\right)
\end{aligned} \tag{4.14}
$$

hold obviously. Using (4.12), (4.13), and (4.14), and noticing (4.4), we conclude from (4.11) that

$$
\|f-g\|_\infty \le \beta_1 \|\lambda - \kappa\|_1 ,
$$

i.e., (4.8), where

$$
\begin{aligned}
\beta_1 = &\left\{1 - \frac{\mu}{\sigma}\left(\frac{a}{b}\right)^{2(1/\sigma-1)} e^{\alpha\bar{d}} - \frac{\mu(\sigma-1)}{\sigma}\left(\frac{a}{b}\right)^{1/\sigma-2} e^{\alpha\bar{d}}\right. \\
&\left. - \frac{(1-\mu)(\sigma-1)}{\sigma} \frac{a^{1/\sigma-2}}{b^{2(1/\sigma-1)}} e^{\alpha\pi}\right\}^{-1} \\
&\times \left\{\mu \frac{a^{\frac{1}{\sigma}-1}}{b^{\frac{1}{\sigma}-2}}\left(e^{2\alpha\bar{d}} + e^{\alpha\bar{d}}\right) + (1-\mu)\frac{a^{\frac{1}{\sigma}-1}}{b^{2\left(\frac{1}{\sigma}-1\right)}} e^{2\alpha\pi}\right\}.
\end{aligned}
$$

□

To construct a local solution to (4.7), we have to introduce an auxiliary problem for (4.7). For a given $\tilde{\lambda} \in \mathcal{M}$, let $\tilde{\Psi}$ be an operator from $E^\infty \times \mathcal{M}$ to E^1 defined by

$$\tilde{\Psi}(w,\lambda)_i = \left[\omega(w,\tilde{\lambda})_i - \sum_{k=1}^{M} \omega(w,\tilde{\lambda})_k \lambda_k\right] \lambda_i, \quad i = 1, \cdots, M. \tag{4.15}$$

For a given $\tilde{\lambda} \in \mathcal{C}([0,\infty);\mathcal{M})$, consider an auxiliary problem

$$\begin{cases} \dfrac{d\lambda}{dt}(t) = \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}(t)), \lambda(t)), & 0 \le t < \infty, \\ \lambda(0) = \lambda_0. \end{cases} \tag{4.16}$$

Proposition 4.3. *Under* (4.1) *and* (4.4)*, let* $\tilde{\lambda}$ *be given as above. Then,* (4.16) *possesses a unique local solution* $\lambda \in \mathcal{C}^1([0,c];\mathcal{M})$*, provided that* $(1 \ge)\ c > 0$ *is sufficiently small, but* c *being independent of the given function* $\tilde{\lambda}$ *and the initial value* λ_0*.*

Proof. Set a closed subset of E^1 given by

$$E^1_1 := \left\{ \lambda \in E^1 \,\middle|\, \sum_{j=1}^{M} \lambda_j = 1 \right\},$$

and define an operator $\tilde{T} : \mathcal{C}([0,c];E^1_1) \to \mathcal{C}([0,c];E^1)$ by

$$\left[\tilde{T}(\lambda)\right](t) = \lambda_0 + \int_0^t \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}(s)), \lambda(s))ds.$$

Using $\tilde{T}$, we rewrite (4.16) into an equivalent problem

$$\lambda(t) = [\tilde{T}(\lambda)](t), \quad 0 \le t < \infty.$$

It is verified that $\tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), \lambda)$ is Lipschitz continuous with respect to $\lambda \in E^1_1$. Indeed, by (4.6) and (4.15), we see that

$$\left\| \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), \lambda) - \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), \kappa) \right\|_1 \le 3 b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} \|\lambda - \kappa\|_1, \quad \lambda, \kappa \in E^1_1. \tag{4.17}$$

Meanwhile, $\tilde{T}$ maps $\mathcal{C}([0,c];E^1_1)$ into itself. To verify this, it is sufficient to see that $\sum_j \tilde{T}(\lambda)_j = 1$, because $\tilde{T}$ obviously maps $\mathcal{C}([0,c];E^1_1)$ into $\mathcal{C}([0,c];E^1)$. Then,

$$\begin{aligned} \sum_j \left[\tilde{T}(\lambda)\right]_j (t) - \sum_j \lambda_{0,j} &= \sum_j \int_0^t \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}(s)), \lambda(s))_j ds \\ &= \int_0^t \sum_j \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}(s)), \lambda(s)) ds = 0 \end{aligned}$$

due to (4.15).

From (4.17),

$$\begin{aligned} \left\| \tilde{T}(\lambda) - \tilde{T}(\kappa) \right\|_{\mathcal{C}([0,c];E^1)} &\le \max_{t \in [0,c]} e^{-t} \int_0^t \left\| \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), \lambda) - \tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), \kappa) \right\|_1 ds \\ &\le 3 b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} \max_{t \in [0,c]} e^{-t} \int_0^t \|\lambda(s) - \kappa(s)\|_1 ds \\ &\le 3 b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} \max_{t \in [0,c]} e^{-t} \int_0^t \|\lambda(s) - \kappa(s)\|_1 e^{-s} e^{s} ds \\ &\le 3 b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} (1 - e^{-c}) \|\lambda - \kappa\|_{\mathcal{C}([0,c];E^1)}. \end{aligned}$$

Therefore, if c is sufficiently small, then $\tilde{T}$ becomes a contraction mapping. Thus, (4.16) has a unique fixed point $\lambda \in \mathcal{C}^1([0,c];E_1^1)$ for sufficiently small $c > 0$.

As a matter of fact, this $\lambda \in \mathcal{C}([0,c];E_1^1)$ is in $\mathcal{C}([0,c];\mathcal{M})$. Indeed, it is sufficient to verify $\lambda_i(t) \geq 0,\ \forall i = 1,\cdots,M$ for $t \in [0,c]$. Since the solution to (4.16) can be written as

$$\lambda_i(t) = \lambda_{0,i} \exp\left[\int_0^t \left\{\omega(\Phi_{\rm f}(\tilde{\lambda}(s)),\tilde{\lambda}(s))_i - \sum_k \omega(\Phi_{\rm f}(\tilde{\lambda}(s)),\tilde{\lambda}(s))_k \lambda_k(s)\right\} ds\right],$$

$\lambda_{0,i} \geq 0,\ i = 1,\cdots,M$, imply that $\lambda_i(t) \geq 0$ for all $i = 1,\cdots,M$.

As seen above, the time $c > 0$ was determined independently of $\tilde{\lambda}$ and λ_0. □

Now, we are ready to construct a local solution to (4.7).

Theorem 4.4. *Under* (4.1) *and* (4.4)*, for each* $\lambda_0 \in \mathcal{M}$*, there exists a unique local solution* $\lambda \in \mathcal{C}^1([0,c];\mathcal{M})$ *to* (4.7)*, provided that* $(1 \geq)\ c > 0$ *is sufficiently small, but* c *being independent of the initial value* λ_0*.*

Proof. By virtue of Proposition 4.3, for each λ_0, we can define an operator F_{λ_0} which corresponds $\tilde{\lambda} \in \mathcal{C}^1([0,c];\mathcal{M})$ to the local solution $\lambda \in \mathcal{C}^1([0,c];\mathcal{M})$ of the auxiliary problem (4.16). By the definition of F_{λ_0}, it immediately follows that

$$\left[F_{\lambda_0}(\tilde{\lambda})\right](t) = \lambda_0 + \int_0^t \tilde{\Psi}\left(\Phi_{\rm f}(\tilde{\lambda}(s)), [F_{\lambda_0}(\tilde{\lambda})](s)\right) ds.$$

If there exists a fixed point of F_{λ_0}, then it is obviously a local solution to (4.7). So, we will prove that F_{λ_0} is a contraction mapping from $\mathcal{C}^1([0,c];\mathcal{M})$ into itself.

For $\tilde{\lambda},\ \tilde{\kappa} \in \mathcal{C}([0,c];\mathcal{M})$,

$$\begin{aligned}
&\left\|\left[F_{\lambda_0}(\tilde{\lambda})\right](t) - \left[F_{\lambda_0}(\tilde{\kappa})\right](t)\right\|_1 \\
&\leq \int_0^t \left\|\tilde{\Psi}(\Phi_{\rm f}(\tilde{\lambda}), F_{\lambda_0}(\tilde{\lambda})) - \tilde{\Psi}(\Phi_{\rm f}(\tilde{\kappa}), F_{\lambda_0}(\tilde{\kappa}))\right\|_1 ds \\
&\leq \int_0^t \left\|\omega(\Phi_{\rm f}(\tilde{\lambda}),\tilde{\lambda})F_{\lambda_0}(\tilde{\lambda}) - \omega(\Phi_{\rm f}(\tilde{\kappa}),\tilde{\kappa})F_{\lambda_0}(\tilde{\kappa})\right\|_1 ds \\
&\quad + \int_0^t \left\|\sum_k \omega(\Phi_{\rm f}(\tilde{\kappa}),\tilde{\kappa})_k F_{\lambda_0}(\tilde{\kappa})_k \cdot F_{\lambda_0}(\tilde{\kappa}) \right. \\
&\qquad \left. - \sum_k \omega(\Phi_{\rm f}(\tilde{\lambda}),\tilde{\lambda})_k F_{\lambda_0}(\tilde{\lambda})_k \cdot F_{\lambda_0}(\tilde{\lambda})\right\|_1 ds.
\end{aligned} \tag{4.18}$$

Note that $\sum_i F_{\lambda_0}(\tilde{\kappa})_i = 1$, then it follows from (4.6) and (4.10) that

$$\begin{aligned}
&\left\|\omega(\Phi_{\rm f}(\tilde{\lambda}),\tilde{\lambda})F_{\lambda_0}(\tilde{\lambda}) - \omega(\Phi_{\rm f}(\tilde{\kappa}),\tilde{\kappa})F_{\lambda_0}(\tilde{\kappa})\right\|_1 \\
&\leq b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} \left\|F_{\lambda_0}(\tilde{\lambda})(t) - F_{\lambda_0}(\tilde{\kappa})(t)\right\|_1 + \beta_3 \left\|\tilde{\lambda} - \tilde{\kappa}\right\|_1
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\sum_k \omega(\Phi_{\rm f}(\tilde{\kappa}),\tilde{\kappa})_k F_{\lambda_0}(\tilde{\kappa})_k \cdot F_{\lambda_0}(\tilde{\kappa}) \right. \\
&\qquad \left. - \sum_k \omega(\Phi_{\rm f}(\tilde{\lambda}),\tilde{\lambda})_k F_{\lambda_0}(\tilde{\lambda})_k \cdot F_{\lambda_0}(\tilde{\lambda})\right\|_1 \\
&\leq 2b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} \left\|F_{\lambda_0}(\tilde{\lambda}) - F_{\lambda_0}(\tilde{\kappa})\right\|_1 + \beta_3 \left\|\tilde{\lambda} - \tilde{\kappa}\right\|_1.
\end{aligned}$$

By applying these estimates to (4.18), it follows that

$$\left\| \left[F_{\lambda_0}(\tilde{\lambda})\right](t) - \left[F_{\lambda_0}(\tilde{\kappa})\right](t) \right\|_1 \le 3b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}} \int_0^t \left\| F_{\lambda_0}(\tilde{\lambda})(s) - F_{\lambda_0}(\tilde{\kappa})(s) \right\|_1 ds$$
$$+ 2\beta_3 \int_0^t \left\| \tilde{\lambda}(s) - \tilde{\kappa}(s) \right\|_1 ds, \quad 0 \le t \le c.$$

As a result, we obtain that

$$\left\| F_{\lambda_0}(\tilde{\lambda}) - F_{\lambda_0}(\tilde{\kappa}) \right\|_{\mathcal{C}([0,c];E^1)} \le k \left\| \tilde{\lambda} - \tilde{\kappa} \right\|_{\mathcal{C}([0,c];E^1)},$$

with

$$k = \frac{2\beta_3(1-e^{-c})}{1 - 3b^{\frac{1}{\sigma}} a^{-\frac{\mu}{\sigma}}(1-e^{-c})}.$$

Therefore, if $c > 0$ is sufficiently small, then $k < 1$, and F_{λ_0} is a contraction mapping in $\mathcal{C}([0,c];\mathcal{M})$. □

4.3 Global Solution. We can now easily extend the local solution of (4.7) to global one.

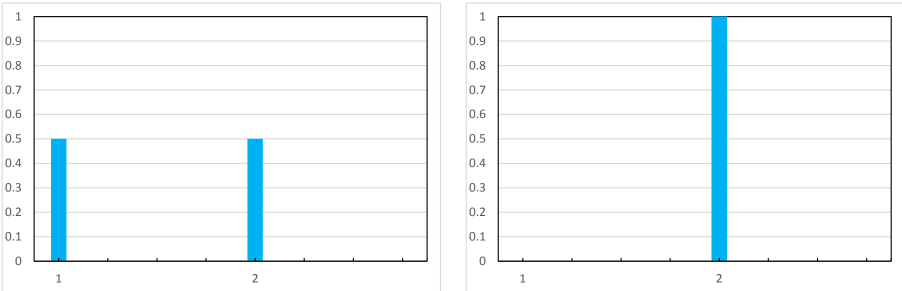
Theorem 4.5. *Under* (4.1) *and* (4.4)*, for each* $\lambda_0 \in \mathcal{M}$*, there exists a unique global solution* $\lambda \in \mathcal{C}^1([0,\infty);\mathcal{M})$ *to* (4.7).

Proof. Note that the interval $[0,c]$ on which we construct a local solution is independent of the initial value λ_0. Then, the uniqueness of the local solution shows that the unique local solution $\lambda \in \mathcal{C}^1([0,2c];\mathcal{M})$ is obtained by repeating the same argument but with the initial value $\lambda(c)$. By repeating this procedure, we finally obtain a unique global solution to (4.7). □

5 Numerical Results. In this section, some examples of numerical computations are illustrated. In Subsection 5.1, the case of $M = 2$, and in Subsection 5.2, the case of $M = 3$ is handled, respectively. Throughout this section, the parameters μ and σ are fixed as $\mu = 0.5$ and $\sigma = 3$. And $\tau > 0$ is changed as a control parameter. The density of agricultural workers is assumed to be constant, i.e., $\phi(x) \equiv \frac{1}{2\pi}$. The initial value for the manufacturing population size $\lambda = (\lambda_1, \cdots, \lambda_M)$ is given by adding small perturbations to the uniform population size $\lambda_i = \overline{\lambda} \equiv 1/M,\ i = 1, \cdots, M$. The circumference S is identified with the interval $[-\pi, \pi]$. In the following, we refer the manufacturing region as the region and the manufacturing population as the population for simplicity.

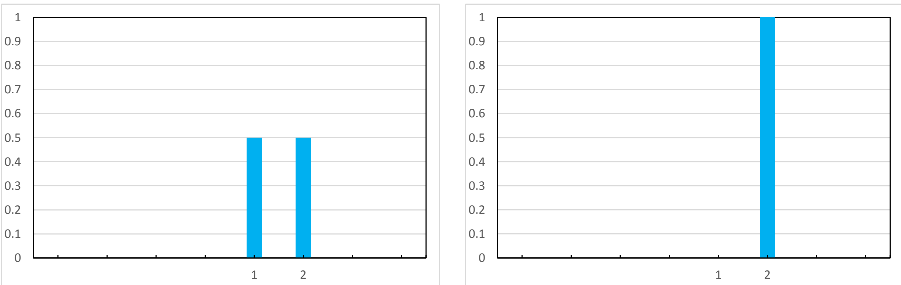
5.1 Case of $M = 2$. We consider two kinds of configurations of two regions such that $|x_1 - x_2| = \pi$ and $|x_1 - x_2| = \pi/4$. Figures 1 and 2 illustrate the stationary solutions $\overline{\lambda}$ to which the solutions $\lambda(t)$ converge as $t \to \infty$ for the cases π and $\pi/4$, respectively. Here, the horizontal axis and the vertical axis denote the interval $[-\pi, \pi]$ and the population size, respectively.

Figure 1(a) shows when $\tau = 1.3$ that the population is separated in the two regions uniformly. However, Figure 1(b) shows when $\tau = 1.2$ that the population is accumulated into a single region. On the other hand, Figure 2(a) shows when $\tau = 1.5$ that the population is separated in the two regions equally, and Figure 2(b) shows when $\tau = 1.45$ that the population is accumulated into a single region. In any case, there exists a threshold $\hat{\tau}$ such that, if $\tau > \hat{\tau}$ the population is equally divided between the regions, and if $\tau < \hat{\tau}$ the population is concentrated in a single region. Moreover, it is observed that the threshold $\widehat{\tau}$ differs by configurations. In fact, Figures 1 and 2 show that $1.2 < \widehat{\tau} < 1.3$ when $|x_1 - x_2| = \pi$ and $1.45 < \widehat{\tau} < 1.5$ when $|x_1 - x_2| = \pi/4$.



(a) $\tau = 1.3$ (b) $\tau = 1.2$

Fig. 1: $\overline{\lambda}$ when $|x_1 - x_2| = \pi$



(a) $\tau = 1.5$ (b) $\tau = 1.45$

Fig. 2: $\overline{\lambda}$ when $|x_1 - x_2| = \pi/4$

5.2 Case of $M = 3$. Let us put nine points on S equidistantly. Then, we select three manufacturing regions from them. By doing so, we consider the seven cases of configurations of the regions as below.

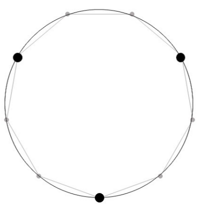
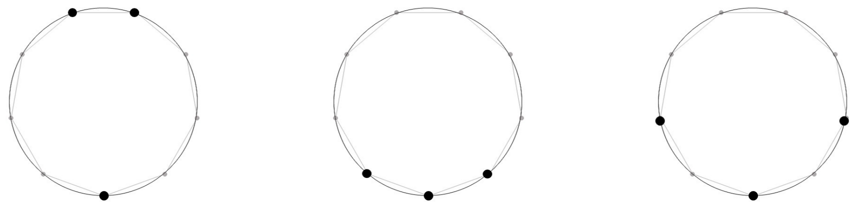
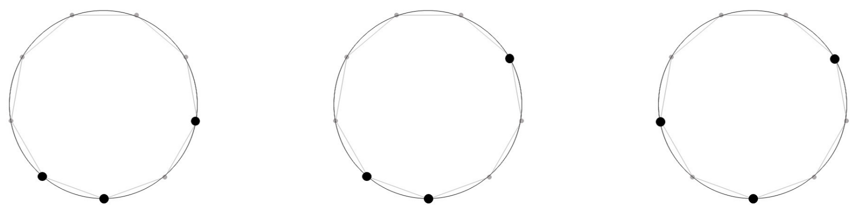


Fig. 3: Equilateral triangle



(a) Isosceles 1 (b) Isosceles 2 (c) Isosceles 3

Fig. 4: Isosceles triangles

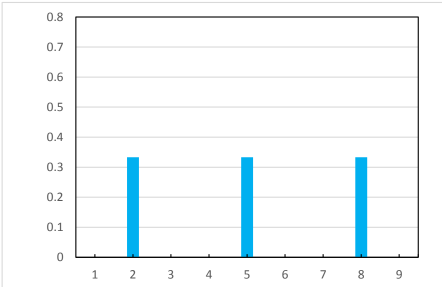


(a) Scalene 1 (b) Scalene 2 (c) Scalene 3

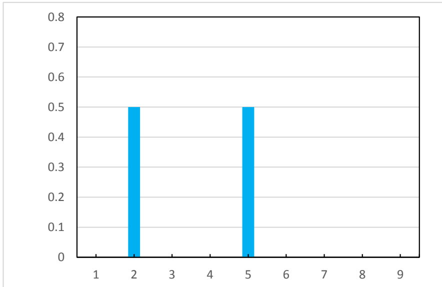
Fig. 5: Scalene triangles

Figures 6 - 12 illustrate the stationary solutions $\overline{\lambda}$ to which the solutions $\overline{\lambda}(t)$ converges as $t \to \infty$ for the seven cases. Here, the horizontal axis and the vertical axis denote the interval $[-\pi, \pi]$ and the population size, respectively.

Figure 6(a) shows under the equilateral configuration, when $\tau = 1.5$ that the population is separated in three regions equally. However, Figure 6(b) shows when $\tau = 1.45$ that the population is accumulated into two regions only. Under the isosceles configuration 1, although the population is dispersed to the three regions when $\tau = 3$, the population is separated into two regions when $\tau = 2.9$ as shown by Figures 7(a) and 7(b). The numerical results illustrated in Figures 8 - 12 are similar, i.e., there exists a threshold $\hat{\tau}$ such that, $\tau > \hat{\tau}$ the population is equally divided among the three regions, and if $\tau < \hat{\tau}$ the population is concentrated in two regions. Moreover, it is observed that the threshold $\widehat{\tau}$ differs by the types of configurations. In fact, there is more than seven times difference in the value of $\widehat{\tau}$ between the equilateral configuration and the isosceles configuration 2.

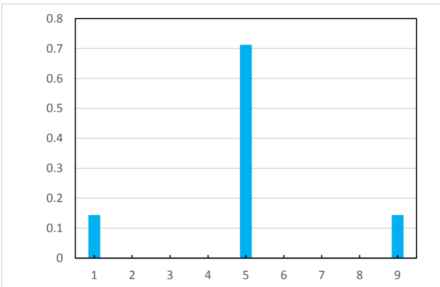


(a) $\tau = 1.5$

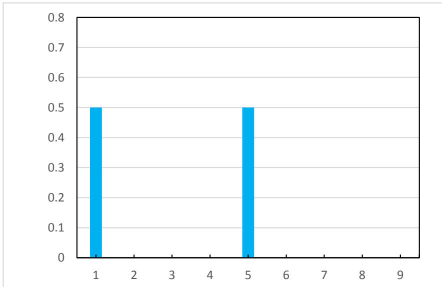


(b) $\tau = 1.45$

Fig. 6: $\overline{\lambda}$ for equilateral triangle

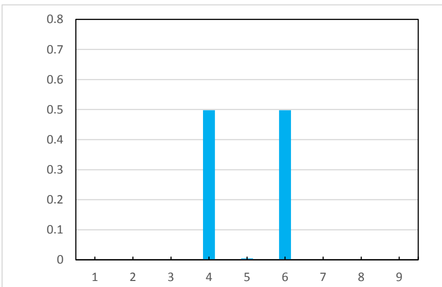


(a) $\tau = 3$

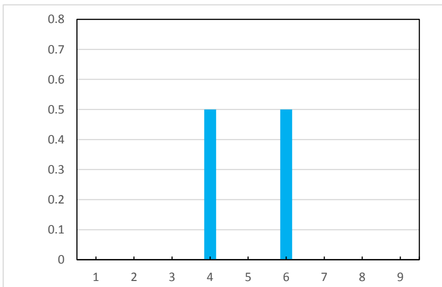


(b) $\tau = 2.9$

Fig. 7: $\overline{\lambda}$ for isosceles 1

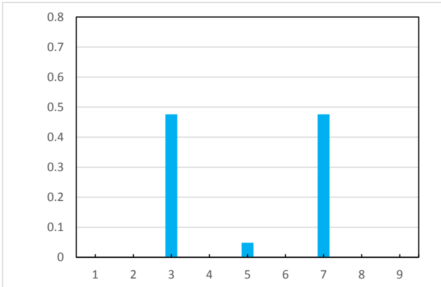


(a) $\tau = 11.5$

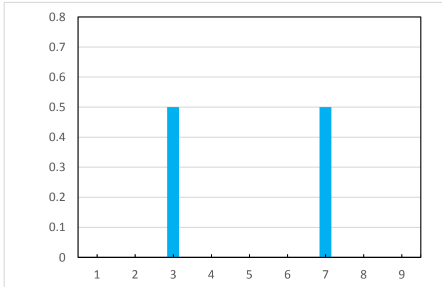


(b) $\tau = 10.9$

Fig. 8: $\overline{\lambda}$ for isosceles 2

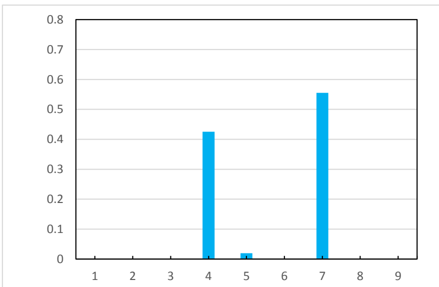


(a) $\tau = 4$

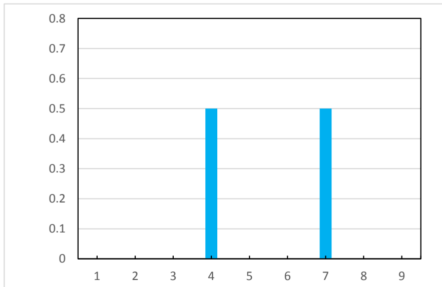


(b) $\tau = 3.9$

Fig. 9: $\overline{\lambda}$ for isosceles 3

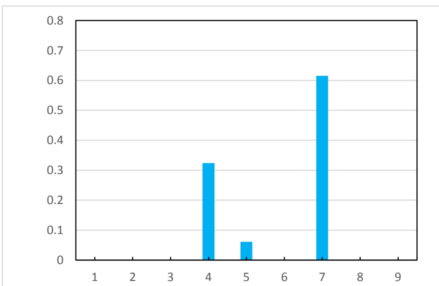


(a) $\tau = 6.8$

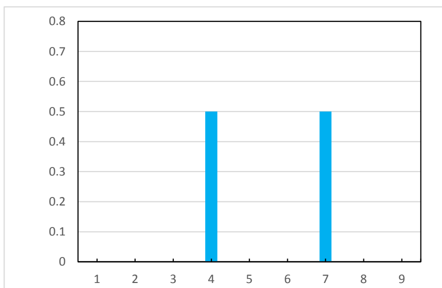


(b) $\tau = 6.7$

Fig. 10: $\overline{\lambda}$ for scalene 1

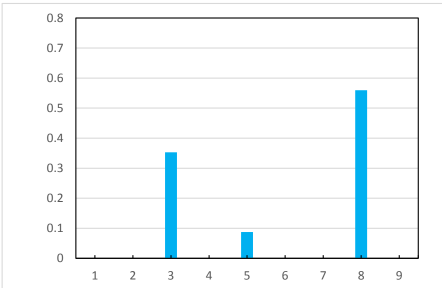


(a) $\tau = 4.8$

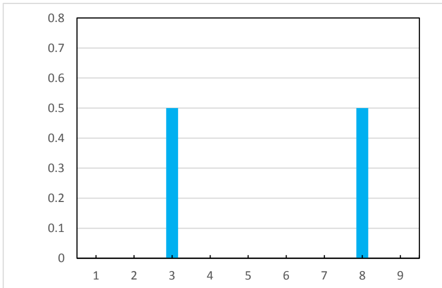


(b) $\tau = 4.7$

Fig. 11: $\overline{\lambda}$ for scalene 2



(a) $\tau = 2.7$



(b) $\tau = 2.6$

Fig. 12: $\overline{\lambda}$ for scalene 3

6 Stability of Stationary Solutions. In this section, we want to investigate stability of stationary solutions for (1.2). As in the previous section, the agricultural population density is assumed to be constant, i.e., $\phi(x) \equiv \overline{\phi}$. After discussing existence of stationary solutions, we investigate their stability in the cases of $M = 2$ and 3.

6.1 Existence of Stationary Solutions. By $(\overline{f}, \overline{\lambda})$ we denote a stationary solution to (3.6), where $\overline{f} = (\overline{f}_1, \cdots, \overline{f}_M) \in \mathbb{R}^M$ and $\overline{\lambda} = (\overline{\lambda}_1, \cdots, \overline{\lambda}_M) \in \mathbb{R}^M$. We also denote $\overline{w} = (\overline{w}_1, \cdots, \overline{w}_M)$, where $\overline{f}_i = \overline{w}_i^\sigma$ for $i = 1, \cdots, M$. Then, the price index and the real wage of stationary state are given by $\overline{G} = \overline{G}(x),\ x \in S$ and $\overline{\omega} = (\overline{\omega}_1, \cdots, \overline{\omega}_M)$, respectively.

From (3.3), the stationary solution must satisfy

$$\begin{aligned} \overline{w}_i^\sigma &= \sum_{j=1}^M \frac{\overline{\lambda}_j \overline{w}_j e^{-\alpha|x_i - x_j|}}{\sum_{k=1}^M \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|x_j - x_k|}} \\ &\quad + (1-\mu)\overline{\phi} \int_S \frac{e^{-\alpha|x_i - y|}}{\sum_{k=1}^M \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|y - x_k|}} dy, \quad i = 1, 2, \cdots, M. \end{aligned} \tag{6.1}$$

Moreover, by the fact that $\Psi(\overline{f}, \overline{\lambda}) = 0$ in (3.6), the following equations

$$\begin{aligned} &\overline{w}_i \left\{ \sum_{k=1}^M \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|x_i - x_k|} \right\}^{\frac{\mu}{\sigma-1}} \\ &\quad = \sum_{j=1}^M \overline{\lambda}_j \left\{ \sum_{k=1}^M \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|x_j - x_k|} \right\}^{\frac{\mu}{\sigma-1}}, \quad i = 1, 2, \cdots, M \end{aligned} \tag{6.2}$$

must be satisfied. Thereby, the number of unknowns $\overline{\lambda}_1, \cdots, \overline{\lambda}_M$ and $\overline{w}_1, \cdots, \overline{w}_M$ is equal to the number of equations. This fact suggests that there may exist a stationary solution to (3.6) under any location of M manufacturing regions on S. But it is very difficult to demonstrate this assertion. We consider only the symmetric solutions.

Definition 6.1. *A stationary solution satisfying the conditions:*

1. *all the distances between adjacent manufacturing regions are equal,*
2. *the population size and the nominal wages are uniform for the regions,*

is called a symmetric stationary solution.

When $M = 2$ or 3, the symmetric stationary solution is obtained analytically.

Theorem 6.1. *When $M = 2$, for any configuration of x_1 and x_2,* (3.6) *has a symmetric stationary solution such that*

$$\begin{aligned} w_i &\equiv \bar{w} = 1, \quad i = 1, 2, \\ \lambda_i &\equiv \bar{\lambda} = 1/2, \quad i = 1, 2. \end{aligned} \tag{6.3}$$

Proof. It is easy to verify that (6.3) is a stationary solution of (3.6) in view of

$$\int_S \frac{e^{-\alpha|x_i-y|}}{e^{-\alpha|x_1-y|} + e^{-\alpha|x_2-y|}} dy = \pi, \qquad i = 1, 2.$$

□

Theorem 6.2. *When $M = 3$, let $|x_2 - x_1|$, $|x_3 - x_2|$ and $|x_1 - x_3|$ be equal to $2\pi/3$. Then,* (1.2) *has a symmetric stationary solution such that*

$$\begin{aligned} w_i &= \overline{w} = 1, \quad i = 1, 2, 3, \\ \lambda_i &= \overline{\lambda} = 1/3, \quad i = 1, 2, 3. \end{aligned} \tag{6.4}$$

Proof. It is easy to verify that (6.4) is a stationary solution of (3.6) due to the fact that

$$\int_S \frac{e^{-\alpha|x_i-y|}}{\sum_{k=1}^3 e^{-\alpha|x_k-y|}} dy = \frac{2\pi}{3}, \qquad i = 1, 2, 3.$$

□

6.2 Linearization Matrix. Let a stationary solution $(\overline{\lambda}, \overline{w})$ be given. We want to linearize (3.6) around it. Let Δw, $\Delta\lambda \in \mathbb{R}^M$ be small perturbations added to $\overline{w}$ and $\overline{\lambda}$, respectively, but satisfying the restriction $\sum_{i=1}^M \Delta\lambda_i = 0$. The linearized equations are given by

$$\begin{cases} (I - A)\,\Delta w = B\Delta\lambda, \\ \dfrac{d}{dt}\Delta\lambda = L\big[I - \overline{\Lambda}\big]\big[(E + FC)\,\Delta w + \big(FD - \overline{R}\big)\,\Delta\lambda\big], \end{cases} \tag{6.5}$$

where I stands for the identity matrix. Here, the $M \times M$ matrices A, B, C, D, E, and F are given by

$$\begin{aligned} A_{ij} = {} & \frac{\mu}{\sigma}\overline{w}_i^{1-\sigma} \frac{\overline{\lambda}_j e^{-\alpha|x_i-x_j|}}{\sum_k \overline{\lambda}_k \overline{w}_k^{1-\sigma} e^{-\alpha|x_j-x_k|}} \\ & + \frac{\mu(\sigma-1)}{\sigma}\overline{w}_i^{1-\sigma}\overline{\lambda}_j\overline{w}_j^{-\sigma} \sum_s \frac{\overline{\lambda}_s\overline{w}_s e^{-\alpha|x_s-x_j|}e^{-\alpha|x_i-x_s|}}{\big[\sum_k \overline{\lambda}_k\overline{w}_k^{1-\sigma}e^{-\alpha|x_s-x_k|}\big]^2} \\ & + \frac{(1-\mu)(\sigma-1)}{2\pi\sigma}\overline{w}_i^{1-\sigma}\overline{\lambda}_j\overline{w}_j^{-\sigma} \int_S \frac{e^{-\alpha|y-x_i|}e^{-\alpha|y-x_j|}}{\big[\sum_k \overline{\lambda}_k\overline{w}_k^{1-\sigma}e^{-\alpha|y-x_k|}\big]^2} dy, \end{aligned} \tag{6.6}$$

$$
\begin{aligned}
B_{ij} = & \frac{\mu}{\sigma}\overline{w}_i^{1-\sigma}\frac{\overline{w}_j e^{-\alpha|x_i-x_j|}}{\sum_k \overline{\lambda}_k \overline{w}_k^{1-\sigma}e^{-\alpha|x_j-x_k|}} \\
& -\frac{\mu}{\sigma}\overline{w}_i^{1-\sigma}\sum_s \frac{\overline{\lambda}_s\overline{w}_s\overline{w}_j^{1-\sigma}e^{-\alpha|x_s-x_j|}e^{-\alpha|x_i-x_s|}}{\left[\sum_k \overline{\lambda}_k\overline{w}_k^{1-\sigma}e^{-\alpha|x_s-x_k|}\right]^2} \\
& -\frac{1-\mu}{2\pi\sigma}\overline{w}_i^{1-\sigma}\overline{w}_j^{1-\sigma}\int_S \frac{e^{-\alpha|y-x_i|}e^{-\alpha|y-x_j|}}{\left[\sum_k \overline{\lambda}_k\overline{w}_k^{1-\sigma}e^{-\alpha|y-x_k|}\right]^2}dy,
\end{aligned} \tag{6.7}
$$

$$
\begin{aligned}
C_{ij} &= \overline{G}_i^{\sigma}\overline{\lambda}_j\overline{w}_j^{-\sigma}e^{-\alpha|x_i-x_j|}, \\
D_{ij} &= -\frac{\overline{G}_i^{\sigma}}{\sigma-1}\overline{w}_j^{1-\sigma}e^{-\alpha|x_i-x_j|},
\end{aligned} \tag{6.8}
$$

$$
\begin{aligned}
E &= \mathrm{diag}(\overline{G}_1^{-\mu}, \cdots, \overline{G}_M^{-\mu}), \\
F &= \mathrm{diag}(-\mu\overline{w}_1\overline{G}_1^{-\mu-1}, \cdots, -\mu\overline{w}_M\overline{G}_M^{-\mu-1})
\end{aligned} \tag{6.9}
$$

respectively. The matrix $\overline{\Lambda}$ denotes

$$
\overline{\Lambda} = \begin{pmatrix} \overline{\lambda}_1 & \cdots & \overline{\lambda}_M \\ \overline{\lambda}_1 & \cdots & \overline{\lambda}_M \\ \vdots & \vdots & \vdots \\ \overline{\lambda}_1 & \cdots & \overline{\lambda}_M \end{pmatrix},
$$

and the matrix L is $L := \mathrm{diag}(\overline{\lambda}_1, \cdots, \overline{\lambda}_M)$. Finally,

$$
\overline{R} = \overline{\omega}\begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.
$$

As a matter of fact, by using the matrix

$$
\Omega := \left[E(I-A)^{-1}B + F\left\{C(I-A)^{-1}B + D\right\}\right],
$$

the linearized equations (6.5) is reduced to

$$
\frac{d}{dt}\Delta\lambda = J\Delta\lambda, \tag{6.10}
$$

where $J = L\left[(I-\overline{\Lambda})\Omega - \overline{R}\right]$.

Since $\sum_{i=1}^M \lambda_i = 1$, it is natural to impose the condition that $\sum_{i=1}^M \Delta\lambda_i = 0$; therefore, $\Delta\lambda_M = -(\Delta\lambda_1 + \cdots + \Delta\lambda_{M-1})$. The M-dimensional ordinary equation (6.10) is actually reduced to an ordinary differential equation for $\Delta\lambda' = (\Delta\lambda_1, \cdots, \Delta\lambda_{M-1})^{\mathrm{T}}$. Introduce an $(M-1)\times M$ matrix P_1 and an $M\times(M-1)$ matrix P_2 as

$$
P_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}
$$

and

$$P_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & -1 \end{pmatrix},$$

respectively. Then, (6.10) is reduced to

$$\frac{d}{dt}\Delta\lambda' = P_1 J P_2 \Delta\lambda'.$$

Hence, stability of the stationary solution is determined by the eigenvalues of the matrix $J' := P_1 J P_2$. In general, it is very complicate to calculate exactly the eigenvalues of J', and we have to rely on numerical computations. However, in the case of symmetric stationary solutions with $M = 2$ or 3, we can compute J' analytically.

Define the $M \times M$ matrices X and Y by

$$X_{ij} = e^{-\alpha|x_i - x_j|}, \tag{6.11}$$

$$Y_{ij} = \int_S \frac{e^{-\alpha|y-x_i|}e^{-\alpha|y-x_j|}}{\sum_{k=1}^M e^{-\alpha|y-x_k|}} dy, \tag{6.12}$$

respectively. Then, from (6.6) and (6.7), A and B are described as

$$\begin{aligned} A = {} & \frac{\mu(\sigma-1)}{\sigma}\overline{\lambda}^2\overline{w}^{2-2\sigma}\overline{G}^{2\sigma-2}X^2 + \frac{\mu}{\sigma}\overline{\lambda}\overline{w}^{1-\sigma}\overline{G}^{\sigma-1}X \\ & + \frac{(1-\mu)(\sigma-1)}{2\pi\sigma}\overline{\lambda}^{-1}\overline{w}^{-1}Y, \end{aligned} \tag{6.13}$$

$$B = -\frac{\mu}{\sigma}\overline{\lambda}\overline{w}^{3-2\sigma}\overline{G}^{2\sigma-2}X^2 + \frac{\mu}{\sigma}\overline{w}^{2-\sigma}\overline{G}^{\sigma-1}X - \frac{1-\mu}{2\pi\sigma}\overline{\lambda}^{-2}Y. \tag{6.14}$$

Similarly, from (6.8) and (6.9), $C,\ D,\ E$ and F are described as

$$\begin{aligned} C &= \overline{G}^{\sigma}\overline{\lambda}\overline{w}^{-\sigma}X, \\ D &= -\frac{\overline{G}^{\sigma}\overline{w}^{1-\sigma}}{\sigma-1}X, \\ E &= \overline{G}^{-\mu}I, \\ F &= \overline{w}\overline{G}^{-\mu-1}I, \end{aligned}$$

respectively. Thereby, Ω is given by

$$\begin{aligned} \Omega = {} & \overline{G}^{-\mu}(I-A)^{-1}B - \mu\overline{\lambda}\overline{w}^{1-\sigma}\overline{G}^{\sigma-\mu-1}X(I-A)^{-1}B \\ & + \frac{\mu}{\sigma-1}\overline{w}^{2-\sigma}\overline{G}^{\sigma-\mu-1}X. \end{aligned} \tag{6.15}$$

Note that all the diagonal components of Ω are equal each other, and all the non-diagonal components are also equal, i.e., Ω takes the form:

$$\text{when } M = 2, \qquad \Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \\ \Omega_2 & \Omega_1 \end{pmatrix};$$

$$\text{when } M = 3, \qquad \Omega = \begin{pmatrix} \Omega_1 & \Omega_2 & \Omega_2 \\ \Omega_2 & \Omega_1 & \Omega_2 \\ \Omega_2 & \Omega_2 & \Omega_1 \end{pmatrix}.$$

We call such a matrix as "*a strong diagonal matrix*". It is easy to see that sum, product, or linear combination of strong diagonal matrices is also a strong diagonal matrix. Moreover, the inverse of a strong diagonal matrix is also a strongly diagonal. By these facts, Ω is seen to be strongly diagonal, because X and Y are strongly diagonal (See (6.11), (6.12)). As a result, the matrix J' is simply given by

$$\text{when } M=2, \qquad J' = \frac{1}{2}\left(\Omega_1 - \Omega_2\right), \tag{6.16}$$
$$\text{when } M=3, \qquad J' = \frac{1}{3}\left(\Omega_1 - \Omega_2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

in the symmetric stationary solution.

6.3 Case of $M = 2$. When $M = 2$, we have the following theorem.

Theorem 6.3. *Let the no black hole condition $(\sigma-1)/\sigma > \mu$ be satisfied. If $\tau > 0$ or $\sigma > 1$ is sufficiently small, then the stationary solution given by* (6.3) *is unstable. On the other hand, if $\tau > 0$ or $\sigma > 1$ is sufficiently large, then the stationary solution given by* (6.3) *is stable.*

Proof. In this proof, the circumference S is identified with the interval $[-\pi, \pi]$ and two regions x_1, x_2 are set as $x_1 = 0$, $x_2 = d \in (0, \pi]$.

First, for sufficiently small $\tau > 0$ or $\sigma > 1$, i.e., for sufficiently small α, we consider the Taylor expansion for J'. Since J' is composed of the matrices X, Y, A, B, we calculate the Taylor expansion for them. As $M = 2$, X is given by

$$X = \begin{pmatrix} 1 & e^{-\alpha d} \\ e^{-\alpha d} & 1 \end{pmatrix},$$

thereby

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - d\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\alpha + O(\alpha^2). \tag{6.17}$$

The matrix Y is given by

$$Y_{11} = Y_{22} = d + \frac{1-e^{\alpha d}}{\alpha(1+e^{\alpha d})} + (\pi - d)\frac{1+e^{2\alpha d}}{(1+e^{\alpha d})^2},$$
$$Y_{12} = Y_{21} = \frac{e^{\alpha d}-1}{\alpha(1+e^{\alpha d})} + (\pi - d)\frac{2e^{\alpha d}}{(1+e^{\alpha d})^2}$$

thereby

$$Y = \frac{\pi}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + O(\alpha^2). \tag{6.18}$$

In addition, $\overline{G}^{1-\sigma}$ is expanded as

$$\overline{G}^{1-\sigma} = 1 - \frac{d}{2}\alpha + O(\alpha^2). \tag{6.19}$$

Hence, A is expanded as

$$A = A_1 + A_2\alpha + O(\alpha^2),$$

where

$$A_1 = \frac{\sigma+\mu-1}{2\sigma}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$A_2 = -\frac{\mu d}{4\sigma}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since $\|A\| < 1$, we have

$$(I-A)^{-1} = I + A + A^2 + A^3 + \cdots .$$

Note that $A_1 A_2 = A_2 A_1$ is the null matrix. It then follows that

$$A^n = A_1^n + O(\alpha^2), \quad \text{for } n = 2, 3, \cdots .$$

So,

$$\begin{aligned}(I-A)^{-1} &= I + A + A^2 + A^3 + \cdots \\ &= I + (A_1 + A_1^2 + A_1^3 + \cdots) + A_2\alpha + O(\alpha^2).\end{aligned}$$

Moreover,

$$\begin{aligned}A_1 + A_1^2 + A_1^3 + \cdots &= \sum_{n=1}^{\infty}\left(\frac{\sigma-1+\mu}{2\sigma}\right)^n 2^{n-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{\mu+\sigma-1}{2(1-\mu)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.\end{aligned}$$

Hence, we obtain that

$$(I-A)^{-1} = I + \frac{\mu+\sigma-1}{2(1-\mu)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{\mu d}{4\sigma}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\alpha + O(\alpha^2). \tag{6.20}$$

Meanwhile, B is expanded as

$$B = -\frac{1-\mu}{\sigma}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{\mu d}{2\sigma}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\alpha + O(\alpha^2). \tag{6.21}$$

By (6.17), (6.18), (6.19), (6.20), and (6.21), it is observed from (6.15) that

$$\begin{aligned}\Omega = &\frac{1+\mu\sigma-\sigma}{\sigma-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &+ \left[-\frac{\mu d(2\sigma-1)}{2\sigma(\sigma-1)}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + \frac{\mu d(1+\mu\sigma-\sigma)}{2(\sigma-1)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right]\alpha + O(\alpha^2).\end{aligned}$$

Thus, J' is given by

$$J' = \frac{\mu d(2\sigma-1)}{\sigma(\sigma-1)}\alpha + O(\alpha^2).$$

The first order term obviously takes positive value for $\alpha > 0$. Therefore, the symmetric stationary solution (6.3) is unstable for sufficiently small $\alpha > 0$.

Next, let us verify that when τ or σ is sufficiently large, i.e., when α is sufficiently large, J' is negative. From (6.3), (6.11) and (6.12), it follows that

$$\begin{aligned}&\lim_{\alpha\to\infty} \overline{G}^{1-\sigma} = 1/2, \\ &\lim_{\alpha\to\infty} X = I, \\ &\lim_{\alpha\to\infty} Y = \pi I.\end{aligned}$$

It follows from these results and (6.13), (6.14) that

$$\lim_{\alpha\to\infty} A = \frac{\sigma - 1 + \mu}{\sigma} I,$$
$$\lim_{\alpha\to\infty} (I - A)^{-1} = \frac{\sigma}{1-\mu} I,$$
$$\lim_{\alpha\to\infty} B = -\frac{2(1-\mu)}{\sigma} I.$$

Therefore, we obtain from (6.15) that

$$\lim_{\alpha\to\infty} \Omega = 2^{\frac{\mu}{1-\sigma}} \frac{-\sigma + 1 + \mu\sigma}{\sigma - 1} I.$$

Then,

$$\lim_{\alpha\to\infty} J' = 2^{\frac{\mu}{1-\sigma}} \frac{-\sigma + 1 + \mu\sigma}{\sigma - 1}.$$

Obviously, this value is negative under the assumption of no black hole $(\sigma - 1)/\sigma > \mu$. □

Figure 13 illustrates the value of J' as a function of α obtained numerically. Here, the horizontal axis and the vertical axis are taken as $\alpha > 0$ and the value of J', respectively. The red line indicates the case when $d = 1$; similarly, the green line $d = 2$, the blue line $d = \pi$. This shows that there exists a threshold $\alpha = \alpha^*$ where the sign of J' changes. Then, smaller α^* means higher degree of stability. Since the longer d results in smaller α^* according to this figure, it follows that the longer distance between two regions is, the higher degree of stability is.

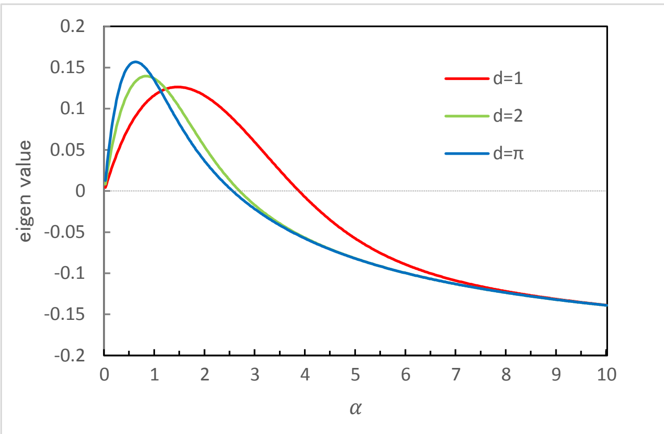


Fig. 13: Value of J'

6.4 Case of $M = 3$. In this subsection we consider the case of $M = 3$.

Theorem 6.4. *Assume the no black hole condition $(\sigma - 1)/\sigma > \mu$. If $\tau > 0$ or $\sigma > 1$ is sufficiently small, then the stationary solution given by* (6.4) *is unstable. On the other hand for sufficiently large τ or σ, the stationary solution given by* (6.4) *is stable.*

Proof. In this proof, S is identified with the interval $[-\pi, \pi]$ and three manufacturing regions x_1, x_2 and x_3 are set as $x_1 = -\frac{2\pi}{3}$, $x_2 = 0$, $x_3 = \frac{2\pi}{3}$.

First, for sufficiently small $\tau > 0$ or $\sigma > 1$, i.e., for sufficiently small α, we consider the Taylor expansion for the matrix J as in the proof of Theorem 6.3. Since the matrix J is composed of the matrices X, Y, A, B, the Taylor expansions for them should be calculated. The matrix X is given by

$$X = \begin{pmatrix} 1 & e^{-\alpha\frac{2\pi}{3}} & e^{-\alpha\frac{2\pi}{3}} \\ e^{-\alpha\frac{2\pi}{3}} & 1 & e^{-\alpha\frac{2\pi}{3}} \\ e^{-\alpha\frac{2\pi}{3}} & e^{-\alpha\frac{2\pi}{3}} & 1 \end{pmatrix},$$

and its Taylor expansion is

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{2\pi}{3} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \alpha + O(\alpha^2). \tag{6.22}$$

In general, the function given by

$$F(\alpha) = \int_{-\pi}^{\pi} f(\alpha, x)dx$$

can be expanded as

$$\begin{aligned} F(\alpha) &= F(0) + F'(0)\alpha + O(\alpha^2) \\ &= \int_{-\pi}^{\pi} f(0, x)dx + \int_{-\pi}^{\pi} \frac{\partial f}{\partial \alpha}(0, x)dx \cdot \alpha + O(\alpha^2). \end{aligned}$$

We then set

$$f(\alpha, x) = \frac{e^{-2\alpha|x-x_1|}}{\left[e^{-\alpha|x-x_1|} + e^{-\alpha|x-x_2|} + e^{-\alpha|x-x_3|}\right]^2}.$$

It is easy to see that

$$f(0, x) = \frac{1}{9},$$

and

$$\frac{\partial f}{\partial \alpha}(0, x) = \frac{-4|x - x_1| + 2|x - x_2| + 2|x - x_3|}{27}.$$

Hence, Y_{11} is expanded as

$$\begin{aligned} Y_{11} &= \frac{1}{9} \int_{-\pi}^{\pi} dy + \frac{1}{27} \int_{-\pi}^{\pi} \left[-4|y - x_1| + 2|y - x_2| + 2|y - x_3|\right] dx \cdot \alpha + O(\alpha^2) \\ &= \frac{2\pi}{9} + O(\alpha^2). \end{aligned}$$

Other diagonal elements are also expanded as

$$\begin{aligned} Y_{22} &= \frac{2\pi}{9} + O(\alpha^2), \\ Y_{33} &= \frac{2\pi}{9} + O(\alpha^2). \end{aligned}$$

As a non-diagonal element, let us consider Y_{12}. If we set

$$f(\alpha, x) = \frac{e^{-\alpha|x-x_1|}e^{-\alpha|x-x_2|}}{\left[e^{-\alpha|x-x_1|} + e^{-\alpha|x-x_2|} + e^{-\alpha|x-x_3|}\right]^2},$$

then it is easy to see that

$$f(0, x) = \frac{1}{9},$$

and

$$\frac{\partial f}{\partial \alpha}(0, x) = \frac{-|x-x_1| - |x-x_2| + 2|x-x_3|}{27}.$$

Hence, Y_{12} is expanded as

$$\begin{aligned} Y_{12} &= \frac{1}{9}\int_{-\pi}^{\pi} dx + \frac{1}{27}\int_{-\pi}^{\pi} \left[-|x-x_1| - |x-x_2| + 2|x-x_3|\right] dx \cdot \alpha + O(\alpha^2) \\ &= \frac{2\pi}{9} + O(\alpha^2). \end{aligned}$$

Other non-diagonal elements are also expanded as

$$\begin{aligned} Y_{13} &= \frac{2\pi}{9} + O(\alpha^2), \\ Y_{23} &= \frac{2\pi}{9} + O(\alpha^2). \end{aligned}$$

After all, Y is expanded as

$$Y = \frac{2\pi}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + O(\alpha^2). \tag{6.23}$$

In addition, $\overline{G}^{1-\sigma}$ is expanded as

$$\overline{G}^{1-\sigma} = 1 - \frac{4\pi}{9}\alpha + O(\alpha^2). \tag{6.24}$$

Hence, A is expanded as

$$A = \frac{\sigma+\mu-1}{3\sigma}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{2\pi\mu}{27\sigma}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\alpha + O(\alpha^2). \tag{6.25}$$

Since $\|A\| < 1$, we have

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots.$$

Repeating the same argument as for the case of $M = 2$, we obtain that

$$\begin{aligned} (I - A)^{-1} &= I + \frac{\sigma+\mu-1}{3(1-\mu)}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &\quad - \frac{2\pi\mu}{27\sigma}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\alpha + O(\alpha^2). \end{aligned} \tag{6.26}$$

Moreover, the matrix B is expanded as

$$B = -\frac{1-\mu}{\sigma}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{2\pi\mu}{9\sigma}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\alpha + O(\alpha^2). \tag{6.27}$$

By (6.22), (6.23), (6.24), (6.26), (6.27), (6.15) provides that

$$\begin{aligned} \Omega = {} & \frac{1+\mu\sigma-\sigma}{\sigma-1}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ & - \left[\frac{2\pi\mu(2\sigma-1)}{9\sigma(\sigma-1)}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} - \frac{4\pi\mu(1+\mu\sigma-\sigma)}{9(\sigma-1)^2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right]\alpha \\ & + O(\alpha^2). \end{aligned}$$

By this and (6.16), it is easy to see that the eigenvalue of J' is expanded as

$$\frac{1}{3}(\Omega_1 - \Omega_2) = \frac{2\pi(2\sigma-1)\mu}{3\sigma(\sigma-1)}\alpha + O(\alpha^2).$$

The first order term obviously takes positive value for $\alpha > 0$. Therefore, the symmetric stationary solution (6.3) is proved to be unstable for sufficiently small $\alpha > 0$.

Next, let us verify that when $\tau \to \infty$ or $\sigma \to \infty$, i.e., when $\alpha \to \infty$, the eigenvalue of J' is negative. From (6.4), (6.11) and (6.12), it follows that

$$\begin{aligned} &\lim_{\alpha\to\infty} \overline{G}^{1-\sigma} = 1/3, \\ &\lim_{\alpha\to\infty} X = I, \\ &\lim_{\alpha\to\infty} Y = \frac{2\pi}{3} I. \end{aligned}$$

It follows from these and (6.13), (6.14) that

$$\begin{aligned} &\lim_{\alpha\to\infty} A = \frac{\sigma-1+\mu}{\sigma} I, \\ &\lim_{\alpha\to\infty} (I-A)^{-1} = \frac{\sigma}{1-\mu} I, \\ &\lim_{\alpha\to\infty} B = -\frac{3(1-\mu)}{\sigma} I. \end{aligned}$$

By these results, (6.15) provides that

$$\lim_{\alpha\to\infty} \Omega = 3^{\frac{\mu}{1-\sigma}} \frac{(-\sigma+1+\mu\sigma)}{\sigma-1} I.$$

Then, as $\alpha \to \infty$, the eigenvalue of J' converges to the limit

$$3^{\frac{\mu}{1-\sigma}} \frac{(-\sigma+1+\mu\sigma)}{\sigma-1}$$

which is obviously negative under the of no black hole condition. □

In fact, Figure 14 illustrates a graph of the eigenvalue of J' as a function of α obtained numerically. Here, the horizontal axis and the vertical axis are taken as $\alpha > 0$ and the eigenvalue of J', respectively. It is observed that the sign of the eigenvalue changes at some threshold $\alpha = \alpha^*$.

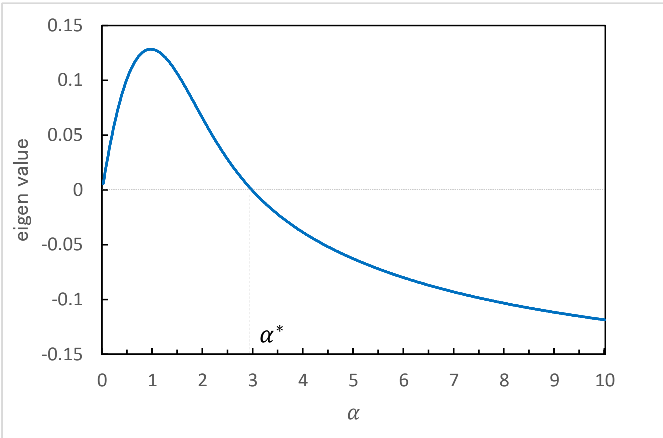


Fig. 14: Eigenvalue of J' when $M = 3$

Even for non-symmetric stationary solutions, we can compute the eigenvalue of J' and investigate its sign. These results show good agreement to the numerical computations performed in Subsection 5.2. But we will omit the details.

References

[1] T. Akamatsu, Y. Takayama, K. Ikeda, *Spatial discounting, Fourier, and racetrack economy: A recipe for the analysis of spatial agglomeration models*, J. Econ. Dyn. Control, 36(11), 1729-1759. 2012.

[2] T. Akamatsu, T. Mori, Y. Takayama, *Agglomerations in a multi-region economy: Poly-centric versus mono-centric patterns*, Discussion Paper 929, Institute of Economic Research, Kyoto University, 2015.

[3] T. Akamatsu, Y. Takayama, *Do polycentric patterns emerge in NEG models?*, Unpublished manuscript. Graduate School of Information Sciences, Tohoku University, 2013.

[4] J. Barbero, J. L. Zofo, *The multiregional core-periphery model: The role of the spatial topology*, Netw. Spat. Econ. **16**(2)(2016), 469-496.

[5] S. B. Castro, J. Correia-da-Silva and P. Mossay, *The core-periphery model with three regions and more*, Pap. Reg. Sci, **91**(2)(2012), 401-418.

[6] P. P. Combes, T. Mayer and J. F. Thisse, *Economic Geography: the Integration of Regions and Nations*, Princeton University Press, 2008.

[7] M. Fabinger, *Cities as solitons: Analytic solutions to models of agglomeration and related numerical approaches*, SSRN: http://ssrn.com/abstract=2630599, 2015.

[8] M. Fujita and J. F. Thisse, *Economics of Agglomeration: Cities, Industrial Location, and Globalization*, Cambridge University Press, 2013.

[9] M. Fujita, P. Krugman and A. Venables, *The Spatial Economy: Cities, Regions, and International Trade* , MIT Press, 2001.

[10] K. Ikeda, T. Akamatsu and T. Kono, *Spatial period-doubling agglomeration of a coreperiphery model with a system of cities*, J. Econ. Dyn. Control **36**(5)(2012), 754-778.

[11] K. Ohtake, A. Yagi, *Asymptotic behavior of solutions to racetrack model in spatial economy*, Sci. Math, Jpn. (2016) (accepted for publication)

[12] M. Tabata and N. Eshima, *A population explosion in an evolutionary game in spatial economics: Blow up radial solutions to the initial value problem for the replicator equation whose growth rate is determined by the continuous Dixit-Stiglitz-Krugman model in an urban setting*, Nonlinear Anal. Real **23**(2015), 26-46.

[13] T. Tabuchi and J. F. Thisse, *A new economic geography model of central places*, J. Urban Econ. **69**(2), 2011, 240-252.

Communicated by *Koichi Osaki*