VELOCITY AND ACCELERATION ON THE PATHS $A \natural_t B$ AND $A \sharp_{t,r} B$

DEDICATED TO THE MEMORY OF PROFESSOR TAKAYUKI FURUTA

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Abstract. Let A and B be strictly positive linear operators on a Hilbert space. The derivative of the path $A \natural_t B$ ($t \in \mathbf{R}$) gives the relative operator entropy, that is, $\frac{d}{dt} A \natural_t B = S_t(A|B)$, which we can regard as the velocity function along $A \natural_t B$. The derivative of velocity function is the acceleration function, so we define the acceleration by $\mathcal{A}_t(A|B) = \frac{d}{dt} S_t(A|B)$. In this paper, we discuss properties of $S_t(A|B)$ and $\mathcal{A}_t(A|B)$. Firstly, we interpret some properties of $S_t(A|B)$ concerning interpolational property and the noncommutative ratio from the viewpoint of velocity. Secondly, we show the properties of $\mathcal{A}_t(A|B)$ similar to those of $S_t(A|B)$.

1 Introduction.

Let A and B be strictly positive linear operators on a Hilbert space $\mathcal{H}$. An operator T on $\mathcal{H}$ is said to be positive (we denote it by $T \geq 0$) if $(T\xi, \xi) \geq 0$ for all $\xi \in \mathcal{H}$ and T is said to be strictly positive (we denote it by $T > 0$) if T is invertible and positive.

For $A, B > 0$, we define a path $A \natural_t B$ as follows ([2, 3, 6, 8, 14] etc.):

$$A \natural_t B \equiv A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}} \quad (t \in \mathbf{R}),$$

which is passing through $A = A \natural_0 B$ and $B = A \natural_1 B$. If $t \in [0,1]$, the path $A \natural_t B$ coincides with the weighted geometric operator mean denoted by $A \sharp_t B$ (cf. [15]). We remark that $A \natural_t B = B \natural_{1-t} A$ holds for $t \in \mathbf{R}$ (cf. [8]).

Fujii and Kamei [1] defined the following relative operator entropy for $A, B > 0$:

$$S(A|B) \equiv A^{\frac{1}{2}} \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

Furuta [7] defined generalized relative operator entropy as follows (see also [9]):

$$\begin{aligned} S_\alpha(A|B) &\equiv A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} \\ &= (A \natural_\alpha B) A^{-1} S(A|B) \ \ (\alpha \in \mathbf{R}). \end{aligned}$$

We know immediately $S_0(A|B) = S(A|B)$. We remark that

$$S(A|B) = \frac{d}{dt} A \natural_t B \bigg|_{t=0} \text{ and } S_\alpha(A|B) = \frac{d}{dt} A \natural_t B \bigg|_{t=\alpha}.$$

Yanagi, Kuriyama and Furuichi [16] introduced the Tsallis relative operator entropy as follows:

$$T_\alpha(A|B) \equiv \frac{A \sharp_\alpha B - A}{\alpha} \ (\alpha \in (0,1]).$$

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Since $\lim_{x\to 0}\dfrac{a^x-1}{x}=\log a$ holds for $a>0$, we have $T_0(A|B)\equiv\lim_{\alpha\to 0}T_\alpha(A|B)=S(A|B)$. The Tsallis relative operator entropy can be defined for any $\alpha\in\mathbf{R}$ by using $\natural_\alpha$ instead of $\sharp_\alpha$.

For $A,B>0$, $t\in[0,1]$ and $r\in[-1,1]$, operator power mean $A\ \sharp_{t,r}\ B$ is defined as follows:

$$A\ \sharp_{t,r}\ B\equiv A^{\frac{1}{2}}\left\{(1-t)I+t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\right\}^{\frac{1}{r}}A^{\frac{1}{2}}\ =\ A\ \natural_{\frac{1}{r}}\left\{A\ \nabla_t\ (A\ \natural_r\ B)\right\}.$$

We remark that $A\ \sharp_{t,r}\ B=B\ \sharp_{1-t,r}\ A$ holds for $t\in[0,1]$ and $r\in[-1,1]$ (cf. [10, 12]). The operator power mean is a path combining $A=A\ \sharp_{0,r}\ B$ and $B=A\ \sharp_{1,r}\ B$, and interpolates the arithmetic operator mean, the geometric operator mean and the harmonic operator mean.

arithmetic operator mean
$A\ \nabla_t\ B=(1-t)A+tB$
$\uparrow_{r=1}$
$A\ \sharp_{t,r}\ B\ \xrightarrow[r\to 0]{}$ geometric operator mean $A\ \sharp_t\ B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^tA^{\frac{1}{2}}$
$\downarrow_{r=-1}$
harmonic operator mean
$A\ \Delta_t\ B=\left(A^{-1}\ \nabla_t\ B^{-1}\right)^{-1}$

For $A,B>0$, $\alpha\in[0,1]$ and $r\in[-1,1]$, expanded relative operator entropy $S_{\alpha,r}(A|B)$ is defined as follows (cf. [10]):

$$\begin{aligned}
S_{\alpha,r}(A|B)\ &\equiv\ \frac{d}{dt}\ A\ \sharp_{t,r}\ B\Big|_{t=\alpha}\\
&=\ A^{\frac{1}{2}}\left[\left\{(1-\alpha)I+\alpha\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{\frac{1}{r}-1}\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r-I}{r}\right]A^{\frac{1}{2}}\\
&=\ (A\ \sharp_{\alpha,r}\ B)(A\ \nabla_\alpha\ (A\ \natural_r\ B))^{-1}S_{0,r}(A|B)\ (r\neq 0),\\
S_{\alpha,0}(A|B)\ &\equiv\ \lim_{r\to 0}S_{\alpha,r}(A|B)\ =\ S_\alpha(A|B).
\end{aligned}$$

We remark that $S_{0,r}(A|B)=T_r(A|B),\ S_{1,r}(A|B)=-T_r(B|A)$ hold for $r\in[-1,1]$.

$S(A|B)$ and $S_{0,r}(A|B)$ are given as follows:

$$S(A|B)\ =\ \frac{d}{dt}\ A\ \natural_t\ B\Big|_{t=0}\quad\text{and}\quad S_{0,r}(A|B)\ =\ \frac{d}{dt}\ A\ \sharp_{t,r}\ B\Big|_{t=0}.$$

We illustrate an image for $S(A|B)$ and $S_{0,r}(A|B)$ in Figure 1.

In [6], $S(A|B)$ and $S_{0,r}(A|B)$ are regarded as the velocities on the paths $A\ \sharp_t\ B$ and $A\ \sharp_{t,r}\ B$ at $t=0$ respectively. According to this viewpoint, it is natural to call $S_\alpha(A|B)$ and $S_{\alpha,r}(A|B)$ the velocities on the paths $A\ \natural_t\ B$ and $A\ \sharp_{t,r}\ B$ respectively. These interpretations inspire us to introduce the accelerations $\mathcal{A}_\alpha(A|B)$ and $\mathcal{A}_{\alpha,r}(A|B)$ on the paths $A\ \natural_t\ B$ and $A\ \sharp_{t,r}\ B$ at $t=\alpha$.

In this paper, we can show that the properties concerning the accelerations $\mathcal{A}_\alpha(A|B)$ and $\mathcal{A}_{\alpha,r}(A|B)$, interpolational property, the behavior of noncommutative ratio and so on, are inherited from those of velocities $S_\alpha(A|B)$ and $S_{\alpha,r}(A|B)$. The contents of this paper are as follows: In section 2, we show properties of the velocity $S_\alpha(A|B)$. In section 3, we introduce the acceleration $\mathcal{A}_\alpha(A|B)$ and we show some properties of $\mathcal{A}_\alpha(A|B)$. In section 4, we introduce the acceleration on the path $A \sharp_{t,r} B$ and we show some results for velocity $S_{\alpha,r}(A|B)$ and acceleration $\mathcal{A}_{\alpha,r}(A|B)$ on the path $A \sharp_{t,r} B$.

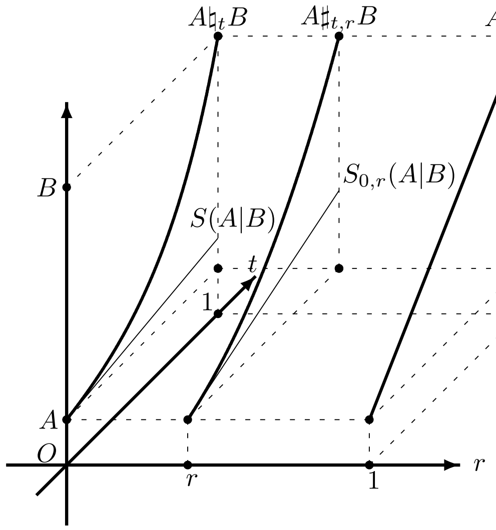

Figure 1: An image of $S(A|B)$ and $S_{0,r}(A|B)$.

2 Velocity on the path $A \natural_t B$**.** As mentioned in section 1, we regard $S_\alpha(A|B)$ as the velocity on the path $A \natural_t B$ at $t = \alpha$. In this section, we show some properties of the velocity $S_\alpha(A|B)$.

The next lemma shows interpolational property of the path $A \natural_t B$. This lemma is fundamental in our discussion.

Lemma 2.1. ([12]) *For* $A, B > 0$ *and* $x, y, \alpha \in \mathbf{R}$,

$$(A \natural_y B) \natural_\alpha (A \natural_x B) \;=\; A \natural_{(1-\alpha)y+\alpha x} B$$

holds.

Let $A \natural_x B$ and $A \natural_y B$ ($x, y \in \mathbf{R}$) be arbitrary points on the path $A \natural_t B$. Concerning the velocity $S_\alpha(A \natural_y B|A \natural_x B)$ at $t = \alpha$, we have the following theorem which was proved in [12].

Theorem 2.2. ([12]) *Let* $A, B > 0$ *and* $\alpha, x, y \in \mathbf{R}$. *Then*

$$S_\alpha(A \natural_y B|A \natural_x B) \;=\; (x - y)S_{(1-\alpha)y+\alpha x}(A|B).$$

In our discussion, for a given path $\gamma(t) = X \natural_t Y$ for $X, Y > 0$ and $t \in \mathbf{R}$, we imagine that an object moves through base points X ($t = 0$) and Y ($t = 1$) on the path $\gamma(t)$. Then $S_\alpha(A \natural_y B|A \natural_x B)$ means the velocity on the path $\gamma_1(t) = (A \natural_y B) \natural_t (A \natural_x B)$ at $t = \alpha$, and also $S_{(1-\alpha)y+\alpha x}(A|B)$ means the velocity on the path $\gamma_2(t) = A \natural_t B$ at $t = (1-\alpha)y + \alpha x$. Note that $\gamma_1(t)$ and $\gamma_2(t)$ represent the same path, and the point on $\gamma_1(t)$ at $t = \alpha$ and the point on $\gamma_2(t)$ at $t = (1-\alpha)y+\alpha x$ are the same point by Lemma 2.1. In this situation, we consider exchanging the path $\gamma_1(t)$ for $\gamma_2(t)$ to change unit length of the path. Then we can regard Theorem 2.2 as the result on the rate of change of velocities at the same point.

The next Corollary 2.3 is an immediate consequence of Theorem 2.2.

Corollary 2.3. *For $A, B > 0$ and $\alpha, x, y \in \mathbf{R}$, the following hold:*

(1) $S_\alpha(B|A) \;=\; -S_{1-\alpha}(A|B)$.

(2) $S_\alpha(A|A \natural_x B) \;=\; xS_{\alpha x}(A|B)$.

(3) $S_\alpha(A \natural_y B|A \natural_{y+1} B) \;=\; S_{\alpha+y}(A|B)$.

Proof. (1) is obtained by putting $x = 0$ and $y = 1$, (2) is obtained by putting $y = 0$ and (3) is obtained by putting $x = y + 1$. □

Next, from the above point of view, we discuss the noncommutative ratio $\mathcal{R}(v; A, B) \equiv (A \natural_v B)A^{-1}$ for $v \in \mathbf{R}$ which is defined in [11]. Note that it is independent of α in $S_\alpha(A|B)$.

Theorem 2.4. ([11]) *For $A, B > 0$ and $v \in \mathbf{R}$,*

$$\mathcal{R}(v; A, B)S_\alpha(A|B) \;=\; S_{\alpha+v}(A|B)$$

for all $\alpha \in \mathbf{R}$.

In particular, by putting $\alpha = 0$ in Theorem 2.4, we have following relation.

Corollary 2.5. ([11]) *For $A, B > 0$ and $v \in \mathbf{R}$, following hold:*

$$\mathcal{R}(v; A, B)S(A|B) \;=\; S_v(A|B).$$

By Theorem 2.4 and (3) in Corollary 2.3, we have

$$(\heartsuit) \qquad \mathcal{R}(v; A, B)S_\alpha(A|B) = S_\alpha(A \natural_v B|A \natural_{v+1} B) \quad \text{for } \alpha, v \in \mathbf{R}.$$

As an extension of this relation, we obtain the following Theorem 2.6. Here, we consider exchanging the path $\gamma_1(t) = (A \natural_y B) \natural_t (A \natural_x B)$ for $\gamma_2(t) = (A \natural_{y+v} B) \natural_t (A \natural_{x+v} B)$, that is, moving base points of the path preserving unit length. Then, Theorem 2.6 shows a relation between velocity on the path $\gamma_1(t)$ at $t = \alpha$ and velocity on the path $\gamma_2(t)$ at $t = \alpha$ by using the noncommutative ratio.

Theorem 2.6. *Let $A, B > 0$ and $\alpha, v, x, y \in \mathbf{R}$. Then*

$$\mathcal{R}(v; A, B)S_\alpha(A \natural_y B|A \natural_x B) \;=\; S_\alpha(A \natural_{y+v} B|A \natural_{x+v} B).$$

Proof. By Theorem 2.2 and Theorem 2.4, we have

$$\begin{aligned}
\mathcal{R}(v; A, B)S_\alpha(A \natural_y B|A \natural_x B) &= (x - y)\mathcal{R}(v; A, B)S_{(1-\alpha)y+\alpha x}(A|B) \\
&= (x - y)S_{(1-\alpha)y+\alpha x+v}(A|B) \\
&= \{(x + v) - (y + v)\}\, S_{(1-\alpha)(y+v)+\alpha(x+v)}(A|B) \\
&= S_\alpha(A \natural_{y+v} B|A \natural_{x+v} B).
\end{aligned}$$

□

Remark. We know that $\mathcal{R}(v(x - y); A, B) \;=\; \mathcal{R}(v; A \natural_y B, A \natural_x B)$ holds for $A, B > 0$ and $v, x, y \in \mathbf{R}$, since

$$\begin{aligned}
\mathcal{R}(v(x - y); A, B) &= (A \natural_{v(x-y)} B)A^{-1} \\
&= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{v(x-y)+y}A^{\frac{1}{2}}A^{-\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-y}A^{-\frac{1}{2}} \\
&= (A \natural_{(1-v)y+vx} B)(A \natural_y B)^{-1} \;=\; ((A \natural_y B) \natural_v (A \natural_x B))\,(A \natural_y B)^{-1} \\
&= \mathcal{R}(v; A \natural_y B, A \natural_x B).
\end{aligned}$$

From this relation and (♡), we can give an alternative proof of Theorem 2.6 as follows: By putting $u = v(x-y)$, we have $\mathcal{R}(u; A, B) = \mathcal{R}(\frac{u}{x-y}; A \natural_y B, A \natural_x B)$. Then

$$\begin{aligned}
\mathcal{R}(u; A, B) S_\alpha(A \natural_y B | A \natural_x B) &= \mathcal{R}\left(\frac{u}{x-y}; A \natural_y B, A \natural_x B\right) S_\alpha(A \natural_y B | A \natural_x B) \\
&= S_\alpha((A \natural_y B) \natural_{\frac{u}{x-y}} (A \natural_x B) | (A \natural_y B) \natural_{\frac{u}{x-y}+1} (A \natural_x B)) \\
&= S_\alpha(A \natural_{(1-\frac{u}{x-y})y + \frac{ux}{x-y}} B | A \natural_{-\frac{uy}{x-y} + (\frac{u}{x-y}+1)x} B) \\
&= S_\alpha(A \natural_{y+u} B | A \natural_{x+u} B).
\end{aligned}$$

Corollary 2.5 means that $\mathcal{R}(v; A, B)$ is the ratio of $S_v(A|B)$ and $S(A|B)$. Related to it, the difference between $S_v(A|B)$ and $S(A|B)$ is as follows:

Proposition 2.7. *For $A, B > 0$ and $v \in \mathbf{R}$,*

$$S_v(A|B) - S(A|B) = v T_v(A|B) A^{-1} S(A|B)$$

holds.

Proof. From Corollary 2.5, we have

$$\begin{aligned}
S_v(A|B) - S(A|B) &= \mathcal{R}(v; A, B) S(A|B) - S(A|B) \\
&= (A \natural_v B - A) A^{-1} S(A|B) = v T_v(A|B) A^{-1} S(A|B).
\end{aligned}$$

□

We remark that the above difference was also represented by using Petz-Bregman divergence (see [13]).

3 Acceleration on the path $A \natural_t B$**.** Since the relative operator entropy $S_\alpha(A|B)$ is regarded as the velocity on the path $A \natural_t B$ at $t = \alpha$, it is natural to call the derivative of $S_t(A|B)$ acceleration on $A \natural_t B$.

Definition 3.1. *For $A, B > 0$ and $\alpha \in \mathbf{R}$, we define the acceleration on the path $A \natural_t B$ at $t = \alpha$ as follows:*

$$\mathcal{A}_\alpha(A|B) \equiv \frac{d}{dt} S_t(A|B)\Big|_{t=\alpha}.$$

The acceleration $\mathcal{A}_\alpha(A|B)$ is represented explicitly as follows:

Theorem 3.2. *Let $A, B > 0$ and $\alpha \in \mathbf{R}$. Then*

$$\mathcal{A}_\alpha(A|B) = S_\alpha(A|B) A^{-1} S(A|B) = S_\alpha(A|B)(A \natural_\alpha B)^{-1} S_\alpha(A|B).$$

In particular,

$$\mathcal{A}_0(A|B) = S(A|B) A^{-1} S(A|B).$$

Proof. For $a > 0$, we have

$$\frac{d}{dt} a^t \log a = a^t (\log a)^2.$$

Then

$$\begin{aligned}\mathcal{A}_\alpha(A|B) &= \frac{d}{dt}S_t(A|B)\Big|_{t=\alpha}\\ &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))^2A^{\frac{1}{2}}\\ &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}A^{-1}A^{\frac{1}{2}}\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}\\ &= S_\alpha(A|B)A^{-1}S(A|B),\end{aligned}$$

which shows the first equality. On the other hand, we have

$$\begin{aligned}S_\alpha(A|B)A^{-1}S(A|B) &= S_\alpha(A|B)(A\ \natural_\alpha\ B)^{-1}(A\ \natural_\alpha\ B)A^{-1}S(A|B)\\ &= S_\alpha(A|B)(A\ \natural_\alpha\ B)^{-1}S_\alpha(A|B).\end{aligned}$$

□

Remark. Theorem 3.2 shows that if we put $\gamma(t) = A\ \natural_t\ B$, then it satisfies the geodesic equation $\ddot{\gamma}(t)-\dot{\gamma}(t)\big(\gamma(t)\big)^{-1}\dot{\gamma}(t) = 0$ since $\dot{\gamma}(t) = S_t(A|B)$ and $\ddot{\gamma}(t) = \mathcal{A}_t(A|B)$. Conversely, $A\ \natural_t\ B$ is given as the solution of the geodesic equation for initial conditions $\gamma(0) = A$ and $\gamma(1) = B$. We show it here according to [5] which treats matrices, but the same arguments are valid for operator valued functions, since, even for a operator valued function $\gamma(t)$, it holds that $(\gamma(t)^{-1})' = -(\gamma(t))^{-1}\gamma(t)'(\gamma(t))^{-1}$ and that $(\log\gamma(t))' = \gamma(t)'(\gamma(t))^{-1}$ if $\gamma(t)\gamma(t)' = \gamma(t)'\gamma(t)$.

By putting $f(t) = \gamma(0)^{-\frac{1}{2}}\gamma(t)\gamma(0)^{-\frac{1}{2}} = A^{-\frac{1}{2}}\gamma(t)A^{-\frac{1}{2}}$, we have

$$f''(t) - f'(t)\big(f(t)\big)^{-1}f'(t) = 0$$

and that $f(0) = A^{-\frac{1}{2}}\gamma(0)A^{-\frac{1}{2}} = I$ and $f(1) = A^{-\frac{1}{2}}\gamma(1)A^{-\frac{1}{2}} = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Since

$$\begin{aligned}\big(f'(t)(f(t))^{-1}\big)' &= f''(t)(f(t))^{-1} - f'(t)(f(t))^{-1}f'(t)(f(t))^{-1}\\ &= f''(t)(f(t))^{-1} - f''(t)(f(t))^{-1} = 0,\end{aligned}$$

then we have $f'(t)(f(t))^{-1} = C$, that is, $f'(t) = Cf(t)$. It is known that $f(t)$ and $f'(t)$ are selfadjoint, so we have

$$C^* = f(0)C^* = (Cf(0))^* = (f'(0))^* = f'(0) = Cf(0) = C.$$

Hence

$$f'(t)(f(t))^{-1} = C = C^* = (f'(t)(f(t))^{-1})^* = (f(t))^{-1}f'(t),$$

and then $f'(t)f(t) = f(t)f'(t)$. So we have $(\log f(t))' = f'(t)(f(t))^{-1} = C$ and then $\log f(t) = Ct + D$. By $f(0) = I$ and $f(1) = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have $\exp C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and $D = 0$, that is, $f(t) = (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t$. Therefore, we obtain

$$\gamma(t) = A^{\frac{1}{2}}f(t)A^{\frac{1}{2}} = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^tA^{\frac{1}{2}} = A\ \natural_t\ B.$$

Through Theorem 3.2, we know that the acceleration $\mathcal{A}_\alpha(A|B)$ has the similar properties to the velocity $S_\alpha(A|B)$. First, we have Theorem 3.3 which corresponds to Theorem 2.2. As mentioned in section 2, the point on the path $\gamma_1(t) = (A\ \natural_y\ B)\ \natural_t\ (A\ \natural_x\ B)$ at $t = \alpha$ and the point on $\gamma_2(t) = A\ \natural_t\ B$ at $t = (1-\alpha)y + \alpha x$ are the same point. Then Theorem 3.3 shows the result on the rate of change of accelerations at the same point.

Theorem 3.3. *Let $A, B > 0$ and $\alpha, x, y \in \mathbf{R}$. Then*

$$\mathcal{A}_\alpha(A \natural_y B|A \natural_x B) = (x-y)^2 \mathcal{A}_{(1-\alpha)y+\alpha x}(A|B).$$

Proof. By Theorem 3.2, Lemma 2.1 and Theorem 2.2, we have

$$\begin{aligned}
&\mathcal{A}_\alpha(A \natural_y B|A \natural_x B) \\
&\quad = S_\alpha(A \natural_y B|A \natural_x B)((A \natural_y B) \natural_\alpha (A \natural_x B))^{-1} S_\alpha(A \natural_y B|A \natural_x B) \\
&\quad = (x-y)^2 S_{(1-\alpha)y+\alpha x}(A|B)(A \natural_{(1-\alpha)y+\alpha x} B)^{-1} S_{(1-\alpha)y+\alpha x}(A|B) \\
&\quad = (x-y)^2 \mathcal{A}_{(1-\alpha)y+\alpha x}(A|B).
\end{aligned}$$

□

Corollary 3.4. *For $A, B > 0$, and $\alpha, x, y \in \mathbf{R}$, the following hold:*

(1) $\mathcal{A}_\alpha(B|A) = \mathcal{A}_{1-\alpha}(A|B)$.

(2) $\mathcal{A}_\alpha(A|A \natural_x B) = x^2 \mathcal{A}_{\alpha x}(A|B)$.

(3) $\mathcal{A}_\alpha(A \natural_y B|A \natural_{y+1} B) = \mathcal{A}_{\alpha+y}(A|B)$.

Secondly, related to the noncommutative ratio, we have Theorem 3.5 which corresponds to Theorem 2.4.

Theorem 3.5. *For $A, B > 0$ and $v \in \mathbf{R}$,*

$$\mathcal{R}(v; A, B)\mathcal{A}_\alpha(A|B) = \mathcal{A}_{\alpha+v}(A|B)$$

for all $\alpha \in \mathbf{R}$. In particular,

$$\mathcal{R}(v; A, B)\mathcal{A}_0(A|B) = \mathcal{A}_v(A|B).$$

Proof. By Theorem 3.2 and Theorem 2.4, we have

$$\begin{aligned}
\mathcal{R}(v; A, B)\mathcal{A}_\alpha(A|B) &= \mathcal{R}(v; A, B) S_\alpha(A|B) A^{-1} S(A|B) \\
&= S_{\alpha+v}(A|B) A^{-1} S(A|B) = \mathcal{A}_{\alpha+v}(A|B).
\end{aligned}$$

□

The following Theorem 3.6 is an extension of Theorem 3.5. Similarly to Theorem 2.6, Theorem 3.6 shows a relation between acceleration on the paths $\gamma_1(t) = (A \natural_y B) \natural_t (A \natural_x B)$ and $\gamma_2(t) = (A \natural_{y+v} B) \natural_t (A \natural_{x+v} B)$ at $t = \alpha$ by using the noncommutative ratio.

Theorem 3.6. *Let $A, B > 0$ and $\alpha, v, x, y \in \mathbf{R}$. Then*

$$\mathcal{R}(v; A, B)\mathcal{A}_\alpha(A \natural_y B|A \natural_x B) = \mathcal{A}_\alpha(A \natural_{y+v} B|A \natural_{x+v} B).$$

Proof. By Theorem 3.3 and Theorem 3.5, we have

$$\begin{aligned}
\mathcal{R}(v; A, B)\mathcal{A}_\alpha(A \natural_y B|A \natural_x B) &= (x-y)^2 \mathcal{R}(v; A, B)\mathcal{A}_{(1-\alpha)y+\alpha x}(A|B) \\
&= (x-y)^2 \mathcal{A}_{(1-\alpha)y+\alpha x+v}(A|B) \\
&= \{(x+v)-(y+v)\}^2 \mathcal{A}_{(1-\alpha)(y+v)+\alpha(x+v)}(A|B) \\
&= \mathcal{A}_\alpha(A \natural_{y+v} B|A \natural_{x+v} B).
\end{aligned}$$

□

Lastly, the difference between $\mathcal{A}_v(A|B)$ and $\mathcal{A}_0(A|B)$ is gotten as follows.

Proposition 3.7. *For $A, B > 0$ and $v \in \mathbf{R}$,*

$$\mathcal{A}_v(A|B) - \mathcal{A}_0(A|B) \ = \ vT_v(A|B)A^{-1}\mathcal{A}_0(A|B).$$

holds.

Proof. By using Theorem 3.5, we have

$$\begin{aligned}\mathcal{A}_v(A|B) - \mathcal{A}_0(A|B) &= \mathcal{R}(v; A, B)\mathcal{A}_0(A|B) - \mathcal{A}_0(A|B) \\ &= (A \natural_v B - A)A^{-1}\mathcal{A}_0(A|B) \\ &= vT_v(A|B)A^{-1}\mathcal{A}_0(A|B).\end{aligned}$$

□

4 Velocity and acceleration on the path $A \sharp_{t,r} B$. In this section, we introduce the velocity and the acceleration on the path $A \sharp_{t,r} B$ and show their properties.

We know that the path $A \sharp_{t,r} B$ has interpolational property. The next lemma is the same property as Lemma 2.1.

Lemma 4.1. ([14]) *For $A, B > 0$, $\alpha, x, y \in [0,1]$ and $r \in [-1,1]$,*

$$(A \sharp_{y,r} B) \sharp_{\alpha,r} (A \sharp_{x,r} B) \ = \ A \sharp_{(1-\alpha)y+\alpha x,r} B$$

holds.

Although the noncommutative ratio discussed in section 2 can not be extended totally to the one concerning $A \sharp_{t,r} B$, the property Corollary 2.5 is extended as follows:

Theorem 4.2. ([12]) *For $A, B > 0$, $\alpha \in [0,1]$ and $r \in [-1,1]$,*

$$S_{\alpha,r}(A|B) \ = \ (A \sharp_{\alpha,r} B)(A \nabla_\alpha (A \natural_r B))^{-1}S_{0,r}(A|B).$$

holds.

We introduce the acceleration on the path $A \sharp_{t,r} B$ as follows:

Definition 4.3. *For $A, B > 0$, $\alpha \in [0,1]$ and $r \in [-1,1]$, we define $\mathcal{A}_{\alpha,r}(A|B)$ as*

$$\mathcal{A}_{\alpha,r}(A|B) \ \equiv \ \frac{d}{dt}S_{t,r}(A|B)\bigg|_{t=\alpha}.$$

We call it the acceleration on the path $A \sharp_{t,r} B$ at $t = \alpha$.

We remark that $\mathcal{A}_{\alpha,0}(A|B) \ = \ \mathcal{A}_\alpha(A|B)$ for $\alpha \in [0,1]$ since $S_{t,0}(A|B) = S_t(A|B)$.

The acceleration $\mathcal{A}_{\alpha,r}(A|B)$ is represented explicitly as follows:

Theorem 4.4. *Let $A, B > 0$, $\alpha \in [0,1]$ and $r \in [-1,1]$. Then*

$$\begin{aligned}\mathcal{A}_{\alpha,r}(A|B) &= (1-r)S_{\alpha,r}(A|B)(A \nabla_\alpha (A \natural_r B))^{-1}S_{0,r}(A|B) \\ &= (1-r)S_{\alpha,r}(A|B)(A \sharp_{\alpha,r} B)^{-1}S_{\alpha,r}(A|B).\end{aligned}$$

In particular,

$$\mathcal{A}_{0,r}(A|B) \ = \ (1-r)S_{0,r}(A|B)A^{-1}S_{0,r}(A|B).$$

Proof. We have shown the case $r = 0$ in Theorem 3.2. Hence, we have only to show the case $r \neq 0$. Since

$$\begin{aligned}
&\frac{d}{dt}S_{t,r}(A|B)\\
&= (1-r)A^{\frac{1}{2}}\left\{(1-t)I + t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{\frac{1}{r}-2}\left(\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I}{r}\right)^2 A^{\frac{1}{2}}\\
&= (1-r)A^{\frac{1}{2}}\left[\left\{(1-t)I + t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{\frac{1}{r}-1}\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I}{r}\right]A^{\frac{1}{2}}\\
&\quad \times A^{-\frac{1}{2}}\left\{(1-t)I + t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{-1}A^{-\frac{1}{2}}A^{\frac{1}{2}}\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I}{r}A^{\frac{1}{2}}\\
&= (1-r)S_{t,r}(A|B)(A \nabla_t (A \natural_r B))^{-1}T_r(A|B)\\
&= (1-r)S_{t,r}(A|B)(A \nabla_t (A \natural_r B))^{-1}S_{0,r}(A|B),
\end{aligned}$$

we have

$$\mathcal{A}_{\alpha,r}(A|B) = (1-r)S_{\alpha,r}(A|B)(A \nabla_\alpha (A \natural_r B))^{-1}S_{0,r}(A|B).$$

On the other hand, by Theorem 4.2, we have

$$(1-r)S_{\alpha,r}(A|B)(A \nabla_\alpha (A \natural_r B))^{-1}S_{0,r}(A|B) = (1-r)S_{\alpha,r}(A|B)(A \sharp_{\alpha,r} B)^{-1}S_{\alpha,r}(A|B).$$

□

By Theorem 4.4 and Lemma 4.1, we give similar properties to those in sections 2 and 3. First, we have the next theorem and corollary.

Theorem 4.5. *Let $A, B > 0$, $\alpha, x, y \in [0,1]$ and $r \in [-1,1]$. Then*

(1) $S_{\alpha,r}(A \sharp_{y,r} B|A \sharp_{x,r} B) = (x-y)S_{(1-\alpha)y+\alpha x,r}(A|B)$.

(2) $\mathcal{A}_{\alpha,r}(A \sharp_{y,r} B|A \sharp_{x,r} B) = (x-y)^2\mathcal{A}_{(1-\alpha)y+\alpha x,r}(A|B)$.

Proof. (1) By using Lemma 4.1, we have

$$\begin{aligned}
&S_{\alpha,r}(A \sharp_{y,r} B|A \sharp_{x,r} B)\\
&= \frac{d}{dt}(A \sharp_{y,r} B) \sharp_{t,r} (A \sharp_{x,r} B)\Big|_{t=\alpha}\\
&= \lim_{v\to 0}\frac{(A \sharp_{y,r} B) \sharp_{\alpha+v,r} (A \sharp_{x,r} B) - (A \sharp_{y,r} B) \sharp_{\alpha,r} (A \sharp_{x,r} B)}{v}\\
&= \lim_{v\to 0}\frac{A \sharp_{(1-(\alpha+v))y+(\alpha+v)x,r} B - A \sharp_{(1-\alpha)y+\alpha x,r} B}{v}\\
&= (x-y)\lim_{v\to 0}\frac{A \sharp_{(1-\alpha)y+\alpha x+v(x-y),r} B - A \sharp_{(1-\alpha)y+\alpha x,r} B}{(x-y)v}\\
&= (x-y)S_{(1-\alpha)y+\alpha x,r}(A|B).
\end{aligned}$$

(2) From Theorem 4.4, (1) in Theorem 4.5 and Lemma 4.1, we obtain

$$\begin{aligned}
&\mathcal{A}_{\alpha,r}(A \sharp_{y,r} B|A \sharp_{x,r} B)\\
&= (1-r)S_{\alpha,r}(A \sharp_{y,r} B|A \sharp_{x,r} B)\big((A \sharp_{y,r} B) \sharp_{\alpha,r}(A \sharp_{x,r} B)\big)^{-1}S_{\alpha,r}(A \sharp_{y,r} B|A \sharp_{x,r} B)\\
&= (x-y)^2(1-r)S_{(1-\alpha)y+\alpha x,r}(A|B)(A \sharp_{(1-\alpha)y+\alpha x,r} B)^{-1}S_{(1-\alpha)y+\alpha x,r}(A|B)\\
&= (x-y)^2\mathcal{A}_{(1-\alpha)y+\alpha x,r}(A|B).
\end{aligned}$$

□

Corollary 4.6. *For $A, B > 0$, $\alpha, x \in [0,1]$ and $r \in [-1,1]$, the following hold:*

(1) $S_{\alpha,r}(B|A) = -S_{1-\alpha,r}(A|B)$ *and* $S_{\alpha,r}(A|A \natural_{x,r} B) = xS_{\alpha x,r}(A|B)$.

(2) $\mathcal{A}_{\alpha,r}(B|A) = \mathcal{A}_{1-\alpha,r}(A|B)$ *and* $\mathcal{A}_{\alpha,r}(A|A \natural_{x,r} B) = x^2\mathcal{A}_{\alpha x,r}(A|B)$.

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