ON DUCCI MATRIX SEQUENCES II

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Abstract. In this paper, we shall consider various properties of Ducci matrix sequences. Those properties are analyzed from the viewpoint of measure theory and of (Baire) category. Considered sets include the set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ whose Ducci matrix sequence expansion contains eventually periodic occurrences of a fixed block, which is of measure zero and meager, and the set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ such that infinitely many i satisfy $j_\alpha(i) = \cdots = j_\alpha(i+l-1) = j_\alpha(i+l)$, which is of full measure and comeager.

It is known that the Ducci matrix sequence expansion is eventually periodic if and only if the continued fraction expansion is [J. Difference Equ. Appl. 22(3) (2016), pp. 411–427]. Inspired by this result, we prove that analogous statements are valid for positive Poisson stability and for the denseness of the orbit, while neither implication is valid for eventual abelian periodicity. For eventual almost periodicity, only one implication is valid.

1 Introduction. A Ducci sequence is a sequence of vectors generated by iterating the following *Ducci map* D to a starting vector:

$$(v_1, v_2, \ldots, v_n) \overset{D}{\longmapsto} (|v_1 - v_2|, |v_2 - v_3|, \ldots, |v_n - v_1|)$$

Ciamberlini and Marengoni attributed a question about the limiting behavior of such sequences to E. Ducci in their paper [3]. Since then, a substantial amount of literature on various generalizations as well as the dynamics of the Ducci map has appeared ([1] provides a large list of references.)

It was Hogenson et al. [8] who introduced the concept *Ducci matrix sequences* for the first time. For each vector in $\mathbb{R}^n$, one can find an $n \times n$ matrix whose application to the vector is equivalent to the application of the Ducci map. This matrix depends, of course, on the chosen vector. Thus, one may associate with a vector $\boldsymbol{v}$ not a single matrix but a sequence $\langle M_{j_1}, M_{j_2}, \ldots \rangle$ of matrices such that the matrix M_{j_n} implements the n-th application of the Ducci map to $\boldsymbol{v}$. By considering those starting vectors in $\mathbb{R}^3$ that lead to unique Ducci matrix sequences, Hogenson et al. [8] established a connection between the Ducci map, the process of forming *mediants* of rational numbers and the *Stern-Brocot tree*. They also showed that a real number α admits a unique Ducci matrix sequence if and only if it is irrational. So for any irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$, we can call the unique Ducci matrix sequence $\langle M_{j_\alpha(1)}, M_{j_\alpha(2)}, M_{j_\alpha(3)}, \ldots \rangle$ associated with it the *Ducci matrix sequence expansion* of α.

Here is an important observation: As has been mentioned above, there are some connections between the Stern-Brocot tree and the Ducci sequences. It is also known [5] that the Stern-Brocot tree has intimate connections with continued fraction. So it is reasonable to anticipate a fundamental role played by continued fractions in understanding the Ducci sequences over $\mathbb{R}^3$. Indeed, it was proved in [6] that for any irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$, one can completely describe the behavior of Ducci map on the starting vector $(0, \alpha, 1)$ in terms of the continued fraction expansion of α.

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This explicit formula proves to be very useful in the study of Ducci matrix sequences. In [6], it was shown that for any irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$, its Ducci matrix sequence expansion is eventually periodic if and only if its continued fraction expansion is eventually periodic. This result, together with Lagrange's theorem, gives a characterization of those $\alpha \in (0,1) \setminus \mathbb{Q}$ having eventually periodic Ducci matrix sequence expansion. The explicit formula finds its applications also in the field of measure theory: Consider the question "Which $\alpha \in (0,1) \setminus \mathbb{Q}$ has uniformly distributed Ducci matrix sequence expansion?" One can take the term uniform distribution in various sense, but one way to formalize it is as follows: $\lim_{n\to\infty} |\{ i \leq n \mid j_\alpha(i) = \jmath \}|/n = 1/6$ for every $\jmath \in \{1, 2, \ldots, 6\}$. This condition, and several other reasonable formalizations, is satisfied by (Lebesgue) almost every $\alpha \in (0,1) \setminus \mathbb{Q}$. (See [7] for more information.)

In this paper, we shall continue our study of Ducci matrix sequences in the same line as [6, 7]. After introducing necessary concepts and reviewing some of the standard facts of them in Section 2, we give in Section 3 several properties of a Ducci matrix sequence that hold almost everywhere. These include "infinitely many i satisfy $j_\alpha(i) = \cdots = j_\alpha(i+l-1) = j_\alpha(i+l)$" and "infinitely many i satisfy $j_\alpha(i)+2l \equiv \cdots \equiv j_\alpha(i+l-1)+2 \equiv j_\alpha(i+l) \pmod 6$". As has been mentioned above, for any irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$, its Ducci matrix sequence expansion is eventually periodic if and only if its continued fraction expansion is eventually periodic. This result has the following consequence: The set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ with eventually periodic Ducci matrix sequence expansion is of measure zero. One of the purposes of Section 4 is to provide a strengthening of this consequence. Specifically, it is proved that the set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ whose Ducci matrix sequence expansion contains eventually periodic occurrences of a fixed block is of measure zero. In contrast, it is shown that for any fixed $l \geq 1$, the Ducci matrix sequence expansion of almost every $\alpha \in (0,1) \setminus \mathbb{Q}$ contains eventually periodic l occurrences of a certain block. Section 5 is devoted to the study of Ducci matrix sequences from the viewpoint of (Baire) category. While measure and category are known to be quite orthogonal, it turns out that for many of the sets we shall treat in this paper, these two concepts do not give rise to big differences. For example, a full-measure set $\{\alpha \in (0,1) \setminus \mathbb{Q} \mid \forall l\, \exists^\infty i\, j_\alpha(i) = \cdots = j_\alpha(i+l-1) = j_\alpha(i+l)\}$ is comeager and a measure-zero set $\{\alpha \in (0,1) \setminus \mathbb{Q} \mid \exists p\, \exists m\, \forall n\, j_\alpha(m+np) = j_\alpha(m+(n+1)p)\}$ is meager. Inspired by the aforementioned equivalence between eventual periodicity of Ducci matrix sequence expansion and of continued fraction expansion, we shall examine in Section 6 whether or not analogous statements are valid for several other combinatorial/dynamical properties. For eventual abelian periodicity, neither implication is true. For eventual almost periodicity, only one implication is true. For both positive Poisson stability and the denseness of the orbit, analogous statements are true. We conclude the paper by presenting a number of problems in Section 7.

2 Preliminary. For the convenience of the reader, we repeat the relevant material from [6, 7] without proofs, thus making our exposition self-contained.

Let us start by fixing certain terminology on continued fractions (as taken from Khinchin's book [9]). We write $[a_0; a_1, a_2, \ldots]$ and $[a_0; a_1, \ldots, a_l]$ for the following infinite and finite continued fraction, respectively:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \qquad \text{and} \qquad a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_l}}}.$$

We assume that a_0 is an integer and $a_1, a_2, \ldots$ are positive integers. We call $a_0, a_1, \ldots$

the *elements* of a continued fraction. For an infinite continued fraction $\alpha = [a_0; a_1, a_2, \dots]$, we call $s_k := [a_0; a_1, \dots, a_k]$ and $r_k := [a_k; a_{k+1}, \dots]$ a *segment* and a *remainder* of α, respectively. Obviously, remainders satisfy the relation $r_k = r_{k+1}^{-1} + a_k$. For finite continued fractions, segments and remainders are defined analogously.

It is well-known that continued fraction can be used as an apparatus for representing real numbers (A proof of the next folklore theorem can be found in, e.g., [9, Theorem 14]):

Theorem 1. *Assume that the last element of any finite continued fraction is greater than* 1. *Then, to every real number α, there corresponds a unique continued fraction with value equal to α. This fraction is finite when α is rational, and is infinite when α is irrational.* □

Using continued fraction expansion, one can completely describe the orbit of $(0, \alpha, 1)$ under the Ducci map D for irrational $\alpha > 0$ as follows. Observe that $\alpha > 0$ implies that the first element a_0 of α's continued fraction expansion is non-negative.

Theorem 2 ([6])**.** *Let $\alpha = [a_0; a_1, a_2, \dots] > 0$. For a given positive integer $n \geq 1$, let k be the least integer satisfying the relation $n \leq \sum_{i=0}^{k} a_i$. Then*

$$D^n(0, \alpha, 1) = \frac{\alpha}{r_0 \cdots r_k} \tau_{n,k} \cdot \begin{pmatrix} 1 \\ r_{k+1}^{-1} + \sum_{i=0}^{k} a_i - n \\ r_{k+1}^{-1} + \sum_{i=0}^{k} a_i - n + 1 \end{pmatrix}^{\mathrm{T}},$$

where $\tau_{n,k} \in \mathfrak{S}_3$ is a permutation that depends only on n if $k = 0$, and n and a segment s_{k-1} if $k > 0$. □

(We write $\tau \cdot \boldsymbol{v} := (v_{\tau(1)}, v_{\tau(2)}, v_{\tau(3)})$ for a permutation $\tau \in \mathfrak{S}_3$ and a vector $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$.) In order to understand the dynamical behavior of $D^n(0, \alpha, 1)$ with α rational, a slight modification of the statement is necessary: For a finite continued fraction $\alpha = [a_0; a_1, \dots, a_l]$, the formula is correct for $n = 1, 2, \dots, \sum_{i=0}^{l-1} a_i$. For n with $\sum_{i=0}^{l-1} a_i < n \leq \sum_{i=0}^{l} a_i$, in order to obtain a correct formula, we have to delete all the occurrences of the term r_{l+1}^{-1} in the entries of the vector, i.e., $D^n(0, \alpha, 1) = (\alpha / r_0 \cdots r_l)\, \tau_{n,l} \cdot (1, \sum_{i=0}^{l} a_i - n, \sum_{i=0}^{l} a_i - n + 1)$.

For convenience, let us introduce one more concept here:

Definition 1. *We say that a real vector $\boldsymbol{v} \in \mathbb{R}^3$ is of*

- type 1 *if it is of the form $\boldsymbol{v}_1\langle c; x; n\rangle := c(1, x+n, x+n+1)$ for some $c > 0, 0 < x < 1$ and a natural number $n \geq 1$;*
- type 2 *if it is of the form $\boldsymbol{v}_2\langle c; x; n\rangle := c(x+n, 1, x+n+1)$ for some $c > 0, 0 < x < 1$ and a natural number $n \geq 1$;*
- type 3 *if it is of the form $\boldsymbol{v}_3\langle c; x; n\rangle := c(x+n, x+n+1, 1)$ for some $c > 0, 0 < x < 1$ and a natural number $n \geq 1$;*
- type 4 *if it is of the form $\boldsymbol{v}_4\langle c; x; n\rangle := c(1, x+n+1, x+n)$ for some $c > 0, 0 < x < 1$ and a natural number $n \geq 1$;*
- type 5 *if it is of the form $\boldsymbol{v}_5\langle c; x; n\rangle := c(x+n+1, 1, x+n)$ for some $c > 0, 0 < x < 1$ and a natural number $n \geq 1$;*
- type 6 *if it is of the form $\boldsymbol{v}_6\langle c; x; n\rangle := c(x+n+1, x+n, 1)$ for some $c > 0, 0 < x < 1$ and a natural number $n \geq 1$.*

In any of these cases, we call n the *integer part* of the vector $\boldsymbol{v}_i\langle c; x; n\rangle$.

Observe that for any irrational number α and $n \geq 1$, the vector $D^n(0, \alpha, 1)$ is of some type. By virtue of Theorem 2, this follows from the observation that the reminders r_i satisfy $0 < r_i^{-1} < 1$, and that we have

$$\frac{\alpha}{r_0 \cdots r_k} \tau_{n,k} \cdot \begin{pmatrix} 1 \\ r_{k+1}^{-1} \\ r_{k+1}^{-1} + 1 \end{pmatrix}^{\mathrm{T}} = \frac{\alpha}{r_0 \cdots r_k r_{k+1}} \tau_{n,k} \cdot \begin{pmatrix} r_{k+2}^{-1} + a_{k+1} \\ 1 \\ r_{k+2}^{-1} + a_{k+1} + 1 \end{pmatrix}^{\mathrm{T}}$$

with $a_{k+1} \geq 1$.

An easy computation shows the following

Proposition 1 ([6])**.** *Let $\alpha = [a_0; a_1, a_2, \dots] > 0$ be irrational. Then for any positive real number $c > 0$ and a natural number $n > 1$, it holds that $D(\boldsymbol{v}_i\langle c; r_k^{-1}; n\rangle) = \boldsymbol{v}_{i+1}\langle c; r_k^{-1}; n-1\rangle$ for every $k \geq 0$ and $i = 1, 2, \dots, 6$, where any subscript greater than 6 is to be understood by modulo 6.*

If the integer part of $\boldsymbol{v}_i$ is 1, then we have the following:

- $D(\boldsymbol{v}_1\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_1\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_2\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_4\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_3\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_3\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_4\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_6\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_5\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_5\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$;
- $D(\boldsymbol{v}_6\langle c; r_k^{-1}; 1\rangle) = \boldsymbol{v}_2\langle c/r_k; r_{k+1}^{-1}; a_k\rangle$ *holds for every* $c > 0$. □

Therefore, an application of the Ducci map D to a vector of the form $\boldsymbol{v}_i\langle c; r_k^{-1}; n\rangle$ with $n \geq 1$ yields the increment of the type by 1 (modulo 6) if and only if the integer part n is greater than 1. This property will play a key role later on.

This proposition will also bring the reader clearer understanding of the computation of the permutation $\tau_{n,k(n)}$.

In order to introduce the concept of Ducci matrix, we still need one more notion.

Definition 2. *The regions $\mathcal{R}_1, \dots, \mathcal{R}_6 \subset \mathbb{R}^3$ are defined as follows:*

- $\mathcal{R}_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \leq x_2 \leq x_3\}$;
- $\mathcal{R}_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \leq x_1 \leq x_3\}$;
- $\mathcal{R}_3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \leq x_1 \leq x_2\}$;
- $\mathcal{R}_4 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \leq x_3 \leq x_2\}$;
- $\mathcal{R}_5 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \leq x_3 \leq x_1\}$;
- $\mathcal{R}_6 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \leq x_2 \leq x_1\}$.

We say that a matrix M *implements* the action of the Ducci map D on $\boldsymbol{v} \in \mathbb{R}^3$ if $D\boldsymbol{v} = \boldsymbol{v}M$ holds. Matrices $M_1, \ldots, M_6$ are defined so that M_i implements the application of the Ducci map to any vector in the region $\mathcal{R}_i$ uniformly, i.e., $D\boldsymbol{v} = \boldsymbol{v}M_i$ holds for every $\boldsymbol{v} \in \mathcal{R}_i$. Specifically,

$$M_1 = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix},$$
$$M_4 = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$

Observe that two distinct regions can overlap each other. For example, $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(x_1, x_2, x_3) \mid x_1 = x_2 \le x_3\} \neq \emptyset$. Consequently, either M_1 or M_2 serves as an implementation of an application of the Ducci map to any vector $\boldsymbol{v} \in \mathcal{R}_1 \cap \mathcal{R}_2$. It is also obvious that when entries of a vector $\boldsymbol{v}$ are pairwise distinct, $\boldsymbol{v}$ belongs to a unique region, and hence has only one implementation.

It would be interesting to consider a sequence of implementations of applications of the Ducci map to a given starting vector. To make this precise, let us introduce one more piece of terminology.

Definition 3 ([8])**.** *For a given vector $\boldsymbol{v} \in \mathbb{R}^3$, a* Ducci matrix sequence associated with $\boldsymbol{v}$ *is a sequence $\langle M_{j_1}, M_{j_2}, \ldots \rangle$ of matrices with $j_1, j_2, \ldots \in \{1, 2, \ldots, 6\}$ such that $D^n\boldsymbol{v} = \boldsymbol{v}M_{j_1} \cdots M_{j_n}$ holds for all $n \ge 1$.*

For a real number $\alpha \in \mathbb{R}$, we define a Ducci matrix sequence associated with α *to be a Ducci matrix sequence associated with the vector $(0, \alpha, 1)$.*

It is natural to ask which α admits a unique Ducci matrix sequence. This question has been addressed in [8] as follows (A different proof can be found in [6].):

Theorem 3 ([8])**.** *α is irrational if and only if there is only one Ducci matrix sequence associated with α.* □

Thus, for a given α, we can call the unique Ducci matrix sequence $\langle M_{j_\alpha(1)}, M_{j_\alpha(2)}, \ldots \rangle$ associated with it the *Ducci matrix sequence expansion* of α. By abuse of terminology, we also call $j(\alpha) := \langle j_\alpha(1), j_\alpha(2), \ldots \rangle$ the Ducci matrix sequence expansion of α.

The next important consequence of Proposition 1 will be used later many times:

Corollary 1. *Let $\alpha = [0; a_1, a_2, \ldots] \in (0, 1) \setminus \mathbb{Q}$ be an arbitrary irrational number.*

(i) If $a_1 = 1$, then $j_\alpha(n) + 1 \not\equiv j_\alpha(n+1) \pmod 6 \iff n = \sum_{i=1}^m a_i$ for some $m \ge 2$;

(ii) If $a_1 > 1$, then $j_\alpha(n) + 1 \not\equiv j_\alpha(n+1) \pmod 6 \iff n = 1$ or $n = \sum_{i=1}^m a_i$ for some $m \ge 1$. □

In the rest of this paper, we shall make use of the following piece of terminology on sequences. If w is a sequence of finite length, we write $\mathrm{lh}(w)$ for its length. The symbol $\frown$ is used for concatenation of two sequences: $\langle x_1, \ldots, x_n \rangle \frown \langle x'_1, x'_2, \ldots \rangle = \langle x_1, x_2, \ldots, x_n, x'_1, x'_2, \ldots \rangle$. For a sequence w of finite length, w^n (resp. w^∞) stands for the concatenation of n-copies (resp. countably many copies) of w. The set of all infinite-length sequences $\langle x_1, x_2, \ldots \rangle$ (resp. all finite-length sequences $\langle x_1, x_2, \ldots, x_n \rangle$ of length ≥ 1) of elements x_i from an alphabet set Σ is denoted by Σ^∞ (resp. Σ^+). The set Σ^∞ is metrized by setting the distance between two distinct points $\langle x_1, x_2, \ldots \rangle$ and $\langle x'_1, x'_2, \ldots \rangle$ to be $2^{-\min\{i \mid x_i \neq x'_i\}}$.

3 Measure theory. According to Proposition 1, for any irrational $\alpha \in (0,1) \setminus \mathbb{Q}$ and $n \geq 1$, there are only three possible relations between $j_\alpha(n+1)$ and $j_\alpha(n)$, i.e., $j_\alpha(n+1) \equiv j_\alpha(n), j_\alpha(n)+1$ or $j_\alpha(n)+2 \pmod 6$. On each of these cases, we shall study various related conditions from the viewpoint of measure theory.

3.1 On the condition $j_\alpha(n)+1 \equiv j_\alpha(n+1) \pmod 6$. We start by citing two results from [7].

Theorem 4 ([7]). *For any $\alpha \in (0,1) \setminus \mathbb{Q}$ and $l \geq 1$, the following are equivalent:*

(i) $\displaystyle \lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i)+1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$

(ii) $\displaystyle \lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i)+l \equiv j_\alpha(i+1)+l-1 \equiv \cdots \equiv j_\alpha(i+l) \pmod 6\}|}{n} = 1;$

(iii) $\displaystyle \lim_{n\to\infty} \frac{\sum_{i=1}^n a_i}{n} = \infty.$ □

Theorem 5 ([7]). *Consider the following three conditions on a given irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$:*

(i) $\displaystyle \lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i)+1 \equiv j_\alpha(i+1) \pmod 6\}|}{n} = 1;$

(ii) $\displaystyle \lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6}$ *holds for every* $\jmath \in \{1,2,\ldots,6\}$;

(iii) $\displaystyle \lim_{n\to\infty} \sqrt[p]{\frac{\sum_{i=1}^n j_\alpha(i)^p}{n}} = \sqrt[p]{\frac{1^p+2^p+\cdots+6^p}{6}}$ *for every positive integer p.*

(i) *implies* (ii) *and* (ii) *implies* (iii). *Neither of these two implications is reversible.* □

In ergodic theory, it is known that for almost every $\alpha = [0; a_1, a_2, \ldots] \in (0,1) \setminus \mathbb{Q}$, we have $\lim_{n\to\infty} \sum_{i=1}^n a_i/n = \infty$ (See, e.g., [4]). Therefore, the above two theorems give a

Corollary 2. *Let $l \geq 1$ be an arbitrary integer. Then the following three conditions are valid almost everywhere:*

(i) $\displaystyle \lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i)+l \equiv j_\alpha(i+1)+l-1 \equiv \cdots \equiv j_\alpha(i+l) \pmod 6\}|}{n} = 1;$

(ii) $\displaystyle \lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) = \jmath\}|}{n} = \frac{1}{6}$ *holds for every* $\jmath \in \{1,2,\ldots,6\}$;

(iii) $\displaystyle \lim_{n\to\infty} \sqrt[p]{\frac{\sum_{i=1}^n j_\alpha(i)^p}{n}} = \sqrt[p]{\frac{1^p+2^p+\cdots+6^p}{6}}$ *for every positive integer p.* □

Item (i) in this corollary has two further consequences. Firstly, note that if we have $j_\alpha(i)+5 \equiv j_\alpha(i+1)+4 \equiv \cdots \equiv j_\alpha(i+5) \pmod 6$, then evidently $\{j_\alpha(i), j_\alpha(i+1), \ldots, j_\alpha(i+5)\} = \{1,2,\ldots,6\}$. So, every $\jmath \in \{1,2,\ldots,6\}$ can be written as $j_\alpha(i+\iota)$ for some $0 \leq \iota \leq 5$. Since Item (i) for $l=5$ says, in particular, that $j_\alpha(i)+5 \equiv j_\alpha(i+1)+4 \equiv \cdots \equiv j_\alpha(i+5) \pmod 6$ holds infinitely often almost everywhere, we conclude that for almost every $\alpha \in (0,1) \setminus \mathbb{Q}$, every Ducci matrix $M_\jmath$ occurs in the Ducci matrix sequence expansion of α infinitely often. In other words, we have a

Corollary 3 ([7]). *The following set is of measure zero:*

$$\{\alpha \in (0,1) \setminus \mathbb{Q} \mid \exists \jmath\, (j_\alpha(n) = \jmath \text{ holds for only finitely many } n)\}. \qquad \square$$

Recall that the conjunction of countably many properties that hold almost everywhere again holds almost everywhere. Hence below is another consequence of Item (i) in the corollary:

Corollary 4. *Almost every $\alpha \in (0,1) \setminus \mathbb{Q}$ satisfies the following property: For each $l \geq 1$, infinitely many n satisfy $j_\alpha(n) + l \equiv \cdots \equiv j_\alpha(n+l-1) + 1 \equiv j_\alpha(n+l) \pmod 6$.* $\square$

We have studied arithmetic mean $\sqrt[p]{\sum_{i=1}^{n} j_\alpha(i)^p/n}$ in [7], but geometric mean can also be dealt with: Indeed, it is not hard to see that Item (ii) in Theorem 5 entails $\lim_{n\to\infty} \sqrt[n]{\prod_{i=1}^{n} j_\alpha(i)} = \sqrt[6]{6!}$. Hence we have a

Corollary 5. $\lim_{n\to\infty} \sqrt[n]{\prod_{i=1}^{n} j_\alpha(i)} = \sqrt[6]{6!}$ *holds almost everywhere.* $\square$

3.2 On the conditions $j_\alpha(n) = j_\alpha(n+1)$ and $j_\alpha(n) + 2 \equiv j_\alpha(n+1) \pmod 6$.

Recall that there are only three possible relations between $j_\alpha(n+1)$ and $j_\alpha(n)$. Namely, $j_\alpha(n+1) \equiv j_\alpha(n), j_\alpha(n)+1$ or $j_\alpha(n)+2 \pmod 6$. Corollary 2 says that for almost every $\alpha \in (0,1) \setminus \mathbb{Q}$, $j_\alpha(n) + 1 \equiv j_\alpha(n+1) \pmod 6$ happens with density 1. Hence,

Corollary 6. *For almost every $\alpha \in (0,1) \setminus \mathbb{Q}$, we have*

$$\lim_{n\to\infty} \frac{|\{i \leq n \mid j_\alpha(i) = j_\alpha(i+1) \ (\textit{resp. } j_\alpha(i) + 2 \equiv j_\alpha(i+1) \pmod 6)\}|}{n} = 0. \qquad \square$$

However, the condition $j_\alpha(i) = j_\alpha(i+1)$ (resp. $j_\alpha(i) + 2 \equiv j_\alpha(i+1) \pmod 6$) still holds infinitely often almost everywhere. In fact, much stronger assertions are valid:

Theorem 6. *For a given fixed integer $l \geq 1$, consider the six conditions below.*

- $j_\alpha(i) = \cdots = j_\alpha(i+l-1) = j_\alpha(i+l) = 1$;
- $j_\alpha(i) = \cdots = j_\alpha(i+l-1) = j_\alpha(i+l) = 3$;
- $j_\alpha(i) = \cdots = j_\alpha(i+l-1) = j_\alpha(i+l) = 5$;
- $j_\alpha(i) + 2l \equiv \cdots \equiv j_\alpha(i+l-1) + 2 \equiv j_\alpha(i+l) \equiv 2 \pmod 6$;
- $j_\alpha(i) + 2l \equiv \cdots \equiv j_\alpha(i+l-1) + 2 \equiv j_\alpha(i+l) \equiv 4 \pmod 6$;
- $j_\alpha(i) + 2l \equiv \cdots \equiv j_\alpha(i+l-1) + 2 \equiv j_\alpha(i+l) \equiv 6 \pmod 6$.

Then almost every $\alpha \in (0,1) \setminus \mathbb{Q}$ has the following property: For each of the above six conditions, there exist infinitely many i which satisfy the condition.

For this result, we need to prepare two lemmata:

Lemma 1. *For each sequence $\langle n_1, \ldots, n_m \rangle \in \mathbb{Z}_{>0}^{+}$ of finite length, there exists a sequence $S(\langle n_1, \ldots, n_m \rangle) \in \mathbb{Z}_{>0}^{+}$ satisfying the following property: For any $\alpha \in (0,1) \setminus \mathbb{Q}$, if we have $S(\langle n_1, \ldots, n_m \rangle) = \langle a_{k+1}, a_{k+2}, \ldots, a_{k+\mathrm{lh}(S(\langle n_1,\ldots,n_m\rangle))} \rangle$ for some $k \geq 2$, then for every $\jmath \in \{1, 2, \ldots, 6\}$, there exists an $l_\jmath$ with $k+1 \leq l_\jmath \leq k + \mathrm{lh}(S(\langle n_1, \ldots, n_m \rangle))$ such that $j_\alpha(1 + \sum_{i=1}^{l_\jmath - 1} a_i) = \jmath$ and $\langle a_{l_\jmath}, a_{l_\jmath+1}, \ldots, a_{l_\jmath+m-1} \rangle = \langle n_1, \ldots, n_m \rangle$.*

Proof. We first claim that if $\langle a_{k+1}, a_{k+2}, \dots, a_{k+6!\cdot m}\rangle = \langle n_1, \dots, n_m\rangle^{6!}$ holds for some $k \geq 2$, then $j_\alpha(1 + \sum_{i=1}^{k} a_i) = j_\alpha(1 + \sum_{i=1}^{k+6!\cdot m} a_i)$. Due to the periodicity of the sequence $\langle a_{k+1}, a_{k+2}, \dots, a_{k+6!\cdot m}\rangle$, it holds that the parity of $j_\alpha(1 + \sum_{i=1}^{k+qm} a_i)$ $(0 \leq q \leq 6!)$ is either always the same or alternating, i.e., $j_\alpha(1 + \sum_{i=1}^{k+qm} a_i) \not\equiv j_\alpha(1 + \sum_{i=1}^{k+(q+1)m} a_i) \pmod 2$ for every $q < 6!$. In both cases, there exist $0 \leq q_1 < q_2 \leq 6$ such that $j_\alpha(1 + \sum_{i=1}^{k+q_1 m} a_i) = j_\alpha(1 + \sum_{i=1}^{k+q_2 m} a_i)$ and their parity is equal to the parity of $j_\alpha(1 + \sum_{i=1}^{k} a_i)$. For this q_1 and q_2, it is not hard to show that $j_\alpha(1 + \sum_{i=1}^{k} a_i) = j_\alpha(1 + \sum_{i=1}^{k+(q_2-q_1)m} a_i)$ with $0 < q_2 - q_1 \leq 6$. By periodicity, this type of equality continues to hold with period $(q_2 - q_1)m$ up to $1 + \sum_{i=1}^{k+6!\cdot m} a_i$. Since $q_2 - q_1$ divides $6!$, we thus get the following verification of our first claim:

$$j_\alpha(1 + \sum_{i=1}^{k} a_i) = j_\alpha(1 + \sum_{i=1}^{k+(q_2-q_1)m} a_i) = j_\alpha(1 + \sum_{i=1}^{k+2(q_2-q_1)m} a_i) = \cdots = j_\alpha(1 + \sum_{i=1}^{k+6!\cdot m} a_i).$$

Now define

$$\begin{aligned} S(\langle n_1, \dots, n_m\rangle) := \langle n_1, \dots, n_m\rangle^{6!} \frown \langle 3\rangle \frown \langle n_1, \dots, n_m\rangle^{6!} \frown \langle 3\rangle \frown \langle n_1, \dots, n_m\rangle^{6!} \\ \frown \langle 2\rangle \frown \langle n_1, \dots, n_m\rangle^{6!} \frown \langle 3\rangle \frown \langle n_1, \dots, n_m\rangle^{6!} \frown \langle 3\rangle \frown \langle n_1, \dots, n_m\rangle^{6!}. \end{aligned}$$

In view of the above claim and Corollary 1, it is not difficult to see that if we have $S(\langle n_1, \dots, n_m\rangle) = \langle a_{k+1}, a_{k+2}, \dots, a_{k+6!\cdot 6m+5}\rangle$ for some $k \geq 2$, then for every $\jmath \in \{1, 2, \dots, 6\}$, there exists an $l_\jmath$ with $k + 1 \leq l_\jmath \leq k + 1 + 5(6! + 1)$ such that $j_\alpha(1 + \sum_{i=1}^{l_\jmath - 1} a_i) = \jmath$ and $\langle a_{l_\jmath}, \dots, a_{l_\jmath + m-1}\rangle = \langle n_1, \dots, n_m\rangle$. This is precisely the assertion of the lemma. □

Although the next result is a standard one in ergodic theory, we shall include a sketch of its proof for the reader's convenience. (Detailed information on ergodic theory can be obtained from, e.g., [4].)

Lemma 2. *Let $w = \langle w_1, w_2, \dots, w_m\rangle \in \mathbb{Z}_{>0}^{+}$ be an arbitrary sequence of finite length. Then for almost every $\alpha \in (0,1) \setminus \mathbb{Q}$, infinitely many copies of w appear in the continued fraction expansion of α. In other words, $\{i \in \mathbb{Z}_{>0} \mid \langle a_i, \dots, a_{i+m-1}\rangle = w\}$ is an infinite set.*

Proof. Let T denote the Gauss map:

$$T(\alpha) := \begin{cases} \frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor & (\alpha \neq 0) \\ 0 & (\alpha = 0) \end{cases}.$$

It is well-known that T preserves Gauss measure ν and is moreover ergodic. On account of Birkhoff's ergodic theorem, we have

$$\begin{aligned} \frac{|\{i \leq n \mid \langle a_i, a_{i+1}, \dots, a_{i+m-1}\rangle = w\}|}{n} &= \frac{1}{n} \sum_{k=0}^{n-1} \chi_{[a,b]}(T^k \alpha) \\ &\to \frac{1}{\log 2} \int_a^b \frac{1}{1+x} dx > 0 \quad (\nu\text{-a.e.}), \end{aligned}$$

where $\{a, b\} = \{[0; w_1, w_2, \dots, w_m], [0; w_1, w_2, \dots, w_m + 1]\}$. Since Gauss measure ν and Lebesgue measure are equivalent, this proves that for Lebesgue almost every $\alpha \in (0,1) \setminus \mathbb{Q}$, the condition $\langle a_i, a_{i+1}, \dots, a_{i+m-1}\rangle = w$ holds with positive density, in particular infinitely often. □

Proof of Theorem 6. Apply Lemma 1 for $\langle 1,\dots,1\rangle$ of length l to get a sequence $S(\langle 1,\dots,1\rangle)$. Lemma 2 then guarantees that the continued fraction expansion of almost every $\alpha \in (0,1)\setminus\mathbb{Q}$ contains $S(\langle 1,\dots,1\rangle)$ infinitely often.

Take such an $\alpha = [0; a_1, a_2, \dots]$ arbitrarily. Then there exist infinitely many $k \geq 2$ such that $S(\langle 1,\dots,1\rangle) = \langle a_{k+1},\dots,a_{k+\mathrm{lh}(S(\langle 1,\dots,1\rangle))}\rangle$. Now it follows from Lemma 1 that for each $\jmath \in \{1,2,\dots,6\}$, there exists infinitely many $l_\jmath$ such that $j_\alpha(1+\sum_{i=1}^{l_\jmath-1} a_i) = \jmath$ and $\langle a_{l_\jmath}, a_{l_\jmath+1},\dots,a_{l_\jmath+l-1}\rangle = \langle 1,\dots,1\rangle$. For each such $l_\jmath$, in view of Proposition 1, it is not hard to conclude from these two properties that

$$\langle j_\alpha(1+\sum_{i=1}^{l_\jmath-1} a_i), j_\alpha(2+\sum_{i=1}^{l_\jmath-1} a_i),\dots,j_\alpha(l+\sum_{i=1}^{l_\jmath-1} a_i)\rangle = u_\jmath,$$

where $u_\jmath$ is the following sequence of length l:

$$\begin{array}{lll} u_1 := \langle 1,1,1,1,1,\dots,1\rangle & u_3 := \langle 3,3,3,3,3,\dots,3\rangle & u_5 := \langle 5,5,5,5,5,\dots,5\rangle \\ u_2 := \langle 2,4,6,2,4,6,\dots\rangle & u_4 := \langle 4,6,2,4,6,2,\dots\rangle & u_6 := \langle 6,2,4,6,2,4,\dots\rangle \end{array}$$

Therefore, all of $u_\jmath$ appear in $j(\alpha)$ infinitely often. □

4 Blocks that occur periodically. Even when a given Ducci matrix sequence is not eventually periodic, it may contain a block that occurs (eventually) periodically. In this section, we analyze Ducci matrix sequences with such properties.

Let us first remind the reader of the following result (A straightforward application of Birkhoff's ergodic theorem gives this result. Another proof can be found in [9, Theorem 29]):

Theorem 7. *The set $\mathbb{B} := \{\alpha \in (0,1)\setminus\mathbb{Q} \mid$ The elements of α are bounded$\}$ is of measure zero.* □

As a first application of this result, we now prove a

Proposition 2. *Almost every $\alpha \in (0,1)\setminus\mathbb{Q}$ satisfies the following property: For every $l, L \geq 1$, there exist $p, m \geq 1$ such that*

$$\langle j_\alpha(m+np),\dots,j_\alpha(L+m+np)\rangle = \langle j_\alpha(m+(n+1)p),\dots,j_\alpha(L+m+(n+1)p)\rangle$$

for $n = 1,2,\dots,l$.

Proof. We see from the above theorem that almost every $\alpha \in (0,1)\setminus\mathbb{Q}$ has infinitely many elements greater than $L+2+6l$. Also, if we have $a_{N+1} > L+2+6l$ for some $N > 1$, then Corollary 1 implies $j_\alpha(\iota) = j_\alpha(\iota+6) = j_\alpha(\iota+12) = \cdots = j_\alpha(\iota+6l)$ for every $\iota = 1+\sum_{i=1}^N a_i, 2+\sum_{i=1}^N a_i,\dots,L+1+\sum_{i=1}^N a_i$. So the assertion is valid with $p = 6$ and $m = -5+\sum_{i=1}^n a_i$. □

Proposition 3. *Almost every $\alpha \in (0,1)\setminus\mathbb{Q}$ satisfies the following property:*

(i) For every $\iota \in \{1,3,5\}$ and $l, L \geq 1$, there exist $p, m \geq 1$ such that

$$\langle j_\alpha(m+np), j_\alpha(m+1+np),\dots,j_\alpha(m+L+np)\rangle = \langle \iota,\iota,\dots,\iota\rangle$$

for $n = 1,2,\dots,l$;

(ii) For every $\iota \in \{2,4,6\}$ and $l, L \geq 1$, there exist $p, m \geq 1$ such that

$$\langle j_\alpha(m+np), j_\alpha(m+1+np),\dots,j_\alpha(m+L+np)\rangle \equiv \langle \iota,\iota+2,\iota+4,\dots,\iota+2L\rangle \pmod 6$$

for $n = 1,2,\dots,l$.

Proof. This is due to Theorem 6; Take l in the statement of the theorem larger than $L+l-1$ for (i) and $L+3l-3$ for (ii). (Period p is 1 and 3, respectively.) $\square$

In [6], we proved that for any irrational number $\alpha > 0$, its Ducci matrix sequence expansion is eventually periodic if and only if its continued fraction expansion is eventually periodic (See Theorem 9). Note that if the sequence $\langle a_0, a_1, a_2, \ldots\rangle$ is eventually periodic then elements a_i of α are bounded. Theorem 7 therefore implies that the set of all α with eventually periodic Ducci matrix sequence expansion is of measure zero.

It turns out that this conclusion is valid under a much weaker assumption, viz., the existence of (eventually) periodic occurrences of a fixed block. Let us prove this strengthening.

Theorem 8. *The set* $\{\alpha \in (0,1) \setminus \mathbb{Q} \mid \exists p\ \exists m\ \forall n\, j_\alpha(m+np) = j_\alpha(m+(n+1)p)\}$ *is of measure zero.*

Proof. It suffices to show that for each fixed p, the set $X_p := \{\alpha \in (0,1) \setminus \mathbb{Q} \mid \exists m\ \forall n\, j_\alpha(m+np) = j_\alpha(m+(n+1)p)\}$ is of measure zero.

Firstly, assume that p is not a multiple of 6. For any $\alpha \in X_p$, take an arbitrary $N > 2$ with $m < \sum_{i=1}^{N} a_i$ and let n be the least integer fulfilling that $\sum_{i=1}^{N} a_i < m+np$. If $a_{N+1} > 3p$, then the minimality of n implies $\sum_{i=1}^{N+1} a_i > m+(n+1)p$ $(> m+np > \sum_{i=1}^{N} a_i)$. Consequently, we have $j_\alpha(m+(n+1)p) \equiv j_\alpha(m+np)+p \pmod 6$, which is not equal to $j_\alpha(m+np) \pmod 6$ because p is not a multiple of 6. This shows that every $\alpha \in X_p$ has some N with $a_{N+1}, a_{N+2}, \ldots \le 3p$. In other words, X_p is a subset of a measure zero set $\mathbb{B}$, proving our claim in this case.

We next turn to the case where p is a multiple of 6. Fix an $\alpha \in (0,1) \setminus \mathbb{Q}$. For a given $m \ge 1$, suppose there exists an $N > 1$ such that $a_{N+1} = a_{N+2} = 2p$ and $m < \sum_{i=1}^{N} a_i$. Then there exists an n such that $\sum_{i=1}^{N} a_i < m+np \le \sum_{i=1}^{N+1} a_i < m+(n+1)p < \sum_{i=1}^{N+2} a_i$. Since we have $j_\alpha(\sum_{i=1}^{N+1} a_i)+1 \not\equiv j_\alpha(1+\sum_{i=1}^{N+1} a_i) \pmod 6$, it holds that

$$\begin{aligned} j_\alpha(m+(n+1)p) &\equiv j_\alpha(1+\sum_{i=1}^{N+1} a_i) + m + (n+1)p - 1 - \sum_{i=1}^{N+1} a_i \\ &\not\equiv j_\alpha(\sum_{i=1}^{N+1} a_i) + m + (n+1)p - \sum_{i=1}^{N+1} a_i \\ &\equiv j_\alpha(m+np) + m + (n+1)p - \sum_{i=1}^{N+1} a_i + \sum_{i=1}^{N+1} a_i - m - np \\ &\equiv j_\alpha(m+np) + p \\ &\equiv j_\alpha(m+np) \pmod 6. \end{aligned}$$

Hence this α is not in the set X_p. Since a straightforward application of Lemma 2 yields, for almost every α, the existence of infinitely many N with $a_{N+1} = a_{N+2} = 2p$, we thus see that almost every $\alpha \in (0,1) \setminus \mathbb{Q}$ is not in X_p, completing the proof of our claim also in this case. $\square$

In particular, for any $L > 1$ and $\iota \in \{1,3,5\}$ (resp. $\iota' \in \{2,4,6\}$), the set of all $\alpha \in (0,1)\setminus\mathbb{Q}$ whose Ducci matrix sequence expansion contains eventually periodic occurrence of the block $\langle \iota, \iota, \ldots, \iota\rangle$ (resp. $\langle \iota', \iota'+2, \ldots, \iota'+2(L-1)\rangle \pmod 6$) of length L is of measure zero.

This corollary can be proved directly by observing that elements of such α are bounded. In fact, it is this type of argument that enables one to prove the following result, which does not follow from Theorem 8: for any $L > 1$ and $\iota \in \{1, 3, 5\}$ (resp. $\iota' \in \{2, 4, 6\}$), the set of all $\alpha \in (0, 1) \setminus \mathbb{Q}$ whose Ducci matrix sequence expansion contains eventually *syndetic* occurrence of the block $\langle \iota, \iota, \dots, \iota\rangle$ (resp. $\langle \iota', \iota' + 2, \dots, \iota' + 2(L - 1)\rangle \pmod 6)$) of length L is of measure zero. (In this setting, syndetic is equivalent to being of bounded gaps. Cf. Subsection 6.2.)

5 Baire category. Although both notions of (Lebesgue) measure and category of sets serve as measurement of size, it is known that these two notions are in fact quite orthogonal. In this section, however, it is shown that for many of the sets that have appeared in the preceding sections, these two notions do not give rise to big differences.

Recall that a *nowhere dense* set is a set whose closure has empty interior. A set that can be written as the union of countably many nowhere dense sets is called *meager* (or *of first category*). Complements of meager sets are referred to as *comeager* (or *residual*). Those sets that are not meager are said to be *non-meager* (or *sets of second category*).

This time, since only subsets of $(0, 1) \setminus \mathbb{Q}$ are studied in this section, both $(0, 1)$ and $(0, 1) \setminus \mathbb{Q}$ have the right to serve as the underlying set. However, we deliberately do not clarify the underlying set because we do not have to do so: For any set $A \subset (0, 1) \setminus \mathbb{Q}$, A is nowhere dense in $(0, 1)$ if and only if it is nowhere dense in $(0, 1) \setminus \mathbb{Q}$. This in turn entails that A is meager in $(0, 1)$ if and only if it is meager in $(0, 1) \setminus \mathbb{Q}$.

Proposition 4. *The set* $\{\alpha \in (0, 1) \setminus \mathbb{Q} \mid \exists \jmath\, (j_\alpha(n) = \jmath \text{ holds for only finitely many } n)\}$ *is meager.*

Proof. It can readily be checked that the set $Y_M := \{\alpha \in (0, 1) \setminus \mathbb{Q} \mid \forall i\ (a_i \le M)\}$ is closed and has no open subset. (In order to verify this assertion, it may be easier to consider not Y_M itself but its homeomorphic image in $\mathbb{Z}_{>0}^\infty$ under the homeomorphism $(0, 1) \setminus \mathbb{Q} \ni [0; a_1, a_2, \dots] \mapsto \langle a_1, a_2, \dots\rangle \in \mathbb{Z}_{>0}^\infty$.) So, Y_M is nowhere dense by definition. Being the union of countably many nowhere dense sets, the set $\bigcup_{M=1}^\infty Y_M$, which is equal to $\mathbb{B}$ $(= \{\alpha \in (0, 1) \setminus \mathbb{Q} \mid \text{The elements of } \alpha \text{ are bounded}\})$, is meager.

Although we have deduced Corollary 3 from Theorems 4 and 5 in this paper, there is another way to prove it: In [7, Theorem 12], it was proved that the set under consideration is a subset of $\mathbb{B}$. Combined with the above observation that $\mathbb{B}$ is meager, this fact establishes our claim. $\square$

In Theorem 8, we saw another example of a measure zero set. Is that set also "small" in terms of Baire category? Here is the answer:

Proposition 5. *The set* $\{\alpha \in (0, 1) \setminus \mathbb{Q} \mid \exists p\ \exists m\ \forall n\, j_\alpha(m + np) = j_\alpha(m + (n + 1)p)\}$ *is meager.*

Proof. As in the proof of Theorem 8, we note that it is sufficient to prove that the set $X_p := \{\alpha \in (0, 1) \setminus \mathbb{Q} \mid \exists m\ \forall n\, j_\alpha(m + np) = j_\alpha(m + (n + 1)p)\}$ is meager for each p.

If p is not a multiple of 6, then it is proved in the second paragraph of the proof of Theorem 8 that X_p is a subset of a meager set $\mathbb{B}$. So we are done in this case.

Then assume that p is a multiple of 6. If only finitely many i satisfy the condition $a_i, a_{i+1} \ge 2p$, then there exists an M such that no $i \in \mathbb{Z}_{>0}$ satisfies $a_i, a_{i+1} > M$, verifying the following inclusion:

$$\{\alpha \in (0,1)\setminus\mathbb{Q} \mid a_i, a_{i+1} \ge 2p \text{ for only finitely many } i\} \subset \bigcup_{M=1}^{\infty} \{\alpha \in (0,1)\setminus\mathbb{Q} \mid \nexists i\, (a_i, a_{i+1} > M)\}.$$

Since the set $\{\alpha \in (0, 1) \setminus \mathbb{Q} \mid \nexists i\, (a_i, a_{i+1} > M)\}$ is clearly closed and without any open subset, it follows at once that this set, and thus the union $\bigcup_{M=1}^\infty \{\alpha \in (0, 1)\setminus\mathbb{Q} \mid \nexists i\, (a_i, a_{i+1} >$

$M)\}$ taken over a countable set $\mathbb{Z}_{>0}$, is meager. Together with the above inclusion, this proves that the set $\{\alpha \in (0,1)\setminus\mathbb{Q} \mid a_i, a_{i+1} \geq 2p$ for only finitely many $i\}$ is meager. Hence the proof is complete by noting the fact that an easy modification of the proof of Theorem 8 yields that the set X_p is contained in a meager set $\{\alpha \in (0,1)\setminus\mathbb{Q} \mid a_i, a_{i+1} \geq 2p$ for only finitely many $i\}$. $\square$

Needless to say, for any $L > 1$ and $\iota \in \{1,3,5\}$ (resp. $\iota' \in \{2,4,6\}$), the set of all $\alpha \in (0,1)\setminus\mathbb{Q}$ whose Ducci matrix sequence expansion contains eventually periodic occurrence of the block $\langle \iota, \iota, \ldots, \iota\rangle$ (resp. $\langle \iota', \iota'+2, \ldots, \iota'+2(L-1)\rangle \pmod 6$) of length L is meager. One can also prove that for any $L > 1$ and $\iota \in \{1,3,5\}$ (resp. $\iota' \in \{2,4,6\}$), the set of all $\alpha \in (0,1)\setminus\mathbb{Q}$ whose Ducci matrix sequence expansion contains eventually syndetic occurrence of the block $\langle \iota, \iota, \ldots, \iota\rangle$ (resp. $\langle \iota', \iota'+2, \ldots, \iota'+2(L-1)\rangle \pmod 6$) of length L is meager.

Several "large" sets in terms of Lebesgue measure are large in terms also of Baire category:

Proposition 6. *All of the following sets are comeager:*

(i) $\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \forall l\, \exists^\infty i\; j_\alpha(i)+l \equiv \cdots \equiv j_\alpha(i+l-1)+1 \equiv j_\alpha(i+l) \pmod 6\}$;

(ii) $\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \forall l\, \exists^\infty i\; j_\alpha(i) = \cdots = j_\alpha(i+l-1) = j_\alpha(i+l) = \iota\}$ *for* $\iota = 1,3,5$;

(iii) $\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \forall l\, \exists^\infty i\; j_\alpha(i)+2l \equiv \cdots \equiv j_\alpha(i+l-1)+2 \equiv j_\alpha(i+l) \equiv \iota \pmod 6\}$ *for* $\iota = 2,4,6$;

(iv) $\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \forall l, L \geq 1 \exists p, m \geq 1 \forall n \in \{1,\ldots,l\}\, \langle j_\alpha(m+np), \ldots, j_\alpha(L+m+np)\rangle = \langle j_\alpha(m+(n+1)p), \ldots, j_\alpha(L+m+(n+1)p)\rangle\}$;

(v) $\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \forall \iota \in \{1,3,5\}, l \geq 1, L \geq 1 \exists p, m \geq 1 \forall n \in \{1,\ldots,l\} \langle j_\alpha(m+np), j_\alpha(m+1+np), \ldots, j_\alpha(m+L+np)\rangle = \langle \iota, \iota, \ldots, \iota\rangle\}$;

(vi) $\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \forall \iota \in \{2,4,6\}, l \geq 1, L \geq 1 \exists p, m \geq 1 \forall n \in \{1,\ldots,l\} \langle j_\alpha(m+np), j_\alpha(m+1+np), \ldots, j_\alpha(m+L+np)\rangle \equiv \langle \iota, \iota+2, \iota+4, \ldots, \iota+2L\rangle \pmod 6\}$.

Here and subsequently, the symbol $\exists^\infty i$ (resp. $\exists^{\leq N} i$) is to be read as "there exist infinitely many (resp. at most N) coordinates i such that".

Proof. (i): If there exists an $l \geq 1$ such that only finitely many n satisfy $j_\alpha(n)+l \equiv \cdots \equiv j_\alpha(n+l-1)+1 \equiv j_\alpha(n+l) \pmod 6$ then, it follows from Corollary 1 that all but finitely many elements a_i are bounded by l. In other words, the set under consideration is a superset of $\mathbb{B}^c$. Since the proof of Proposition 4 shows that the set $\mathbb{B}$ is meager, the set $\mathbb{B}^c$ is comeager; so also are its supersets.

(iv): It is clear from the proof of Proposition 2 that the set contains $\mathbb{B}^c$. So the assertion is again a consequence of the fact that $\mathbb{B}$ is a meager set.

(ii), (iii), (v) & (vi): In Theorem 6 and Proposition 3, it was proved that each of these sets contains the intersection of countably many sets of the form $\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \exists^\infty i\, (\langle a_i, a_{i+1}, \ldots, a_{i+l}\rangle = \langle m_1, \ldots, m_{l+1}\rangle)\}$ for some appropriate $l \geq 1$ and $m_1, \ldots, m_{l+1} \in \mathbb{Z}_{>0}$. Clearly, this set is equal to $(\bigcup_N W_N)^c$, where $W_N := \{\alpha \in (0,1)\setminus\mathbb{Q} \mid \exists^{\leq N} i\, (\langle a_i, a_{i+1}, \ldots, a_{i+l}\rangle = \langle m_1, \ldots, m_{l+1}\rangle)\}$. Since for each N, the set W_N is easily checked to be closed and without open subset, it is obvious that W_N, and so the union $\bigcup_N W_N$ taken over a countable set $\mathbb{Z}_{>0}$, is meager. Since the intersection of countably many comeager sets is still comeager, the sets under consideration are all comeager. $\square$

6 Other combinatorial/dynamical properties of Ducci matrix sequences. As an answer to a question posed by Hogenson et al. [8], the author gave a characterization of those irrational numbers $\alpha \in (0,1) \setminus \mathbb{Q}$ whose Ducci matrix sequence expansion is eventually periodic [6, Theorem 7.1]: An irrational number $\alpha > 0$ is quadratic if and only if its Ducci matrix sequence expansion is eventually periodic. This result was obtained by combining the following theorem with Legendre's theorem, which states that an irrational number α admits an eventually periodic continued fraction expansion if and only if it is quadratic:

Theorem 9 ([6])**.** *For a positive irrational number $\alpha > 0$, its Ducci matrix sequence expansion is eventually periodic $\iff$ its continued fraction expansion is eventually periodic.* □

Inspired by this result, we shall examine the validity of analogous statements for several other combinatorial/dynamical properties. The reader will not get confused by the phrase "analogous statements for other *combinatorial* properties", but a word is in order about what we mean by *dynamical* properties. When we say dynamical properties of a Ducci matrix sequence, we are viewing the sequence as a point of the shift dynamical system over $\{1, 2, \ldots, 6\}$. Likewise, when we say dynamical properties of a continued fraction $[0; a_1, a_2, \ldots]$, we are regarding it as a point $\langle a_1, a_2, \ldots \rangle$ of the shift dynamical system over $\mathbb{Z}_{>0}$. (Recall that $(0,1) \setminus \mathbb{Q}$ and $\mathbb{Z}_{>0}^{\infty}$ are homeomorphic under the mapping $(0,1) \setminus \mathbb{Q} \ni [0; a_1, a_2, \ldots] \mapsto \langle a_1, a_2, \ldots \rangle \in \mathbb{Z}_{>0}^{\infty}$.)

Topologically, there is a clear distinction between $\mathbb{Z}_{>0}^{\infty}$ and $\{1, 2, \ldots, 6\}^{\infty}$ in terms of compactness. Dynamically, the behavior of shift maps on respective space is different because the composition of j with the shift map on $\mathbb{Z}_{>0}^{\infty}$ is not equal to the composition of the shift map on $\{1, 2, \ldots, 6\}^{\infty}$ with j. Given these differences, the type of equivalence that we are concerned with does not seem trivial.

6.1 Abelian periodicity. In this subsection, we shall provide examples of irrational numbers $\alpha \in (0,1) \setminus \mathbb{Q}$ witnessing that the eventual abelian periodicity of elements of continued fraction expansion and of Ducci matrix sequence expansion are independent.

Say that a sequence $\langle x_1, x_2, x_3, \ldots \rangle$ of elements x_i of an alphabet set Σ is *abelian periodic* if there exist countably many sequences $b_1, b_2, b_3, \ldots \in \Sigma^+$ such that

- $\langle x_1, x_2, x_3, \ldots \rangle = b_1{}^\frown b_2{}^\frown b_3{}^\frown \cdots;$
- b_i and b_j are *abelian equivalent*, i.e., for each symbol $x \in \Sigma$, the number of occurrences of x in b_i and in b_j are the same.

It necessarily follows that the length of b_i is always the same, which we call the *period* of the abelian periodic sequence $\langle x_1, x_2, x_3, \ldots \rangle$. A sequence $\langle x_1, x_2, x_3, \ldots \rangle$ is *eventually abelian periodic* if there exists an $n \geq 1$ such that $\langle x_n, x_{n+1}, x_{n+2}, \ldots \rangle$ is abelian periodic.

Example 1. *Define an irrational number $\alpha = [0; a_1, a_2, a_3, \ldots] \in (0,1) \setminus \mathbb{Q}$ by describing its elements $a_1, a_2, \ldots$ as follows:*

$$\begin{aligned} \langle a_1, a_2, a_3, \ldots \rangle &:= \langle 1, 6 \rangle {}^\frown \mathsf{a}_1{}^\frown \mathsf{a}_2{}^\frown \mathsf{a}_1^2{}^\frown \mathsf{a}_2^2{}^\frown \mathsf{a}_1^3{}^\frown \mathsf{a}_2^3{}^\frown \cdots {}^\frown \mathsf{a}_1^n{}^\frown \mathsf{a}_2^n{}^\frown \cdots, \quad \textit{where} \\ \mathsf{a}_1 &= \langle 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 5 \rangle \quad \textit{and} \\ \mathsf{a}_2 &= \langle 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 5 \rangle. \end{aligned}$$

Since a_1 and a_2 are clearly abelian equivalent, the continued fraction expansion of α is eventually abelian periodic with period 11.

An easy computation shows that the Ducci matrix sequence expansion $j(\alpha)$ of this α is

$$\begin{aligned} j(\alpha) &= \langle 1,2,3,4,5,6,1\rangle ^\frown \mathrm{j}_1{}^\frown \mathrm{j}_2{}^\frown \mathrm{j}_1^2{}^\frown \mathrm{j}_2^2{}^\frown \cdots {}^\frown \mathrm{j}_1^n{}^\frown \mathrm{j}_2^n{}^\frown \cdots, \quad \text{where} \\ \mathrm{j}_1 &= \langle 1,1,1,1,1,1,1,1,1,1,1,2,3,3,4,5,6,1\rangle \quad \text{and} \\ \mathrm{j}_2 &= \langle 1,2,3,3,3,3,3,3,3,3,3,3,3,4,5,6,1\rangle. \end{aligned}$$

We claim that this sequence $j(\alpha)$ is not eventually abelian periodic. To see this, we shall show that for any $m, p \geq 1$, if $\langle j_\alpha(m), j_\alpha(m+1), j_\alpha(m+2), \ldots\rangle = b_1{}^\frown b_2{}^\frown b_3{}^\frown \cdots$ with $\mathrm{lh}(b_i) = p$, then there exist two blocks b_m and $b_{m'}$ ($m \neq m'$) which are not abelian equivalent each other.

If p is less than 17, then, since the length 17 of j_1 and j_2 is prime, for every sufficiently large n, there exists a block b_m (resp. $b_{m'}$) in j_1^n (resp. j_2^n) which starts with $\min\{11, p\}$ consecutive 1's (resp. 3's). Since b_m (resp. $b_{m'}$) cannot contain $\min\{11, p\}$ consecutive 3's (resp. 1's), this prevents b_m and $b_{m'}$ from being abelian equivalent each other.

Next consider the case where $p \geq 17$. Take n and n' sufficiently large so that there exist blocks b_m and $b_{m'}$ which is contained in j_1^n and $\mathrm{j}_2^{n'}$, respectively. Express p as $p = 17i + i'$ for $i \geq 1$ and $0 \leq i' < 17$. Then the number of occurrences of 1 in b_m and in $b_{m'}$ is at least $11i$ and at most $2(i+1)$, respectively. Since $i \geq 1$, this indicates that b_m and $b_{m'}$ are not abelian equivalent each other.

Example 2. *Let*

$$\alpha := [0; 6,4,3,2,2,20, \underline{5,4,3,2,2}, 38, \underline{5,4,3,2,2}, 56, \ldots, \underline{5,4,3,2,2}, 20+18n, \underline{5,4,3,2,2}, \ldots].$$

Since the elements of α are unbounded, the continued fraction expansion of this α is not eventually abelian periodic. However, the Ducci matrix sequence expansion is eventually abelian periodic. Indeed, an easy computation shows that

$$\begin{aligned} j(\alpha) &= b_1{}^\frown b_2{}^\frown b_1{}^\frown b_2^2{}^\frown b_1{}^\frown b_2^3{}^\frown \cdots {}^\frown b_1{}^\frown b_2^n{}^\frown b_1{}^\frown \cdots, \quad \text{where} \\ b_1 &= \langle 1,1,2,3,4,5,5,6,1,2,4,5,6,2,3,3,4,6\rangle \\ b_2 &= \langle 1,2,3,4,5,6,1,2,3,4,5,6,1,2,3,4,5,6\rangle. \end{aligned}$$

In both b_1 and b_2, every $\jmath \in \{1,2,\ldots,6\}$ occurs exactly three times. Hence $j(\alpha)$ is abelian periodic with period 18.

By changing the above definition of α to

$$\begin{aligned} &[0; 6,4,3,2,2+6p, 20+6p, \underline{5,4,3,2,2+6p}, 38+6p, \underline{5,4,3,2,2+6p}, 56+6p, \ldots \\ &\qquad \underline{5,4,3,2,2+6p}, 20+18n+6p, \underline{5,4,3,2,2+6p}, \ldots], \end{aligned}$$

one obtains an example of a non-abelian periodic continued fraction whose Ducci matrix sequence expansion is abelian periodic with period $18 + 6p$.

Now we ask a few questions:

Question 1. *Characterize those $\alpha \in (0,1) \setminus \mathbb{Q}$ whose Ducci matrix sequence expansion is (eventually) abelian periodic.*

Question 2. *Is the set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ having (eventually) abelian periodic Ducci matrix sequence expansion of measure zero?*

Clearly, the intersection of that set with the measure zero set $\mathbb{B}$ is of measure zero. So the above question is equivalent to asking whether or not the set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ whose Ducci matrix sequence expansion is (eventually) abelian periodic and whose continued fraction expansion has unbounded elements is of measure zero. It is not so difficult to observe that the period of eventually abelian periodic Ducci matrix sequence expansion of such an $\alpha \in (0,1) \setminus \mathbb{Q}$ is a multiple of 6. As has already been observed, there actually exists such a $j(\alpha)$ with period $18+6p$ for each $p \geq 0$.

For any sequence $\langle a_1, a_2, \ldots, a_n \rangle \in \mathbb{Z}^+_{>0}$ and periodic sequence $\langle c_1, c_2, c_3, \ldots \rangle \in \mathbb{Z}^\infty_{>0}$, the continued fraction $[0; a_1, \ldots, a_n, c_1, c_2, c_3, \ldots]$ is evidently eventually periodic. Since it is clear that every open subset of $(0,1) \setminus \mathbb{Q}$ contains an element of this form, this proves that the set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ with eventually periodic continued fraction expansion is dense. Combined with Theorem 9, this observation entails that the set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ with eventually periodic Ducci matrix sequence expansion, and hence eventually abelian periodic Ducci matrix sequence expansion, is dense. Unfortunately though, this observation does not help us solving Question 2, because, while being of full measure entails denseness, denseness does not imply having non-zero measure (as witnessed by any countable dense subset of $(0,1) \setminus \mathbb{Q}$).

6.2 Almost periodicity. A point in a dynamical system is said to be *almost periodic* (aka *syndetically recurrent*, *uniformly recurrent*) if for every its neighborhood, the set of return times is syndetic. Observe that over acting (semi)groups $\mathbb{Z}_{\geq 0}, \mathbb{R}_{\geq 0}, \mathbb{Z}$ and $\mathbb{R}$, being syndetic, *relatively dense* and having bounded gaps are all equivalent.

It is then clear that a point $j(\alpha) = \langle j_\alpha(1), j_\alpha(2), \ldots \rangle$ is almost periodic if and only if there exists a function $l : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for every $n \geq 1$, each block $\langle j_\alpha(m+1), j_\alpha(m+2), \ldots, j_\alpha(m+l(n)+n-1) \rangle$ $(m \geq 1)$ of length $l(n)+n-1$ contains the sequence $\langle j_\alpha(1), j_\alpha(2), \ldots, j_\alpha(n) \rangle$. We say that $j(\alpha) = \langle j_\alpha(1), j_\alpha(2), \ldots \rangle$ is *eventually almost periodic* if $\langle j_\alpha(n), j_\alpha(n+1), \ldots \rangle$ is almost periodic for some $n \geq 1$. Note that if $\langle j_\alpha(n), j_\alpha(n+1), \ldots \rangle$ is almost periodic, then so is $\langle j_\alpha(n+m), j_\alpha(n+m+1), \ldots \rangle$ for every $m \geq 0$.

It is convenient here to prepare a

Lemma 3. *Let* $\alpha = [0; a_1, a_2, \ldots]$ *and* $\alpha' = [0; a'_1, a'_2, \ldots]$ *be irrational numbers. Suppose we have*

$$\begin{aligned}(1)\quad &\langle j_\alpha(1+\sum_{i=1}^{m} a_i), j_\alpha(2+\sum_{i=1}^{m} a_i), \ldots, j_\alpha(1+\sum_{i=1}^{m+M} a_i)\rangle \\ &\qquad = \langle j_{\alpha'}(1+\sum_{i=1}^{k} a'_i), j_{\alpha'}(2+\sum_{i=1}^{k} a'_i), \ldots, j_{\alpha'}(1+\sum_{i=m+1}^{m+M} a_i + \sum_{i=1}^{k} a'_i)\rangle\end{aligned}$$

for some $m, M, k \geq 1$*. Then* $a_{m+l} = a'_{k+l}$ *holds for* $l = 1, 2, \ldots, M$*. In other words, we have* $\langle a_{m+1}, \ldots, a_{m+M} \rangle = \langle a'_{k+1}, \ldots, a'_{k+M} \rangle$*.*

Proof. The least $\iota \geq 1$ satisfying $j_\alpha(\iota + \sum_{i=1}^m a_i) + 1 \not\equiv j_\alpha(1 + \iota + \sum_{i=1}^m a_i) \pmod 6$ (resp. $j_{\alpha'}(\iota + \sum_{i=1}^k a'_i) + 1 \not\equiv j_{\alpha'}(1 + \iota + \sum_{i=1}^k a'_i) \pmod 6$) is a_{m+1} (resp. a'_{k+1}). By equation (1), we get $a_{m+1} = a'_{k+1}$. The second smallest $\iota \geq 1$ fulfilling $j_\alpha(\iota + \sum_{i=1}^m a_i) + 1 \not\equiv j_\alpha(1 + \iota + \sum_{i=1}^m a_i) \pmod 6$ (resp. $j_{\alpha'}(\iota + \sum_{i=1}^k a'_i) + 1 \not\equiv j_{\alpha'}(1 + \iota + \sum_{i=1}^k a'_i) \pmod 6$) is $a_{m+1} + a_{m+2}$ (resp. $a'_{k+1} + a'_{k+2}$). Consequently, $a_{m+2} = a'_{k+2}$ by equation (1) and $a_{m+1} = a'_{k+1}$. We continue in this fashion to obtain $a_{m+l} = a'_{k+l}$ for $l = 1, 2, \ldots, M$. □

Proposition 7. *If the Ducci matrix sequence expansion of an* $\alpha \in (0,1) \setminus \mathbb{Q}$ *is eventually almost periodic, then so is its continued fraction expansion.*

Proof. By assumption, there exists an $N' \geq 1$ such that the sequence $\langle j_\alpha(N'+1), j_\alpha(N'+2), \ldots\rangle$ is almost periodic. In view of the observation made earlier, we may assume without loss of generality that $N' = \sum_{i=1}^{n_0-1} a_i$ for some sufficiently large $n_0 > 2$. Select a function $l' : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ so that for every $n' \geq 1$, each block $\langle j_\alpha(m'+1), j_\alpha(m'+2), \ldots, j_\alpha(m'+l'(n')+n'-1)\rangle$ $(m' \geq N')$ contains the sequence $\langle j_\alpha(N'+1), \ldots, j_\alpha(N'+n')\rangle$.

Fix an $n \geq 1$. Then the sequence $\langle j_\alpha(N'+1), \ldots, j_\alpha(1+\sum_{i=1}^{n_0+n} a_i)\ (= j_\alpha(N'+1+\sum_{i=n_0}^{n_0+n} a_i))\rangle$ is contained in $\langle j_\alpha(1+\sum_{i=1}^{m} a_i), j_\alpha(2+\sum_{i=1}^{m} a_i), \ldots, j_\alpha(\sum_{i=1}^{m} a_i + l'(1+\sum_{i=n_0}^{n_0+n} a_i) + \sum_{i=n_0}^{n_0+n} a_i)\rangle$ for each $m \geq n_0$. Therefore, for each $m \geq n_0$, one can find an integer $M(m)$ from the set $\{\sum_{i=1}^{m} a_i, 1+\sum_{i=1}^{m} a_i, \ldots, \sum_{i=1}^{m} a_i + l'(1+\sum_{i=n_0}^{n_0+n} a_i) - 1\}$ fulfilling that

$$j_\alpha(N'+\iota) = j_\alpha(M(m)+\iota) \quad \text{for } \iota = 1, 2, \ldots, 1+\sum_{i=n_0}^{n_0+n} a_i.$$

By virtue of these equalities, one can deduce the relation $j_\alpha(M(m)+a_{n_0})+1 \not\equiv j_\alpha(M(m)+a_{n_0}+1) \pmod 6$ from $j_\alpha(N'+a_{n_0})+1 \not\equiv j_\alpha(N'+a_{n_0}+1) \pmod 6$. In view of Corollary 1, one sees that $M(m)+a_{n_0}$ may be written as $\sum_{i=1}^{k} a_i$ for some k. Now we can apply Lemma 3 to get $\langle a_{n_0+1}, a_{n_0+2}, \ldots, a_{n_0+n}\rangle = \langle a_{k+1}, a_{k+2}, \ldots, a_{k+n}\rangle$.

From $\sum_{i=1}^{m} a_i + a_{n_0} \leq M(m)+a_{n_0} = \sum_{i=1}^{k} a_i$, it follows that $m+1 \leq k$. Also from $\sum_{i=1}^{k} a_i = M(m)+a_{n_0} \leq \sum_{i=1}^{m} a_i + l'(1+\sum_{i=n_0}^{n_0+n} a_i) + a_{n_0} - 1$, it follows that $k-m \leq \sum_{i=m+1}^{k} a_i \leq l'(1+\sum_{i=n_0}^{n_0+n} a_i)+a_{n_0}-1$ and thus $k+n \leq m+l'(1+\sum_{i=n_0}^{n_0+n} a_i)+a_{n_0}+n-1$. Since $\langle a_{n_0+1}, a_{n_0+2}, \ldots, a_{n_0+n}\rangle$ and $\langle a_{k+1}, a_{k+2}, \ldots, a_{k+n}\rangle$ are identical, this argument proves that the sequence $\langle a_{n_0+1}, a_{n_0+2}, \ldots, a_{n_0+n}\rangle$ is contained in the sequence $\langle a_{m+1}, a_{m+2}, \ldots, a_{m+l'(1+\sum_{i=n_0}^{n_0+n} a_i)+a_{n_0}+n-1}\rangle$. Since both $n \geq 1$ and $m \geq n_0$ were chosen arbitrarily, the function $l : \mathbb{Z}_{>0} \ni n \mapsto l'(1+\sum_{i=n_0}^{n_0+n} a_i)+a_{n_0} \in \mathbb{Z}_{>0}$ witnesses the almost periodicity of $\langle a_{n_0+1}, a_{n_0+2}, \ldots\rangle$. $\square$

The converse of this proposition is not true in general. Here is a witness:

Example 3. (*In this example, we shall identify a sequence* $\langle x_1, \ldots, x_n\rangle$ $(x_i \geq 5)$ *with a rational number* $[0; x_1, \ldots, x_n]$.) *Let a sequence* $\{\mathsf{a}_n\}$ *of rational numbers be given by*

$$\begin{aligned} &\mathsf{a}_1 := \langle 5\rangle \\ &\mathsf{a}_{n+1} := \mathsf{a}_n{}^\frown \langle \mathsf{a}_{n,1}, \mathsf{a}_{n,2}, \ldots, \mathsf{a}_{n,2^{n-1}-1}, 5^{\mathsf{a}_{n,2^{n-1}}}\rangle, \quad \textit{where } \mathsf{a}_n = \langle \mathsf{a}_{n,1}, \mathsf{a}_{n,2}, \ldots, \mathsf{a}_{n,2^{n-1}}\rangle. \end{aligned}$$

Then, define an irrational number $\alpha = [0; a_1, a_2, a_3, \ldots] \in (0,1) \setminus \mathbb{Q}$ *as their limit:*

$$\alpha := \lim_{n\to\infty} \mathsf{a}_n \ (= [0; 5, 5^5, 5, 5^{5^5}, 5, 5^5, 5, 5^{5^{5^5}}, \ldots]).$$

Let us first verify the almost periodicity of the continued fraction expansion of this α. *To this end, we need to show that for every* N, *there exists an* $l(N) \in \mathbb{Z}_{>0}$ *such that the block* $\langle a_1, a_2, \ldots, a_N\rangle$ *appears in the sequence* $\langle a_{m+1}, a_{m+2}, \ldots, a_{m+l(N)+N}\rangle$ *for each* m.

For simplicity, set $M := \lceil 2 + \log_2 N\rceil$. *Then the block* $\langle a_1, a_2, \ldots, a_N\rangle$ *appears as an initial segment of the first half of* a_M. *For arbitrarily given* $m \geq 0$, *pick a* $k > 1$ *sufficiently large so that* a_{M+k} *contains the sequence* $\langle a_{m+1}, a_{m+2}, \ldots, a_{m+2^{M-1}+N-1}\rangle$. *Clearly, at least one of* $m+1, m+2, \ldots, m+2^{M-1}$, *say* $m+\iota$, *can be written as* $1 + i \cdot 2^{M-1}$ *for some* $i \in \{0, 1, \ldots, 2^k - 1\}$. *As an easy induction on* k *shows that every subsequence of* a_{M+k} *of length* N *starting at position* $1 + i \cdot 2^{M-1}$ $(i = 0, 1, \ldots, 2^k - 1)$ *is identical with* $\langle a_1, \ldots, a_N\rangle$, *this proves that the sequence* $\langle a_{m+1}, a_{m+2}, \ldots, a_{m+2^{M-1}+N-1}\rangle$ *contains the* *sequence* $\langle a_1, \ldots, a_N\rangle$ $(= \langle a_{m+\iota}, \ldots, a_{m+\iota+N-1}\rangle)$. *By putting* $l(N) := 2^{M-1}$, *we thus see that our first claim is valid.*

The Ducci matrix sequence expansion of this α *is not eventually almost periodic: Fix an* M *and consider* $\langle j_\alpha(M+1), j_\alpha(M+2), j_\alpha(M+3), \ldots\rangle$. *Let* $k > 0$ *be the least integer satisfying* $M + 1 \le \sum_{i=1}^{k} a_i$. *In the sequence* $\langle j_\alpha(M+1), j_\alpha(M+2), \ldots, j_\alpha(1 + \sum_{i=1}^{k+1} a_i)\rangle$, *the relation* $j_\alpha(n) + 1 \equiv j_\alpha(n+1) \pmod 6$ *is not always fulfilled, e.g.,* $j_\alpha(\sum_{i=1}^{k+1} a_i) + 1 \not\equiv j_\alpha(1 + \sum_{i=1}^{k+1} a_i) \pmod 6$. *On the other hand, if* l' *is such that* $a_{l'} > 1 + \sum_{i=1}^{k+1} a_i$ *then the relation* $j_\alpha(n) + 1 \equiv j_\alpha(n+1) \pmod 6$ *is always fulfilled in the sequence* $\langle j_\alpha(1 + \sum_{i=1}^{l'-1} a_i), j_\alpha(2 + \sum_{i=1}^{l'-1} a_i), \ldots, j_\alpha(\sum_{i=1}^{l'} a_i)\rangle$ *of length* $a_{l'}$. *From these observations, we conclude that if such an* l' *exists, then there exists a sequence of length* $a_{l'}$ *which does not contain* $\langle j_\alpha(M+1), j_\alpha(M+2), \ldots, j_\alpha(1 + \sum_{i=1}^{k+1} a_i)\rangle$ *as a subsequence. Since it is evident from the construction that continued fraction expansion of this* α *contains an arbitrary large element* $5, 5^5, 5^{5^5}, 5^{5^{5^5}}, \ldots$, *one can find an arbitrary long sequences without containing* $\langle j_\alpha(M+1), j_\alpha(M+2), \ldots, j_\alpha(1+\sum_{i=1}^{k+1} a_i)\rangle$, *proving that the Ducci matrix sequence expansion of this* α *is not eventually almost periodic.*

Suppose that a point $j(\alpha)$ is eventually almost periodic. Take an N sufficiently large so that $\langle j_\alpha(1 + \sum_{i=1}^{N-1} a_i), j_\alpha(2 + \sum_{i=1}^{N-1} a_i), \ldots\rangle$ is almost periodic. Then it follows that for a sequence $\langle j_\alpha(1 + \sum_{i=1}^{N-1} a_i), j_\alpha(2 + \sum_{i=1}^{N-1} a_i), \ldots, j_\alpha(1 + \sum_{i=1}^{N} a_i)\rangle$ of length $1 + a_N$, there exists an l such that every sequence $\langle j_\alpha(m+1), j_\alpha(m+2), \ldots, j_\alpha(m+l+a_N)\rangle$ $(m \ge \sum_{i=1}^{N-1} a_i)$ contains it. Since we have $j_\alpha(\sum_{i=1}^{N} a_i) + 1 \not\equiv j_\alpha(1 + \sum_{i=1}^{N} a_i) \pmod 6$, we see that every interval $\{m+1, m+2, \ldots, m+l+a_N\}$ $(m \ge \sum_{i=1}^{N-1} a_i)$ contains an i which satisfies the condition $j_\alpha(i) + 1 \not\equiv j_\alpha(i+1) \pmod 6$. By virtue of Corollary 1, we conclude that the elements $a_N, a_{N+1}, \ldots$, and consequently $a_1, a_2, a_3, \ldots$, are bounded. Therefore, any $\alpha \in (0,1) \setminus \mathbb{Q}$ with eventually almost periodic Ducci matrix sequence expansion is in a measure zero set $\mathbb{B}$. We have thus proved a

Proposition 8. *The set of all* $\alpha \in (0,1) \setminus \mathbb{Q}$ *with eventually almost periodic Ducci matrix sequence expansion is of measure zero.* □

Question 1 for (eventual) almost periodicity is of interest. Nevertheless, as in the case of eventual abelian periodicity, the set under consideration is dense in $(0,1) \setminus \mathbb{Q}$.

6.3 Positive Poisson stability. A point in a dynamical system is called *positively Poisson stable* if its orbit intersects its ω-limit set. (More information on Poisson stability in topological dynamics is available from, e.g., [11].) In the setting of the shift dynamical system over an alphabet set, we can rephrase this concept as follows: A point $\langle x_1, x_2, \ldots\rangle$ is positively Poisson stable if and only if there exists an $M \ge 1$ such that for every (resp. for some) $m \ge M$, the point $\langle x_{m+1}, x_{m+2}, \ldots\rangle$ belongs to its own ω-limit set, which can further be paraphrased as: For each (resp. for some) $m \ge M$, there exists a strictly increasing function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $\langle x_{m+1}, x_{m+2}, \ldots, x_{m+n}\rangle = \langle x_{m+f(n)}, x_{m+f(n)+1}, \ldots, x_{m+f(n)+n-1}\rangle$ holds for every n.

Before proving the main result of this subsection, we state the following consequence of Proposition 1, which is easily proved by induction (So its proof is omitted here):

Proposition 9. *Suppose we have for* $n > 1$ *and* $m, N \ge 1$ *that* $\langle a_n, a_{n+1}, \ldots, a_{n+m}\rangle = \langle a_{n+N}, a_{n+N+1}, \ldots, a_{n+N+m}\rangle$. *Set* $\iota := j_\alpha(1 + \sum_{i=1}^{n+N-1} a_i) - j_\alpha(1 + \sum_{i=1}^{n-1} a_i) \pmod 6$.

(i) If ι *is even, then we have, for* $k = 1 + \sum_{i=1}^{n-1} a_i, 2 + \sum_{i=1}^{n-1} a_i, \ldots, 1 + \sum_{i=1}^{n+m} a_i$, *that*

$$j_\alpha(k) + \iota \equiv j_\alpha(k + \sum_{i=n}^{n+N-1} a_i) \pmod 6.$$

(ii) If ι is odd, then we have, for $k = 1 + \sum_{i=1}^{n-1} a_i, 2 + \sum_{i=1}^{n-1} a_i, \ldots, 1 + \sum_{i=1}^{n+m} a_i$, that

$$j_\alpha(k) \not\equiv j_\alpha(k + \sum_{i=n}^{n+N-1} a_i) \pmod 2,$$

i.e., their parity is always different. □

Here is the main result of this subsection:

Theorem 10. *For a positive irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$, its Ducci matrix sequence expansion is positively Poisson stable $\Longleftrightarrow$ its continued fraction expansion is positively Poisson stable.*

Proof. ($\Longleftarrow$) : By assumption, for each sufficiently large $N > 1$, there exists a strictly increasing function $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $\langle a_{N+1}, \ldots, a_{N+n} \rangle = \langle a_{N+g(n)}, \ldots, a_{N+g(n)+n-1} \rangle$ holds for every $n \geq 1$. Fix such an $N > 1$. We shall show that for $M := \sum_{i=1}^{N} a_i > 1$, there exists a strictly increasing function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ satisfying $\langle j_\alpha(M+1), \ldots, j_\alpha(M+m) \rangle = \langle j_\alpha(M + f(m)), \ldots, j_\alpha(M + f(m) + m - 1) \rangle$ for every $m \geq 1$. To construct such an f, it is sufficient to prove that for every $m, l \geq 1$, there exists an $e > l$ such that $\langle j_\alpha(M+1), \ldots, j_\alpha(M+m) \rangle = \langle j_\alpha(M+e), \ldots, j_\alpha(M+e+m-1) \rangle$ holds.

Now fix $m, l \geq 1$ and let q_0 be the least integer such that $l < 1 + \sum_{i=N+1}^{q_0-1} a_i$ and $N \leq q_0$. For $n > q_0$, define

$$I(n) := \{q \in \mathbb{Z}_{>0} \mid q_0 < q \leq n \text{ and } \langle a_{N+1}, \ldots, a_{N+m} \rangle = \langle a_q, \ldots, a_{q+m-1} \rangle\}, \quad \text{and}$$
$$V(n) := \Big\{ j_\alpha(1 + \sum_{i=1}^{q-1} a_i) \Bigm| q \in I(n) \Big\}.$$

The next lemma plays a key role in this proof:

Lemma 4. *For every $n > q_0$, there exists an $n' > n$ such that either $V(n') \ni j_\alpha(1+\sum_{i=1}^{N} a_i)$ or $V(n) \subsetneqq V(n')$ holds.*

Before proving this lemma, we proceed to see how to find an $e > l$ with the desired property using this lemma:

Since the function g is strictly increasing, there exists an $m' \geq m$ such that $g(m') > \max\{l, q_0\}$. Since m is less than or equal to m', it is evident that $\langle a_{N+1}, \ldots, a_{N+m} \rangle = \langle a_{N+g(m')}, \ldots, a_{N+g(m')+m-1} \rangle$. Set $h_0 := N + g(m')$. (So $V(h_0) \neq \emptyset$.) If $V(h_0)$ does not contain $j_\alpha(1 + \sum_{i=1}^{N} a_i)$, then apply the above lemma to get an h_1 satisfying either $V(h_1) \ni j_\alpha(1 + \sum_{i=1}^{N} a_i)$ or $V(h_0) \subsetneqq V(h_1)$. If $V(h_1)$ does not contain $j_\alpha(1 + \sum_{i=1}^{N} a_i)$, choose an h_2 so that either $V(h_2) \ni j_\alpha(1 + \sum_{i=1}^{N} a_i)$ or $V(h_1) \subsetneqq V(h_2)$ holds. Repeating this argument, since $V(n)$ is a subset of a finite set $\{1, 2, \ldots, 6\}$, we get an h (within at most six iterations) for which $j_\alpha(1 + \sum_{i=1}^{N} a_i) \in V(h)$ holds. This means that there exists a $q > q_0$ such that

$$\langle a_{N+1}, \ldots, a_{N+m} \rangle = \langle a_q, \ldots, a_{q+m-1} \rangle \quad \text{and} \quad j_\alpha(1 + \sum_{i=1}^{N} a_i) = j_\alpha(1 + \sum_{i=1}^{q-1} a_i).$$

Proposition 9 (i) proves that we have $j_\alpha(k) = j_\alpha(k + \sum_{i=N+1}^{q-1} a_i)$ for $k = 1 + \sum_{i=1}^{N} a_i, 2 + \sum_{i=1}^{N} a_i, \ldots, 1 + \sum_{i=1}^{N+m} a_i$. Since $M = \sum_{i=1}^{N} a_i$ (by definition) and $1 + \sum_{i=N+1}^{N+m} a_i > m$, it follows that

$$\langle j_\alpha(M+1), \ldots, j_\alpha(M+m)\rangle = \langle j_\alpha(M+1+\sum_{i=N+1}^{q-1} a_i), \ldots, j_\alpha(M+m+\sum_{i=N+1}^{q-1} a_i)\rangle.$$

Also, since $q > q_0$, we see that $1+\sum_{i=N+1}^{q-1} a_i > l$. Therefore, we can take $e := 1+\sum_{i=N+1}^{q-1} a_i$.

What remains to be done now is to prove the lemma:

Proof of Lemma 4. We first claim that there exists an $h \geq n+m$ which satisfies both

$$j_\alpha(1+\sum_{i=1}^{N} a_i) \equiv j_\alpha(1+\sum_{i=1}^{h-1} a_i) \pmod 2, \text{ i.e., they have the same parity, and} \tag{2}$$

$$\langle a_{N+1}, \ldots, a_{n+m}\rangle = \langle a_h, \ldots, a_{h+n+m-N-1}\rangle. \tag{3}$$

By the definition of g, we have

$$\langle a_{N+1}, \ldots, a_{n+m}\rangle = \langle a_x, \ldots, a_{x+n+m-N-1}\rangle, \tag{4}$$

where $x := N+g(n+m-N)$. Hence if $j_\alpha(1+\sum_{i=1}^{N} a_i)$ and $j_\alpha(1+\sum_{i=1}^{x-1} a_i)$ have the same parity, then, by setting $h := x$, we see that the claim is correct. Otherwise, equation (4) and

$$\langle a_{N+1}, \ldots, a_{n+m}, \ldots, a_x, \ldots, a_{x+n+m-N-1}\rangle = \langle a_{x'}, \ldots, a_{x'+x+n+m-2N-2}\rangle, \tag{5}$$

where $x' := N+g(x+n+m-2N-1)$, imply that

$$\begin{aligned} \langle a_{N+1}, \ldots, a_{n+m}\rangle &= \langle a_{x'}, \ldots, a_{x'+n+m-N-1}\rangle \qquad (6)\\ &= \langle a_{x'+x-N-1}, \ldots, a_{x'+x+n+m-2N-2}\rangle. \qquad (7)\end{aligned}$$

Therefore if $j_\alpha(1+\sum_{i=1}^{N} a_i)$ and $j_\alpha(1+\sum_{i=1}^{x'-1} a_i)$ have the same parity, then one can take x' as h. Suppose otherwise. Then $\iota := j_\alpha(1+\sum_{i=1}^{x'-1} a_i) - j_\alpha(1+\sum_{i=1}^{N} a_i)$ is odd. Also, a trivial verification shows that

$$1+\sum_{i=1}^{x'+x-N-2} a_i = 1+\sum_{i=1}^{x'-1} a_i + \sum_{i=x'}^{x'+x-N-2} a_i \overset{\text{Eq. (5)}}{=\!=} 1+\sum_{i=1}^{x'-1} a_i + \sum_{i=N+1}^{x-1} a_i = 1+\sum_{i=1}^{x-1} a_i + \sum_{i=N+1}^{x'-1} a_i.$$

Now an application of Proposition 9 (ii) to equation (6) tells us that the parity of $j_\alpha(1+\sum_{i=1}^{x-1} a_i)$ and $j_\alpha(1+\sum_{i=1}^{x'+x-N-2} a_i)$ $(= j_\alpha(1+\sum_{i=1}^{x-1} a_i + \sum_{i=N+1}^{x'-1} a_i))$ are different. On the other hand, we have assumed that the parity of $j_\alpha(1+\sum_{i=1}^{x-1} a_i)$ is different from that of $j_\alpha(1+\sum_{i=1}^{N} a_i)$. Therefore the parity of $j_\alpha(1+\sum_{i=1}^{x'+x-N-2} a_i)$ and of $j_\alpha(1+\sum_{i=1}^{N} a_i)$ are the same. This and equation (7) show that $x'+x-N-1$ works as h in this case. This completes the proof of our first claim.

Now, for a given n, take an h as in the claim above. Then it follows from equation (2) that $\iota := j_\alpha(1+\sum_{i=1}^{h-1} a_i) - j_\alpha(1+\sum_{i=1}^{N} a_i)$ is even. We then claim the following three things: $j_\alpha(1+\sum_{i=1}^{h-1} a_i) \in V(h+n+m)$, $V(n) \subset V(h+n+m)$ and $V(n)+\iota := \{j_\alpha(1+\sum_{i=1}^{q-1} a_i)+\iota \mid q \in I(n)\} \subset V(h+n+m)$. First two statements are checked readily. For the third, observe that we have, by Proposition 9 (i), $j_\alpha(k)+\iota \equiv j_\alpha(k+\sum_{i=N+1}^{h-1} a_i) \pmod 6$ for $k = 1+\sum_{i=1}^{N} a_i, 2+\sum_{i=1}^{N} a_i, \ldots, 1+\sum_{i=1}^{n+m} a_i$. Since every $q \in I(n)$ satisfies $1+\sum_{i=1}^{N} a_i \leq 1+\sum_{i=1}^{q_0} a_i \leq 1+\sum_{i=1}^{q-1} a_i \leq 1+\sum_{i=1}^{n} a_i \leq 1+\sum_{i=1}^{n+m} a_i$, we have in particular,

$$j_\alpha(1+\sum_{i=1}^{q-1} a_i)+\iota \equiv j_\alpha(1+\sum_{i=1}^{q-1} a_i + \sum_{i=N+1}^{h-1} a_i) \pmod 6 \tag{8}$$

for any $q \in I(n)$. Also, $q \in I(n)$ implies $h+q-N-1 \in I(h+n+m)$, which is due to equation (3). Having these in mind, we get

$$\begin{aligned} z \in V(n)+\iota &\Longleftrightarrow z \equiv j_\alpha(1+\sum_{i=1}^{q-1} a_i)+\iota \pmod 6 \text{ for some } q \in I(n) \\ &\overset{\text{Eq. (8)}}{\Longleftrightarrow} z = j_\alpha(1+\sum_{i=1}^{q-1} a_i + \sum_{i=N+1}^{h-1} a_i) \text{ for some } q \in I(n) \\ &\overset{\text{Eq. (3)}}{\Longleftrightarrow} z = j_\alpha(1+\sum_{i=1}^{h+q-N-2} a_i) \text{ for some } q \in I(n) \\ &\Longrightarrow z = j_\alpha(1+\sum_{i=1}^{q'-1} a_i) \text{ for some } q' \in I(h+n+m). \end{aligned}$$

This proves our second claim.

Our final claim is that $h+n+m$ has the desired property for n'. Suppose $j_\alpha(1+\sum_{i=1}^{N} a_i) \notin V(h+n+m)$. We assume that $j_\alpha(1+\sum_{i=1}^{N} a_i)$ is odd. (A proof for the other case is obtained simply by replacing all occurrences of the word "odd" below by "even".) Since $\iota = j_\alpha(1+\sum_{i=1}^{h-1} a_i) - j_\alpha(1+\sum_{i=1}^{N} a_i)$ is even by equation (2), we see that $j_\alpha(1+\sum_{i=1}^{h-1} a_i)$ is also odd. $j_\alpha(1+\sum_{i=1}^{h-1} a_i) \in V(h+n+m)$ and $j_\alpha(1+\sum_{i=1}^{N} a_i) \notin V(h+n+m)$ implies that the even number ι (mod 6) is nonzero. If $V(n)$ has no odd element, then $j_\alpha(1+\sum_{i=1}^{h-1} a_i) \in V(h+n+m)$ implies that $V(n) \subsetneqq V(h+n+m)$. Next, assume that $V(n)$ has only one odd element. As ι is nonzero, if $j_\alpha(1+\sum_{i=1}^{q-1} a_i) \in V(n)$ is odd, then the odd element $j_\alpha(1+\sum_{i=1}^{q-1} a_i)+\iota \in V(h+n+m)$ is different from $j_\alpha(1+\sum_{i=1}^{q-1} a_i)$, proving the proper inclusion $V(n) \subsetneqq V(h+n+m)$ again. If $V(n)$ has two odd elements, say $j_\alpha(1+\sum_{i=1}^{q_1-1} a_i)$ and $j_\alpha(1+\sum_{i=1}^{q_2-1} a_i)$, then $\iota \neq 0$ implies that one of $j_\alpha(1+\sum_{i=1}^{q_k-1} a_i)+\iota$ ($k=1$ or 2) is equal to $j_\alpha(1+\sum_{i=1}^{N} a_i)$, contradicting the assumption that $j_\alpha(1+\sum_{i=1}^{N} a_i) \notin V(h+n+m)$. $V(n)$ does not have three odd elements, because a superset $V(h+n+m)$ of $V(n)$ does not contain an odd element $j_\alpha(1+\sum_{i=1}^{N} a_i)$. The proof is now complete. $\square$

($\Longrightarrow$) : By taking $m>1$ large enough, we can assume that the point $\langle j_\alpha(1+\sum_{i=1}^{m-1} a_i), j_\alpha(2+\sum_{i=1}^{m-1} a_i), \ldots\rangle$ belongs to its ω-limit set. Set $N := \sum_{i=1}^{m-1} a_i$ and pick a strictly increasing function $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ satisfying

$$\langle j_\alpha(N+1), \ldots, j_\alpha(N+n)\rangle = \langle j_\alpha(N+g(n)), \ldots, j_\alpha(N+g(n)+n-1)\rangle \tag{9}$$

for every $n \geq 1$. Then, since we have $j_\alpha(N+a_m)+1 \not\equiv j_\alpha(N+a_m+1) \pmod 6$, it follows from equation (9) that $j_\alpha(N+g(n)+a_m-1)+1 \not\equiv j_\alpha(N+g(n)+a_m) \pmod 6$ for every $n \geq a_m+1$. So, for each such n, we can write $N+g(n)+a_m-1$ as $\sum_{i=1}^{k} a_i$ for some $k>m$, which is due to Corollary 1. Choose an $M \geq 1$ arbitrarily. By applying Lemma 3 to equation (9) for $n = 1+\sum_{i=m}^{m+M} a_i$, we obtain $\langle a_{m+1}, a_{m+2}, \ldots, a_{m+M}\rangle = \langle a_{k+1}, a_{k+2}, \ldots, a_{k+M}\rangle$. Define a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ by the equation $\sum_{i=m+1}^{m+f(M)-1} a_i = g(1+\sum_{i=m}^{m+M} a_i) - 1$. Since g is strictly increasing by assumption, so is f. For this function f, we have $\langle a_{m+1}, a_{m+2}, \ldots, a_{m+M}\rangle = \langle a_{m+f(M)}, \ldots, a_{m+f(M)+M-1}\rangle$. As this holds for every $M \geq 1$, we see that the sequence $\langle a_{m+1}, a_{m+2}, \ldots\rangle$ is in its ω-limit set and thus positively Poisson stable. $\square$

It is worth investigating Question 2 also for positive Poisson stability. We note that, as in the final paragraph of Subsection 6.1, one can show that the set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ with positively Poisson stable Ducci matrix sequence expansion is dense.

6.4 Dense orbit. Recall that a point in a given dynamical system is said to *have dense orbit* if the point enters every open subset of the phase space. Then it is immediate that a point $\langle a_1, a_2, \dots\rangle \in \mathbb{Z}_{>0}^{\infty}$ has dense orbit if and only if for any sequence $\langle n_1, \dots, n_N\rangle \in \mathbb{Z}_{>0}^{+}$ of finite length, there exists an $m \geq 1$ such that $\langle a_m, a_{m+1}, \dots, a_{m+N-1}\rangle = \langle n_1, n_2, \dots, n_N\rangle$ holds. For the next result, we need to relativize denseness of an orbit on the whole space to a given set: The orbit of a point x in a dynamical system *is dense over* a (not necessarily invariant) subset A of the phase space if its intersection with A is a dense subset of A. So the orbit of the Ducci matrix sequence expansion $j(\alpha)$ is dense over $\{j(\alpha') \mid \alpha' \in (0,1) \setminus \mathbb{Q}\}$ if and only if for every $\alpha' \in (0,1) \setminus \mathbb{Q}$ and $N \geq 1$, there exists an $m \geq 1$ such that we have $\langle j_\alpha(m), j_\alpha(m+1), \dots, j_\alpha(m+N-1)\rangle = \langle j_{\alpha'}(1), j_{\alpha'}(2), \dots, j_{\alpha'}(N)\rangle$.

Theorem 11. *For a positive irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$, its Ducci matrix sequence expansion has dense orbit over $\{j(\alpha') \mid \alpha' \in (0,1) \setminus \mathbb{Q}\}$ $\Longleftrightarrow$ its continued fraction has dense orbit.*

Proof. ($\Longleftarrow$) : Choose an $\alpha' = [0; a'_1, a'_2, \dots] \in (0,1) \setminus \mathbb{Q}$ arbitrarily. It suffices to show that for every N', there exists an $m \geq 1$ such that $\langle j_\alpha(m), \dots, j_\alpha(m + \sum_{i=1}^{N'} a'_i - 1)\rangle = \langle j_{\alpha'}(1), \dots, j_{\alpha'}(\sum_{i=1}^{N'} a'_i)\rangle$.

If $a'_1 = 1$ (resp. $a'_1 > 1$), then apply Lemma 1 to the sequence $\langle a'_1 + a'_2, a'_3, a'_4, \dots, a'_{N'}\rangle$ (resp. $\langle 1, a'_1 - 1, a'_2, a'_3, \dots, a'_{N'}\rangle$) to get a sequence $S(\langle a'_1 + a'_2, a'_3, a'_4, \dots, a'_{N'}\rangle)$ (resp. $S(\langle 1, a'_1 - 1, a'_2, a'_3, \dots, a'_{N'}\rangle)$). Since our assumption guarantees that the sequence $S(\langle a'_1 + a'_2, a'_3, a'_4, \dots, a'_{N'}\rangle)$ (resp. $S(\langle 1, a'_1 - 1, a'_2, a'_3, \dots, a'_{N'}\rangle)$) appears in $\langle a_3, a_4, \dots\rangle$ as a subsequence, it follows from Lemma 1 that there exists an N such that

- $j_\alpha(1 + \sum_{i=1}^{N-1} a_i) = 1$ and
- $\langle a_N, a_{N+1}, \dots, a_{N+N'-2}\rangle = \langle a'_1 + a'_2, a'_3, a'_4, \dots, a'_{N'}\rangle$ (resp. $\langle a_N, a_{N+1}, \dots, a_{N+N'}\rangle = \langle 1, a'_1 - 1, a'_2, a'_3, \dots, a'_{N'}\rangle$).

It is easy to check that these two conditions give $j_\alpha(k + \sum_{i=1}^{N-1} a_i) = j_{\alpha'}(k)$ for $k = 1, 2, \dots, \sum_{i=1}^{N'} a'_i$ (cf. Corollary 1).

($\Longrightarrow$) : Take a sequence $\langle n_1, \dots, n_N\rangle \in \mathbb{Z}_{>0}^{+}$ arbitrarily. By assumption, there exists an arbitrary large $M \geq a_1 + a_2$ such that we have

$$\langle j_\alpha(M), j_\alpha(M+1), \dots, j_\alpha(M + 2 + \sum_{i=1}^{N} n_i)\rangle = \langle j_{\alpha'}(1), j_{\alpha'}(2), \dots, j_{\alpha'}(3 + \sum_{i=1}^{N} n_i)\rangle, \tag{10}$$

where $\alpha' := [0; 2, n_1, n_2, \dots, n_N, 5, 5, 5, \dots]$. On account of Corollary 1, we have $j_{\alpha'}(1) = j_{\alpha'}(2) = j_{\alpha'}(3) = 1$. Together with equation (10) and Corollary 1, this ensures that $M+1$ can be written as $\sum_{i=1}^{m-1} a_i$ for some $m \geq 3$. Now one can proceed in much the same way as Lemma 3 to obtain $a_{m+i-1} = n_i$ $(i = 1, 2, \dots, N)$, i.e., $\langle a_m, a_{m+1}, \dots, a_{m+N-1}\rangle = \langle n_1, n_2, \dots, n_N\rangle$. $\square$

The anonymous referee made the following observation: For each sequence $w \in \mathbb{Z}_{>0}^{+}$ of finite length, Lemma 2 shows that the set $\mathcal{I}(w) := \{\alpha \in (0,1) \setminus \mathbb{Q} \mid \langle a_i, \dots, a_{i+\mathrm{lh}(w)-1}\rangle = w$ holds for infinitely many $i\}$ is of full measure. Being the intersection of countably many full-measure sets, $\bigcap_{w \in \mathbb{Z}_{>0}^{+}} \mathcal{I}(w)$ is still of full measure. Since the continued fraction expansion

M_1	Eigenvalue	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
	Corresponding eigenvector	$(1,1,1)$	$(1, \frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$	$(1, \frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2})$
M_2	Eigenvalue	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
	Corresponding eigenvector	$(1,1,1)$	$(1, \frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2})$	$(1, \frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$
M_3	Eigenvalue	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
	Corresponding eigenvector	$(1,1,1)$	$(1, \frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$	$(1, \frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2})$
M_4	Eigenvalue	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
	Corresponding eigenvector	$(1,1,1)$	$(1, \frac{3+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2})$	$(1, \frac{3-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$
M_5	Eigenvalue	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
	Corresponding eigenvector	$(1,1,1)$	$(1, \frac{3-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$	$(1, \frac{3+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2})$
M_6	Eigenvalue	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
	Corresponding eigenvector	$(1,1,1)$	$(1, \frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2})$	$(1, \frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$

Table 1: Eigenvalues and corresponding eigenvectors of Ducci matrices

of any element of $\bigcap_{w\in\mathbb{Z}^+_{>0}} \mathcal{I}(w)$ has dense orbit, one can conclude from the above theorem that the set of all $\alpha \in (0,1)\setminus\mathbb{Q}$ whose Ducci matrix sequence expansion have dense orbit over $\{j(\alpha') \mid \alpha' \in (0,1)\setminus\mathbb{Q}\}$ is of full measure.

7 Future work. In this section, we collect problems that have not been mentioned yet.

7.1 Linear algebraic problems. One can study Ducci matrices of course from the viewpoint of linear algebra: To begin with, let us list eigenvalues and corresponding eigenvectors of Ducci matrices $M_1,\dots,M_6$ as Table 1. Another easily checked fact is that for any $\jmath_1,\dots,\jmath_n \in \{1,2,\dots,6\}$ $(n\geq 1)$, the sum of each column of the matrix $M_{\jmath_1}\cdots M_{\jmath_n}$ is zero, i.e., $\sum_{i=1}^{3}(M_{\jmath_1}\cdots M_{\jmath_n})_{i,j} = 0$ for $j=1,2,3$.

Let us now ask a few questions concerning Ducci matrices:

Question 3. *If $M_{j_\alpha(1)}\cdots M_{j_\alpha(n)} = M_{j_{\alpha'}(1)}\cdots M_{j_{\alpha'}(m)}$ for $\alpha,\alpha'\in(0,1)\setminus\mathbb{Q}$ and $n,m\geq 1$, must it be the case that $n=m$ and $j_\alpha(i) = j_{\alpha'}(i)$ $(1\leq i\leq n)$?*

Say that a sequence $\langle M_{\jmath_1},\dots,M_{\jmath_m}\rangle$ of finitely many Ducci matrices is *legal* if there exist an $\alpha\in(0,1)\setminus\mathbb{Q}$ and an n such that $\jmath_i = j_\alpha(n+i-1)$ for $i=1,2,\dots,m$. Using this terminology, we can formulate a more general question as follows:

Question 4. *If two legal matrix sequences $\langle M_{\jmath_1},\dots,M_{\jmath_n}\rangle$ and $\langle M_{\imath_1},\dots,M_{\imath_m}\rangle$ satisfy $M_{\jmath_1}\cdots M_{\jmath_n} = M_{\imath_1}\cdots M_{\imath_m}$, must it be the case that $n=m$ and $\jmath_i = \imath_i$ $(1\leq i\leq n)$?*

Questions arise also from the field of computability theory:

Question 5. *Do all $\alpha\in(0,1)\setminus\mathbb{Q}$ admit an effective algorithm that, given an input matrix $M\in M_3(\mathbb{Z})$, checks whether or not M is in the set $\{M_{j_\alpha(1)}\cdots M_{j_\alpha(n)} \mid n\geq 1\}$? In other words, is the set $\{M_{j_\alpha(1)}\cdots M_{j_\alpha(n)} \mid n\geq 1\}$ decidable for every $\alpha\in(0,1)\setminus\mathbb{Q}$? If not, which $\alpha\in(0,1)\setminus\mathbb{Q}$ admits such an algorithm?*

Question 6. *Is the subset $\{M_{j_\alpha(1)}\cdots M_{j_\alpha(n)} \mid n\geq 1 \text{ and } \alpha\in(0,1)\setminus\mathbb{Q}\}$ of $M_3(\mathbb{Z})$ decidable? More generally, is the set $\{M_{\jmath_1}\cdots M_{\jmath_n} \mid \langle M_{\jmath_1},\dots,M_{\jmath_n}\rangle$ is legal$\}$ a decidable subset of $M_3(\mathbb{Z})$?*

7.2 Characterizing conditions in terms of elements of the continued fraction expansion. In Theorem 4, we saw that the conditions $\lim_{n\to\infty}|\{i \le n \mid j_\alpha(i)+1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$ and $\lim_{n\to\infty}\sum_{i=1}^{n} a_i/n = \infty$ are equivalent. What about other conditions? Using only elements a_i of the continued fraction expansion of α, is it possible to characterize, for example, the condition "$\lim_{n\to\infty}|\{i \le n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ holds for every $\jmath \in \{1,2,\dots,6\}$" or "$\lim_{n\to\infty}\sqrt[p]{\sum_{i=1}^{n} j_\alpha(i)^p/n} = \sqrt[p]{(1^p+2^p+\cdots+6^p)/6}$ for every positive integer p"?

We know from Theorem 5 that the condition "$\lim_{n\to\infty}|\{i \le n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ holds for every $\jmath \in \{1,2,\dots,6\}$" is strictly weaker than the condition $\lim_{n\to\infty}|\{i \le n \mid j_\alpha(i)+1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$. On the other hand, there is an obvious weakening of the condition "$\lim_{n\to\infty}\sum_{i=1}^{n} a_i/n = \infty$", i.e., the unboundedness of the arithmetic mean. Given the aforementioned equivalence, one may expect that there still exist some relationships even after weakening each of the two equivalent conditions in that way. These two weakened conditions are in fact independent, as the next two examples show.

Example 4. *Consider the following eventually periodic continued fraction*

$$\alpha = [0;1,2,\underline{3,3,2,1,1,5,6,3},\underline{3,3,2,1,1,5,6,3},\underline{3,3,2,1,1,5,6,3},\dots].$$

The Ducci matrix sequence expansion of this α is

$$\langle M_1, M_2, M_3, M_3, M_4, M_5, M_5, M_6, M_1, M_1, M_2, M_4, M_6, \\ M_2, M_3, M_4, M_5, M_6, M_2, M_3, M_4, M_5, M_6, M_1\rangle^\infty.$$

In the periodic part of this matrix sequence expansion, each $M_\jmath$ appears exactly four times. From this, it is easy to conclude that we have $\lim_{n\to\infty}|\{i \le n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ for every $\jmath \in \{1,2,\dots,6\}$. On the other hand, the elements $a_1, a_2, \dots$ of α (and hence the number $\sum_{i=1}^{n} a_i/n$) is bounded by 6.

This example demonstrates that "$\lim_{n\to\infty}|\{i \le n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ for every $\jmath \in \{1,2,\dots,6\}$" does not imply unboundedness of the elements of α (and also of $\sum_{i=1}^{n} a_i/n$).

Example 5. *Let*

$$\alpha = [0;1,6,\underbrace{7,7,\dots,7}_{n_1},\underbrace{1,1,\dots,1}_{m_1},\underbrace{13,13,\dots,13}_{n_2},\underbrace{1,1,\dots,1}_{m_2}, \\ \dots,\underbrace{6k+1,6k+1,\dots,6k+1}_{n_k},\underbrace{1,1,\dots,1}_{m_k},\dots],$$

where

$$n_k = \min\left\{ n \in \mathbb{Z}_{>0} \;\middle|\; \frac{1+\sum_{i=1}^{k-1} i\cdot n_i + kn}{7+\sum_{i=1}^{k-1}\{(6i+1)n_i+m_i\}+(6k+1)n} > \frac{k}{6k+1} - \frac{1}{2^k} \quad \text{and} \right. \\ \left. \frac{7+\sum_{i=1}^{k-1}\{(6i+1)n_i+m_i\}+(6k+1)n}{2+\sum_{i=1}^{k-1}(n_i+m_i)+n} > 6k \right\};$$

$$m_k = \min\left\{ n \in \mathbb{Z}_{>0} \;\middle|\; \frac{1+\sum_{i=1}^{k} i\cdot n_i}{7+\sum_{i=1}^{k-1}\{(6i+1)\cdot n_i+m_i\}+(6k+1)n_k+n} < \frac{1}{2^k} \right\}.$$

(One can show that the above induction indeed defines sequences $\{n_k\}$ and $\{m_k\}$.) By the definition of $\{n_k\}$, we get $\sum_{i=1}^{L(k)} a_i/L(k) > 6k$ for every k, where $L(k) = 2+\sum_{i=1}^{k-1}(n_i + m_i) + n_k$. In particular, we see that the arithmetic mean is unbounded.

Next, consider the condition $j_\alpha(i) = 3$. In view of the definition of $\{n_k\}$ and $\{m_k\}$, it is not hard to see

$$\frac{|\{i \leq N_k \mid j_\alpha(i) = 3\}|}{N_k} = \frac{1 + \sum_{i=1}^{k} i \cdot n_i}{N_k} > \frac{k}{6k+1} - \frac{1}{2^k} \quad \textit{and}$$
$$\frac{|\{i \leq M_k \mid j_\alpha(i) = 3\}|}{M_k} = \frac{1 + \sum_{i=1}^{k} i \cdot n_i}{M_k} < \frac{1}{2^k},$$

where divergent sequences $\{N_k\}$ and $\{M_k\}$ satisfying $N_1 < M_1 < N_2 < M_2 < \cdots$ are given by $N_k := 7 + \sum_{i=1}^{k-1}\{(6i+1)n_i + m_i\} + (6k+1)n_k$ and $M_k := 7 + \sum_{i=1}^{k}\{(6i+1)n_i + m_i\}$. By letting $k \to \infty$, we see that the number $|\{i \leq n \mid j_\alpha(i) = 3\}|/n$ cannot have any limit. Hence unboundedness of $\sum_{i=1}^{n} a_i/n$ does not imply "$\lim_{n\to\infty}|\{i \leq n \mid j_\alpha(i) = \jmath\}|/n = 1/6$ for every $\jmath \in \{1, 2, \ldots, 6\}$".

7.3 Complexity. In this paper, we have considered various properties of Ducci matrix sequences. Although our analysis of these properties mainly focused on the size of the set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ with respective property, these properties can be studied also from the viewpoint of complexity.

For some properties of Ducci matrix sequences, we have obtained a characterization in terms of the elements of continued fraction expansion. Using these, one can readily see, for example

- "$\lim_{n\to\infty}|\{i \leq n \mid j_\alpha(i) + 1 \equiv j_\alpha(i+1) \pmod 6\}|/n = 1$" is $\mathbf{\Pi^0_3}$ over $(0,1) \setminus \mathbb{Q}$;
- "$\jmath(\alpha)$ is positively Poisson stable" is $\mathbf{\Sigma^0_3}$ over $(0,1) \setminus \mathbb{Q}$.

But what about other properties for which it is difficult to obtain such characterizations (cf. Subsection 7.2)? For each of such properties, one can investigate whether or not the set of all $\alpha \in (0,1) \setminus \mathbb{Q}$ with that property is a Borel subset of $(0,1) \setminus \mathbb{Q}$. If so, it is worth thinking about its class in the *Borel hierarchy*. It is also possible to investigate their complexity not over $(0,1) \setminus \mathbb{Q}$ but over $(0,1)$.

One can ask similar questions also for other notions of hierarchy, including *Wadge* and *Lipschitz hierarchy*.

7.4 Ducci map on $\mathbb{R}^5$ or $\mathbb{R}^6$. It is known [2] that for general n, any starting vector in $\mathbb{R}^n$ converges asymptotically to a periodic sequence, not necessarily to the sequence of zero vectors. Indeed, a vector which converges asymptotically to a non-trivial periodic sequence is constructed in [2] for $n = 7$. For $n = 3$, it is known [1] that any Ducci sequence is either eventually periodic or contains no periodic vectors but approaches the zero vector asymptotically. (Another proof of this fact can be found in [6].) For $n = 4$, while every vector in $\mathbb{Z}^4$ converges to the zero vector in finite time, Lotan [10] constructed vectors in $\mathbb{R}^4$ whose Ducci sequence never reach the zero vector. However, not many vectors exhibit such asymptotic behavior — A vector does not reach the zero vector if and only if it reaches a trivial transformation of the vector $(1, q, q^2, q^3)$ after finite time, where $1 < q < 2$ is the unique positive solution of the equation $x^3 - x^2 - x - 1 = 0$ [10].

There are natural questions concerning the Ducci map on $\mathbb{R}^5$ and $\mathbb{R}^6$, including the problem of the existence of a starting vector which is not periodic but converges asymptotically to a *non-trivial* periodic sequence. It might be possible to utilize the explicit description of the behavior of Ducci sequences for $n = 3$ given in [6] to tackle this question for $n = 6$.

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References

[1] G. Brockman and R.J. Zerr, Asymptotic behavior of certain Ducci sequences, Fibonacci Quart. 45(2) (2007), pp. 155–163.

[2] R. Brown and J.L. Merzel, Limiting behavior in Ducci sequences, Period. Math. Hungar. 47(1-2) (2003), pp. 45–50.

[3] C. Ciamberlini and A. Marengoni, Su una interessante curiosità numerica, Period. Mat., IV. Ser. 17(4) (1937), pp. 25–30.

[4] M. Einsiedler and T. Ward, Ergodic theory. With a view towards number theory. Springer, London, 2011.

[5] R.L. Graham, D.E. Knuth and O. Patashnik, Concrete mathematics: A foundation for computer science, Addison-Wesley, Reading, MA, 1989.

[6] T. Hida, Ducci sequences of triples and continued fractions, J. Difference Equ. Appl. 22(3) (2016), pp. 411–427.

[7] T. Hida, On Ducci matrix sequences, Sci. Math. Jpn. (2016), The electronic version is available as 2016-3 (e-2016).

[8] K. Hogenson, S. Negaard and R.J. Zerr, Matrix sequences associated with the Ducci map and the mediant construction of the rationals, Linear Algebra Appl. 437(1) (2012), pp. 285–293.

[9] A.Ya. Khinchin, Continued fractions, Dover, Mineola, NY, 1997.

[10] M. Lotan, A problem in difference sets, Amer. Math. Monthly 56(8) (1949), pp. 535–541.

[11] K.S. Sibirsky, Introduction to topological dynamics, Noordhoff International Publishing, Leyden, The Netherlands, 1975.

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