CLASS p-$wA(s,t)$ OPERATORS AND RANGE KERNEL ORTHOGONALITY

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Abstract. Let $T = U|T|$ be a polar decomposition of a bounded linear operator T on a complex Hilbert space with $\ker U = \ker |T|$. T is said to be class p-$wA(s,t)$ if $\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$ and $|T|^{2sp} \geq \left(|T|^s|T^*|^{2t}|T|^s\right)^{\frac{sp}{s+t}}$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. This is a generalization of p-hyponormal or class A operators. In this paper we prove following assertions. (i) If T is class p-$wA(s,t)$, then T is normaloid and isoloid. (ii) If T is class p-$wA(s,t)$ and $\sigma(T) = \{\lambda\}$, then $T = \lambda$. (iii) If T is class p-$wA(s,t)$, then T is finite and the range of generalized derivation $\delta_T : B(\mathcal{H}) \ni X \to TX - XT \in B(\mathcal{H})$ is orthogonal to its kernel. (iv) If S is class p-$wA(s,t), T^*$ is an invertible p-$wA(t,s)$ operator and X is a Hilbert-Schmidt operator such that $SX = XT$, then $S^*X = XT^*$.

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1. Introduction and Preliminaries

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and let $\ker(T)$, $\mathrm{ran}(T)$ and $\sigma(T)$ denote the kernel, the range and the spectrum of $T \in B(\mathcal{H})$, respectively. Recall that an operator T is said to be hyponormal if $T^*T \geq TT^*$. Aluthge [1] defined p-hyponormal operator as $(T^*T)^p \geq (TT^*)^p$ with $p \in (0,1]$, and he proved many interesting properties of p-hyponormal operators by using Furuta's inequality [9]. An invertible operator T is said to be log-hyponormal if $\log(T^*T) \geq \log(TT^*)$. It is known that invertible p-hyponormal operator is log-hyponormal, but the reverse does not hold by [16]. Moreover, by using Furuta's inequality, Furuta, Ito and Yamazaki [10] define class A operator as

$$|T^2| \geq |T|^2$$

and class $A(k)$ operator as

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2.$$

These classes are an extension of p-hyponormal, log-hyponormal operators, and moreover, class A and class $A(k)$ operator are extended to class $wA(s,t)$ operators with $0 < s, t$ as

$$\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{t}{s+t}} \geq |T^*|^{2t} \tag{1.1}$$

and

$$|T|^{2s} \geq \left(|T|^s|T^*|^{2t}|T|^s\right)^{\frac{s}{s+t}}. \tag{1.2}$$

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In [8], an operator T is said to be class $A(s,t)$ if T satisfies (1.1). However Ito and Yamazaki [12] proved that (1.1) implies (1.2). This is a striking result. An operator T is said to be class $A(s,t)$ if T satisfies (1.1). Hence Ito and Yamazaki proved that class $wA(s,t)$ coincides with class $A(s,t)$. It is known that every invertible p-hyponormal operator is log-hyponormal, every p-hyponormal, log-hyponormal operator is class $A(s,t)$ for all $0 < s,t$ and if T is invertible and class $A(s,t)$ for all $0 < s,t$ then T is log-hyponormal ([8], [11], [12], [16]).

It is well known that class $A(s,t)$ operators enjoy many interesting properties as hyponormal operators, for example, Fuglede-Putnam type theorem, Weyl type theorem, subscalarity and Putnam's inequality. Although there are many outstanding problems which are still open for hyponormal operators, for example, the invariant subspace problem, investigating new generalizations of hyponormal operators is one of recent interest in operator theory.

For $T \in B(\mathcal{H})$, set $|T| = (T^*T)^{\frac{1}{2}}$ as usual. By taking $U|T|x = Tx$ for $x \in \mathcal{H}$ and $Ux = 0$ for $x \in \ker|T|$, T has a unique polar decomposition $T = U|T|$ with $\ker U = \ker|T|$. An operator T is said to be class p-$wA(s,t)$ [15] if

$$\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{tp}{s+t}} \geq |T^*|^{2tp} \tag{1.3}$$

and

$$|T|^{2sp} \geq \left(|T|^s|T^*|^{2t}|T|^s\right)^{\frac{sp}{s+t}} \tag{1.4}$$

where $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. In [5], the authors proved that a set of class p-$wA(s,t)$ operators are increasing for $0 < s,t$ and decreasing for $0 < p \leq 1$.

Lemma 1.1. [5] *If $T \in B(\mathcal{H})$ is class p-$wA(s,t)$ and $0 < s \leq s_1, 0 < t \leq t_1, 0 < p_1 \leq p \leq 1$, then T is class p_1-$wA(s_1,t_1)$.*

Ito and Yamazaki [12] proved that (1.1) implies (1.2). However it is not known that whether (1.3) implies (1.4) or not. Class $A(1,1)$ is said to be class A and class $A(\frac{1}{2},\frac{1}{2})$ is said to be w-hyponormal (see [8, 11, 12, 20]). It is known that an operator T of class A is normaloid, i.e., its spectral radius $r(T)$ coincides with its norm $\|T\|$. Also, class A operator T are isoloid, i.e., its isolated point of spectrum $\sigma(T)$ is a point spectrum of T. The first aim of this paper is to prove that class p-$wA(s,t)$ operator is normaloid and isoloid.

Following [19], we say that an operator $T \in B(\mathcal{H})$ is finite if

$$\|I - (TX - XT)\| \geq 1$$

holds for all $X \in B(\mathcal{H})$. The above inequality is the starting point of the study of commutator approximations, a topic with roots in quantum theory [18]. Let $\mathcal{B}$ denote a Banach algebra. Recall that $b \in \mathcal{B}$ is said to be orthogonal to $a \in \mathcal{B}$, written $b \perp a$, if the inequality

$$\|a\| \leq \|a + \mu b\|$$

holds for all $\mu \in \mathbb{C}$. The above definition of orthogonality has natural geometric meaning, namely, $b \perp a$ if and only if the line $\{a + \mu b : \mu \in \mathbb{C}\}$ is tangent to the ball of center zero and radius $\|a\|$. If $\mathcal{B} = \mathcal{H}$, then the orthogonality means usual sense $\langle a, b\rangle = 0$.

The generalized derivation $\delta_{S,T} : B(\mathcal{H}) \to B(\mathcal{H})$ for $S,T \in B(\mathcal{H})$ is defined by $\delta_{S,T}(X) = SX - XT$ for $X \in B(\mathcal{H})$, and we note $\delta_{T,T} = \delta_T$. If the following inequality

$$\|S-(TX-XT)\| \geq \|S\|$$

holds for all $S \in \ker \delta_T$ and for all $X \in B(\mathcal{H})$, then we say that the range of δ_T is orthogonal to the kernel of δ_T .

Let $T \in B(\mathcal{H})$ and let $\{e_n\}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. The Hilbert-Schmidt norm is given by

$$\|T\|_2 = \left(\sum_{n=1}^{\infty} \|Te_n\|^2\right)^{\frac{1}{2}}.$$

An operator T is said to be a Hilbert-Schmidt operator if $\|T\|_2 < \infty$ (see [7] for details). $C_2(\mathcal{H})$ denotes a set of all Hilbert-Schmidt operators. For $S, T \in B(\mathcal{H})$, the operator $\Gamma_{S,T}$ defined as $\Gamma_{S,T} : C_2(\mathcal{H}) \ni X \to SXT \in C_2(\mathcal{H})$ has been studied in [3]. It is known that $|\Gamma| \leq \|S\|\|T\|$ and $(\Gamma_{S,T})^* X = S^* X T^* = \Gamma_{S^*,T^*} X$. For more information see [3].

In [19], J. P. Williams proved that normal operators and operators with a compact direct summand are finite. S. Mecheri ([13],[14]) extended Williams's results to more general classes of operators containing the classes of hyponormal operators and paranormal operators and studied range kernel orthogonality for these classes.

The second aim of this paper is to prove that (1) class p-$wA(s,t)$ operators with $0 < s+t \leq 1, 0 < p \leq 1$ are finite, and (2) if $T \in B(\mathcal{H})$ is class p-$wA(s,t)$, then the range of generalized derivation δ_T is orthogonal to its kernel, and (3) if $S \in B(\mathcal{H})$ is class p-$wA(s,t)$ and if $T^* \in B(\mathcal{H})$ is an invertible class p-$wA(t,s)$ operator and X is a Hilbert-Schmidt operator such that $SX = XT$, then $S^*X = XT^*$.

2. Main Results

We begin with the definition of generalized Aluthge transformation.

Definition 2.1. Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T with $\ker U = \ker |T|$. For $s, t > 0$, the generalized Aluthge transformation $T(s,t)$ of T is defined by

$$T(s,t) = |T|^s U |T|^t.$$

Hence, we have

$$T(s,t)^* = |T|^t U^* |T|^s.$$

In [15], the authors proved that if $T \in B(\mathcal{H})$ is class p-$wA(s,t)$, then $T(s,t)$ is $\frac{\rho p}{s+t}$-hyponormal for any $\rho \in (0, \min\{s,t\}]$.

Proposition 2.2. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then*

$$|T(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$$

and

$$|T|^{2sp} \geq |T(s,t)^*|^{\frac{2sp}{s+t}}.$$

Hence

$$|T(s,t)|^{\frac{2\rho p}{s+t}} \geq |T|^{2\rho p} \geq |T(s,t)^*|^{\frac{2\rho p}{s+t}} \tag{2.1}$$

for any $\rho \in (0, \min\{s,t\}]$.

A complex number λ is said to be an approximate eigenvalue of T if there exists a sequence $\{x_n\}$ of unit vectors such that

$$(T-\lambda)x_n \to 0 \quad (n \to \infty).$$

Also λ is said to be a joint approximate eigenvalue of T if there exists a sequence $\{x_n\}$ of unit vectors such that

$$(T-\lambda)x_n \to 0 \ \text{ and } \ (T-\lambda)^* x_n \to 0 \quad (n \to \infty).$$

We denote the set of all approximate eigenvalues of T by $\sigma_a(T)$ and denote the set of all joint approximate eigenvalues of T by $\sigma_{ja}(T)$. We say that $\lambda \in \sigma(T)$ belongs to the (Xia's) residual spectrum $\sigma_r^X(T)$ of T if $(T-\lambda)\mathcal{H} \neq \mathcal{H}$ and there exists a positive number $c > 0$ such that

$$\|(T-\lambda)x\| \geq c\|x\| \quad \text{ for } \quad x \in \mathcal{H}.$$

By the definition, $\sigma(T)$ is a disjoint union of $\sigma_a(T)$ and $\sigma_r^X(T)$.

Recently, the following result was proved by M. Chō, M.H.M. Rashid, K. Tanahashi and A. Uchiyama [5].

Proposition 2.3. [5] *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Let $re^{i\theta} \in \mathbb{C}$ with $0 < r$ and $(T - re^{i\theta})x_n \to 0$. Then $(|T|-r)x_n, (U-e^{i\theta})x_n, (U-e^{i\theta})^* x_n, (T-re^{i\theta})^* x_n \to 0$.*

Lemma 2.4. *Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T with $\ker U = \ker |T|$ and let $T_\alpha = U|T|^\alpha$ with $0 < \alpha$. Then*

$$0 \in \sigma_a(T) \Longleftrightarrow 0 \in \sigma_a(T_\alpha),$$
$$0 \in \sigma_r^X(T) \Longleftrightarrow 0 \in \sigma_r^X(T_\alpha),$$
$$0 \in \sigma(T) \Longleftrightarrow 0 \in \sigma(T_\alpha).$$

Proof. Let $0 \in \sigma_a(T)$. Then there exist unit vectors x_n such that $Tx_n \to 0$. Then $|T|x_n = U^*U|T|x_n = U^*Tx_n \to 0$. Hence $T_\alpha x_n = U|T|^\alpha x_n \to 0$ and $0 \in \sigma_a(T_\alpha)$. The converse is similar. Let $0 \notin \sigma(T)$. Then $|T|$ is invertible and U is unitary. Hence $T_\alpha = U|T|^\alpha$ is invertible and $0 \notin \sigma(T_\alpha)$. The converse is similar. Since $\sigma(T)$ is a disjoint union of $\sigma_a(T)$ and $\sigma_r^X(T)$, the proof is completed. □

Theorem 2.5. *If $T = U|T| \in B(\mathcal{H})$ is class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ and if $T_\alpha = U|T|^\alpha$ with $s+t \leq \alpha$, then*

$$\sigma_a(T_\alpha) = \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma_a(T)\}, \tag{2.2}$$

$$\sigma_r^X(T_\alpha) = \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma_r^X(T)\}, \tag{2.3}$$

$$\sigma(T_\alpha) = \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma(T)\}. \tag{2.4}$$

Proof. Let $T = U|T|$ be class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Let $\lambda = re^{i\theta} \in \sigma_a(T)\backslash\{0\}$ with $0 < r$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(T-re^{i\theta})x_n \to 0$. Hence $(T-re^{i\theta})^* x_n \to 0, (|T|-r)x_n \to 0, (U-e^{i\theta})x_n \to 0$ and $(U-e^{i\theta})^* x_n \to 0$ by Proposition 2.3. Hence $\lambda_\alpha := r^\alpha e^{i\theta} \in \sigma_{ja}(T_\alpha) \subset \sigma_a(T_\alpha)$. Conversely, let $\mu = r'e^{\phi} \in \sigma_a(T_\alpha) \setminus \{0\}$ with $0 < r'$. Then there exists a sequence unit vectors $\{x_n\}$ such that $(T_\alpha - r'e^{\phi})x_n \to 0$. Since T_α is p-$wA(s/\alpha, t/\alpha)$ and $0 < s/\alpha + t/\alpha \leq 1$, we have that $\mu = r'e^{\phi} \in \sigma_{ja}(T_\alpha)$ by Proposition 2.3. Hence $\mu_{1/\alpha} = (r')^{1/\alpha}e^{i\phi} \in \sigma_{ja}(T) \subset \sigma_a(T)$. Therefore

$$\sigma_a(T_\alpha) \setminus \{0\} = \{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma_a(T)\} \setminus \{0\}. \tag{2.5}$$

Hence we have (2.2) by Lemma 2.4.

Next we show (2.3). Let $\lambda = re^{i\theta} \in \sigma_r^X(T)\setminus\{0\}$ with $0 < r$. We claim $\lambda_\alpha = r^\alpha e^{i\theta} \in \sigma(T_\alpha)$.

Assume that $\lambda_\alpha = r^\alpha e^{i\theta} \notin \sigma(T_\alpha)$. Let J be a closed interval $[1,\alpha]$ (or $[\alpha,1]$) and let f be an operator valued continuous function $f(x) := T_x - r^x e^{i\theta}$ $(x \in J)$. Then $f(1)$ is a semi-Fredholm operator with the Fredholm index

$$\mathrm{ind}(f(1)) = \dim(\ker(T - re^{i\theta})) - \dim(\ker(T - re^{i\theta})^*) \le -1,$$

and $f(\alpha)$ is invertible (so, it is Fredholm with index 0).

We claim that there exists a real number $x_0 \in J$ such that $f(x_0)$ is not semi-Fredholm. Assume that there exists no such $x \in J$. Since $F(J) = \{f(x) | x \in J\}$ is connected in the set of all semi-Fredholm operators of $\mathcal{H}$ and every operator in $F(J)$ has the same Fredholm index, we have that $f(1)$ and $f(\alpha)$ have same Fredholm index. But this is a contradiction.

Since there exists $x_0 \in J$ such that $f(x_0)$ is not semi-Fredholm, we have

$$r^{x_0}e^{i\theta} \in \sigma(T_{x_0}) \setminus \sigma_r^X(T_{x_0}) = \sigma_a(T_{x_0}).$$

Since $s + t \le x_0$ and $0 < r$, we have $\lambda = re^{i\theta} \in \sigma_a(T)$ by (2.2). But it is a contradiction. Hence $\lambda_\alpha = r^\alpha e^{i\theta} \in \sigma(T_\alpha)$.

We claim $\lambda_\alpha = r^\alpha e^{i\theta} \notin \sigma_a(T_\alpha)$. Assume $\lambda_\alpha = r^\alpha e^{i\theta} \in \sigma_a(T_\alpha)$. Then $\lambda = re^{i\theta} \in \sigma_a(T)$ by (2.2). But it is a contradiction. Hence

$$\{r^\alpha e^{i\theta} \mid re^{i\theta} \in \sigma_r^X(T)\setminus\{0\}\} \subset \sigma_r^X(T_\alpha)\setminus\{0\}.$$

Similarly we have

$$\{(r')^{1/\alpha} e^{i\theta} \mid r'e^{i\theta} \in \sigma_r^X(T_\alpha)\setminus\{0\}\} \subset \sigma_r^X(T)\setminus\{0\}.$$

Hence (2.3) holds by Lemma 2.4. Since $\sigma(T)$ is a disjoint union of $\sigma_a(T)$ and $\sigma_r^X(T)$, the proof of (2.4) is completed,

□

The following result was proved by [5] if $s + t = 1$ and $\rho \ne 0$.

Theorem 2.6. *Let $T \in B(\mathcal{H})$ be class p-$wA(s,t)$ with $0 < p \le 1$ and $0 < s, t, s+t \le 1$. Let $re^{i\theta} \in \mathbb{C}$ with $0 \le r$. Then*

$$\ker(T - re^{i\theta}) = \ker(T(s,t) - r^{s+t}e^{i\theta}).$$

Proof. Assume $0 < r$. Let $x \in \ker(T - re^{i\theta})$. Then $|T|x = rx, Ux = e^{i\theta}x$ by Theorem 2.2 of [5]. Hence $T(s,t)x = |T|^s U |T|^t x = r^{s+t}e^{i\theta}x$ and $x \in \ker(T(s,t) - r^{s+t}e^{i\theta})$.

Conversely, let $x \in \ker(T(s,t) - r^{s+t}e^{i\theta})$. Since

$$|T(s,t)|^{\frac{2\rho p}{s+t}} \ge |T|^{2\rho p} \ge |T(s,t)^*|^{\frac{2\rho p}{s+t}} \tag{2.6}$$

and $T(s,t)$ is ρp-hyponormal for any $\rho \in (0, \min\{s,t\}]$ by Proposition 2.2, we have

$$T(s,t)^* x = r^{s+t}e^{-i\theta}x$$

and

$$|T(s,t)|x = |T(s,t)^*|x = r^{s+t}x$$

by Theorem 4 of [4]. Then

$$0 \le |T(s,t)|^{\frac{2\rho p}{s+t}} - |T|^{2\rho p} \le |T(s,t)|^{\frac{2\rho p}{s+t}} - |T(s,t)^*|^{\frac{2\rho p}{s+t}},$$

and we have

$$\begin{aligned}\| \left(|T(s,t)|^{\frac{2\rho p}{s+t}} - |T|^{2\rho p}\right)^{\frac{1}{2}} x\|^2 &= \langle \left(|T(s,t)|^{\frac{2\rho p}{s+t}} - |T|^{2\rho p}\right) x, x\rangle \\ &\leq \langle \left(|T(s,t)|^{\frac{2\rho p}{s+t}} - |T(s,t)^*|^{\frac{2\rho p}{s+t}}\right) x, x\rangle = 0.\end{aligned}$$

Hence $\left(|T(s,t)|^{\frac{2\rho p}{s+t}} - |T|^{2\rho p}\right)^{\frac{1}{2}} x = 0$ and

$$|T|^{2\rho p}x = |T(s,t)|^{\frac{2\rho p}{s+t}} x = r^{2\rho p}x.$$

This implies $|T|x = rx$. Since

$$r^{s+t}e^{-i\theta}x = T(s,t)^*x = |T|^tU^*|T|^sx = r^s|T|^tU^*x,$$

we have

$$T^*x = |T|^{1-t}|T|^tU^*x = |T|^{1-t}r^te^{-i\theta}x = re^{-i\theta}x.$$

Then

$$\begin{aligned}\|(T - re^{i\theta})x\|^2 &= \|Tx\|^2 - re^{i\theta}\langle x, Tx\rangle - re^{-i\theta}\langle Tx, x\rangle + r^2\|x\|^2 \\ &= \||T|x\|^2 - re^{i\theta}\langle T^*x, x\rangle - re^{-i\theta}\langle x, T^*x\rangle + r^2\|x\|^2 \\ &= (r^2 - r^2 - r^2 + r^2)\|x\|^2 = 0.\end{aligned}$$

Hence $x \in \ker(T - re^{i\theta})$.

Assume $r = 0$. Let $x \in \ker(T)$. Then $|T|x = 0$ and $T(s,t)x = |T|^sU|T|^tx = 0$.

Conversely, let $x \in \ker(T(s,t))$. Then $|T(s,t)|x = 0$ and $|T|x = 0$ by (2.6). Thus $x \in \ker(T)$.

□

Corollary 2.7. *If T is class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, then T is normaloid.*

Proof. Since $T(s,t)$ is $\frac{\rho p}{s+t}$-hyponormal and satisfies

$$|T(s,t)|^{\frac{2\rho p}{s+t}} \geq |T|^{2\rho p} \geq |T(s,t)^*|^{\frac{2\rho p}{s+t}} \tag{2.7}$$

for all $\rho \in (0, \min\{s,t\}]$ by Proposition 2.2, we have

$$\sigma(T(s,t)) = \sigma(|T|^sU|T|^t) = \sigma(U|T|^{s+t}) = \{r^{s+t}e^{i\theta} \mid re^{i\theta} \in \sigma(T)\}$$

by Lemma 6 of [17] and Theorem 2.5. Since $T(s,t)$ is normaloid, we have

$$\begin{aligned}\||T(s,t)|^{\frac{2\rho p}{s+t}}\| &= \||T(s,t)|\|^{\frac{2\rho p}{s+t}} = \|T(s,t)\|^{\frac{2\rho p}{s+t}} \\ &= r\left(T(s,t)\right)^{\frac{2\rho p}{s+t}} = \left(r(T)^{s+t}\right)^{\frac{2\rho p}{s+t}} = r(T)^{2\rho p},\end{aligned}$$

and

$$\|T\|^{2\rho p} = \||T|\|^{2\rho p} = \||T|^{2\rho p}\| \leq \||T(s,t)|^{\frac{2\rho p}{s+t}}\| = r(T)^{2\rho p}$$

by (2.7). Hence $\|T\| \leq r(T)$ and therefore $\|T\| = r(T)$. □

Corollary 2.8. *If T is class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, then T is isoloid.*

Proof. Let $re^{i\theta}$ be an isolated point of $\sigma(T)$ with $0 \le r$. Since

$$\sigma(T(s,t)) = \sigma(|T|^s U |T|^t) = \sigma(U|T|^{s+t})$$

by Lemma 6 of [17] and

$$\sigma(U|T|^{s+t}) = \{r^{s+t}e^{i\theta} \mid re^{i\theta} \in \sigma(T)\}$$

by Theorem 2.5, we have $r^{s+t}e^{i\theta}$ is an isolated point of $\sigma(T(s,t))$. We remark $T(s,t)$ is $\frac{\rho p}{s+t}$-hyponormal for any $\rho \in (0, \min\{s,t\}]$ by Proposition 2.2.

Assume $re^{i\theta} = 0$. Since $T(s,t)$ is $\frac{\rho p}{s+t}$-hyponormal, we have $E_0(s,t) = \ker T(s,t)$ where $E_0(s,t)$ is the Riesz idempotent of $T(s,t)$ for $0 \in \operatorname{iso}\sigma(T(s,t))$ by Theorem 5 of [6]. Hence there exists non-zero vector $x \in \mathcal{H}$ such that $T(s,t)x = 0$. Hence $Tx = 0$ by (2.7).

Assume $re^{i\theta} \neq 0$. Then

$$E_{r^{s+t}e^{i\theta}}(s,t) = \ker(T(s,t) - r^{s+t}e^{i\theta}) = \ker((T(s,t) - r^{s+t}e^{i\theta})^*)$$

where $E_{r^{s+t}e^{i\theta}}(s,t)$ is the Riesz idempotent of $T(s,t)$ for $r^{s+t}e^{i\theta} \in \operatorname{iso}\sigma(T(s,t))$ by Theorem 5 of [6]. Hence there exists non-zero vector $x \in \ker(T(s,t) - r^{s+t}e^{i\theta})$ such that $T(s,t)^*x = r^{s+t}e^{-i\theta}x$ and $|T(s,t)|x = |T(s,t)^*|x = r^{s+t}x$ by Theorem 5 of [6]. Then we have

$$\begin{aligned} 0 &= \langle \left(|T(s,t)|^{\frac{2\rho p}{s+t}} - r^{2\rho p} \right) x, x \rangle \ge \langle \left(|T|^{2\rho p} - r^{2\rho p} \right) x, x \rangle \\ &\ge \langle \left(|T(s,t)^*|^{\frac{2\rho p}{s+t}} - r^{2\rho p} \right) x, x \rangle = 0 \end{aligned}$$

by (2.7). Hence $\langle \left(|T|^{2\rho p} - r^{2rp} \right) x, x \rangle = 0$. Since $0 < \rho \le \min\{s,t\}$ is arbitrary, we have $\langle (|T|^{\rho p} - r^{\rho p}) x, x \rangle = 0$ by the same arguement. Then

$$\begin{aligned} \| (|T|^{\rho p} - r^{\rho p}) x \|^2 &= \langle (|T|^{\rho p} - r^{\rho p})^2 x, x \rangle \\ &= \langle \left(|T|^{2\rho p} - r^{2\rho p} \right) x, x \rangle - 2r^{\rho p} \langle (|T|^{\rho p} - r^{\rho p}) x, x \rangle = 0. \end{aligned}$$

Hence $(|T|^{\rho p} - r^{\rho p}) x = 0$ and this implies $|T|x = rx$. Then $U^*Ux = U^*U|T|r^{-1}x = |T|r^{-1}x = x$. Since $r^{s+t}e^{-i\theta}x = T(s,t)^*x = |T|^t U^* |T|^s x = |T|^t U^* r^s x$, we have $|T|^t U^* x = r^t e^{-i\theta} x = |T|^t e^{-i\theta} x$. Hence $\left(U^* - e^{-i\theta} \right) x \in \ker |T|^t = \ker |T| = \ker U$. Hence $U \left(U^* - e^{-i\theta} \right) x = 0$ and $UU^*x = e^{-i\theta}Ux$. Then

$$U^*x = U^*UU^*x = e^{-i\theta}U^*Ux = e^{-i\theta}x$$

because $U^*Ux = x$. Then

$$\begin{aligned} \| \left(U - e^{i\theta} \right) x \|^2 &= \langle \left(U - e^{i\theta} \right) x, \left(U - e^{i\theta} \right) x \rangle \\ &= \langle \left(U - e^{i\theta} \right)^* \left(U - e^{i\theta} \right) x, x \rangle \\ &= \langle U^*Ux - e^{-i\theta} \left(U - e^{i\theta} \right) x - e^{i\theta} \left(U^* - e^{-i\theta} \right) x - x, x \rangle \\ &= \langle -e^{-i\theta}x, \left(U - e^{i\theta} \right)^* x \rangle = 0. \end{aligned}$$

Hence $Ux = e^{i\theta}x$. Thus $Tx = U|T|x = re^{i\theta}x$ and the proof is completed. □

Theorem 2.9. *Let $T \in B(\mathcal{H})$ be a class p-$wA(s,t)$ operator with $0 < p \le 1$ and $0 < s,t,s+t \le 1$ and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$.*

Proof. Let $\lambda = 0$. Since T is normaloid by Corollary 2.7, we have $\|T\| = r(T) = 0$. Hence $T = 0$. Let $\lambda \neq 0$. Then $S := T/\lambda$ is class p-$wA(s,t)$ and $\sigma(S) = \{1\}$. Hence $\|S\| = r(S) = 1$ by Corollary 2.7. Since S^{-1} is class p-$wA(t,s)$ by [17], we have $\|S^{-1}\| = r(S^{-1}) = 1$ by Corollary 2.7. This implies $S = 1$. Hence $T = \lambda$. □

Theorem 2.10. *Let $T \in B(\mathcal{H})$ be a class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then T is finite.*

Proof. We may assume $T \neq 0$. If $\sigma(T) = \{0\}$, then $T = 0$ by Theorem 2.9. Hence $\sigma(T) \neq \{0\}$. Hence T has an approximate point spectrum $\mu \neq 0$. Hence there exists a sequence $\{x_n\}$ of unit vectors such that $(T-\mu)x_n \to 0$. Then $(T-\mu)^* x_n \to 0$ by Proposition 2.3. Hence $\sigma_{ja}(T) \neq \emptyset$ and $T \in \overline{\mathcal{R}_1}$ where $\mathcal{R}_1$ is a class of all operators with a one-dimensional reducing subspace. Thus T is finite by Theorem 6 of [19]. □

Remark. The referee pointed us a simple proof of Theorem 2.10, that is, since T is normaloid by Corollary 2.7, T is finite by Theorem 5 of [19].

Next we consider a generalization of Theorem 2.10; in other words, we show the range kernel orthogonality of class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ by the method of [14]. We begin with the following lemma.

Lemma 2.11. *If $T \in B(\mathcal{H})$ is a class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ and if S is a normal operator such that $TS = ST$, then we have*

$$\|S - (TX - XT)\| \geq |\mu|$$

for all $\mu \in \sigma_p(S)$ and for all $X \in B(\mathcal{H})$.

Proof. Let $\mathcal{M}_\mu$ be an eigen space of $\mu \in \sigma_p(S)$. Since S is normal, the Fuglede-Putnam theorem ensures $TS = ST$ implies $S^*T = ST^*$. Hence $\mathcal{M}_\mu$ reduces both T and S. Now we write matrix representations of T, S and X as

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, S = \begin{pmatrix} \mu & 0 \\ 0 & S_2 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

on $\mathcal{H} = \mathcal{M}_\mu \oplus \mathcal{M}_\mu^\perp$. Hence we have

$$S - (TX - XT) = \begin{pmatrix} \mu - (T_1X_1 - X_1T_1) & A \\ B & C \end{pmatrix}.$$

for some operators A, B and C and so

$$\|S - (TX - XT)\| \geq \|\mu - (T_1X_1 - X_1T_1)\|. \tag{2.8}$$

Since T is a class p-$wA(s,t)$ operator and $\mathcal{M}_\mu$ is a reducing subspace of T, the restriction $T_1 = T|_{\mathcal{M}_\mu}$ is a class p-$wA(s,t)$ operator. Since T_1 is finite by Theorem 2.10, we have

$$\|(T_1X_1 - X_1T_1) - \mu\| \geq \|T_1(\frac{X_1}{\mu}) + (\frac{X_1}{\mu})T_1 - 1\||\mu| \geq |\mu|. \tag{2.9}$$

From (2.8) and (2.9), we have

$$\|S - (TX - XT)\| \geq |\mu|$$

for all $X \in B(\mathcal{H})$. □

The following result due to S.K. Berberian [2] is well known.

Proposition 2.12. [2] [Berberian technique] *Let $\mathcal{H}$ be a complex Hilbert space. Then there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\psi : B(\mathcal{H}) \to B(\mathcal{K})$ such that ψ is an $*$-isometric isomorphism preserving the order satisfying*

(i) $\psi(T^*) = \psi(T)^*$, $\psi(I_{\mathcal{H}}) = I_{\mathcal{K}}$, $\psi(\alpha T + \beta S) = \alpha\psi(T) + \beta\psi(S)$, $\psi(TS) = \psi(T)\psi(S)$, $\|\psi(T)\| = \|T\|$, $\psi(T) \le \psi(S)$ *if* $T \le S$ *for all* $T, S \in B(\mathcal{H})$ *and for all* $\alpha, \beta \in \mathbb{C}$.

(ii) $\sigma(T) = \sigma(\psi(T)), \sigma_a(T) = \sigma_a(\psi(T)) = \sigma_p(\psi(T))$, *where* $\sigma_p(T)$ *is the point spectrum of* T.

Theorem 2.13. *Let $T \in B(\mathcal{H})$ be a class p-$wA(s,t)$ operator operator with $0 < p \le 1$ and $0 < s, t, s+t \le 1$, and let S be a normal operator such that $TS = ST$. Then*

$$\|S\| \le \|S - (TX - XT)\|$$

for all $X \in B(\mathcal{H})$.

Proof. By Proposition 2.12, it follows that $\psi(S)$ is normal, $\psi(T)$ is p-$wA(s,t)$ and $\psi(T)\psi(S) = \psi(S)\psi(T)$. Since $\sigma_p(\psi(S)) = \sigma_a(\psi(S)) = \sigma_a(S) = \sigma(S)$, we have

$$|\mu| \le \|\psi(S) - \psi(T)\psi(X) - \psi(X)\psi(T)\| = \|S - (TX - XT)\|$$

for all $\mu \in \sigma(S)$ and for all $X \in B(\mathcal{H})$ by Lemma 2.11. Hence

$$\sup_{\mu\in\sigma(S)}|\mu| = r(S) = \|S\| \le \|S - (TX - XT)\|.$$

This completes the proof. ☐

Now we prove if $S \in B(\mathcal{H})$ is a class p-$wA(s,t)$ operator, $T^* \in B(\mathcal{H})$ is an invertible class p-$wA(t,s)$ operator and $X \in B(\mathcal{H})$ is a Hilbert-Schmidt operator such that $SX = XT$, then $S^*X = XT^*$. The following key lemma is necessary for the proof of theorem 2.15.

Lemma 2.14. *Let $S, T^* \in B(\mathcal{H})$ be class p-$wA(s,t)$ operators with $0 < p \le 1$ and $0 < s, t, s+t \le 1$ and let $X \in B(\mathcal{H})$ be a Hilbert-Schmidt operator. Then the operator $\Gamma = \Gamma_{S,T} : C_2(\mathcal{H}) \ni X \to SXT \in C_2(\mathcal{H})$ is class p-$wA(s,t)$.*

Proof. Since $\Gamma^*X = S^*XT^*, |\Gamma|X = |S|X|T^*|, |\Gamma^*|X = |S^*|X|T|$, we have

$$\begin{aligned}
&\left(\left(|\Gamma^*|^t|\Gamma|^{2s}|\Gamma^*|^t\right)^{\frac{tp}{s+t}} - |\Gamma^*|^{2tp}\right)X \\
&= \left(|S^*|^t|S|^{2s}|S^*|^t\right)^{\frac{tp}{s+t}} X \left(|T|^t|T^*|^{2t}|T|^t\right)^{\frac{tp}{s+t}} - |S^*|^{2tp}X|T|^{2tp} \\
&= \left(\left(|S^*|^t|S|^{2s}|S^*|^t\right)^{\frac{tp}{s+t}} - |S^*|^{2tp}\right) X \left(|T|^t|T^*|^{2t}|T|^t\right)^{\frac{tp}{s+t}} \\
&\quad + |S^*|^{2tp}X\left(\left(|T|^t|T^*|^{2t}|T|^t\right)^{\frac{tp}{s+t}} - |T|^{2tp}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(|\Gamma|^{2sp} - \left(|\Gamma|^s|\Gamma^*|^{2t}|\Gamma|^s\right)^{\frac{sp}{s+t}}\right)X \\
&= |S|^{2sp}X|T^*|^{2sp} - \left(|S|^s|S^*|^{2t}|S|^s\right)^{\frac{sp}{s+t}} X \left(|T^*|^s|T|^{2t}|T^*|^s\right)^{\frac{sp}{s+t}} \\
&= \left(|S|^{2sp} - \left(|S|^s|S^*|^{2t}|S|^s\right)^{\frac{sp}{s+t}}\right) X|T^*|^{2sp} \\
&\quad + \left(|S|^s|S^*|^{2t}|S|^s\right)^{\frac{sp}{s+t}} X \left(|T^*|^{2sp} - \left(|T^*|^s|T|^{2t}|T^*|^s\right)^{\frac{sp}{s+t}}\right).
\end{aligned}$$

Hence $|\Gamma|^{2sp} - (|\Gamma|^s|\Gamma^*|^{2t}|\Gamma|^s)^{\frac{sp}{s+t}} \geq 0$ and $(|\Gamma^*|^t|\Gamma|^{2t}|\Gamma^*|^t)^{\frac{tp}{s+t}} - |\Gamma^*|^{2tp} \geq 0$. Thus Γ is class p-$wA(s,t)$. □

Theorem 2.15. *Let $S \in B(\mathcal{H})$ be a class p-$wA(s,t)$ operator, $T^* \in B(\mathcal{H})$ be an invertible class p-$wA(t,s)$ operator and $X \in B(\mathcal{H})$ be a Hilbert-Schmidt operator such that $SX = XT$. Then $S^*X = XT^*$.*

Proof. Let $\Gamma_{S,T^{-1}} : C_2(\mathcal{H}) \ni X \to SXT^{-1} \in C_2(\mathcal{H})$. Since S and $(T^*)^{-1}$ are class p-$wA(s,t)$ operators by Corollary 2.4 of [15], Lemma 2.14 ensures that $\Gamma_{S,T^{-1}}$ is class p-$wA(s,t)$. Since $SX = XT$, we have $\Gamma_{S,T^{-1}}X = SXT^{-1} = X$. Applying Proposition 2.3, it follows that $(\Gamma_{S,T^{-1}})^*X = X$. Hence $S^*X(T^{-1})^* = X$ and $S^*X = XT^*$.

□

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