POSITIVE DEFINITE SEQUENCES WITH CONSTANT MODULUS

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Abstract. Let $a_0, a_1, ..., a_N$ be complex numbers. We consider the Toeplitz matrix T_N, where the (i, j)-th component is a_{i-j} if $i \geq j$ and $\overline{a_{j-i}}$ if $i < j$. If T_N is positive and $|a_0| = |a_1| \neq 0$, then $a_2, a_3, ..., a_N$ can be represented in terms of a_0 and a_1 and there exists a unique positive definite sequence f such that $f(i) = a_i$ for any $i = 0, 1, 2, ..., N$. In particular, it holds $|f(n)| = |a_0|$ for any n. We also provides some applications related to this fact.

1 Introduction

Let $\mathbb{N} = \{0, 1, 2, ...\}$ and f is a complex-valued function on $\mathbb{N}$. An $n \times n$ matrix $A = (a_{ij})$ with complex entries is said to be positive and it is denoted by $A \geq 0$ if

$$\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j a_{ij} \geq 0 \text{ for all } \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}.$$

It is well-known that $A \geq 0$ if and only if there exists a $k \times n$ matrix B in which $A = B^*B$ for some $k \in \mathbb{N} \setminus \{0\}$. We call that f is a positive definite sequence if, for any positive integer N, the following $(N+1) \times (N+1)$ Toeplitz matrix

$$T_N = \begin{pmatrix} f(0) & \overline{f(1)} & \cdots & \overline{f(N)} \\ f(1) & f(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{f(1)} \\ f(N) & \cdots & f(1) & f(0) \end{pmatrix}$$

is positive, where the (i, j)-th component of T_N is $f(i-j)$ if $i \geq j$ and $\overline{f(j-i)}$ if $i < j$. We remark that the positivity of T_N implies $|f(i)| \leq f(0)$ for any $i = 1, 2, ..., N$.

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For any $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$, the function f given by $f(n) = e^{n\theta\sqrt{-1}}$ is a positive definite sequence. In fact, for any positive integer N, T_N is positive since

$$T_N = \begin{pmatrix} 1 & e^{-\theta\sqrt{-1}} & \cdots & e^{-N\theta\sqrt{-1}} \\ e^{\theta\sqrt{-1}} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e^{-\theta\sqrt{-1}} \\ e^{N\theta\sqrt{-1}} & \cdots & e^{\theta\sqrt{-1}} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ e^{\theta\sqrt{-1}} \\ \vdots \\ e^{N\theta\sqrt{-1}} \end{pmatrix} \begin{pmatrix} 1 & e^{-\theta\sqrt{-1}} & \cdots & e^{-N\theta\sqrt{-1}} \end{pmatrix} \geq 0.$$

This function is a typical example of positive definite sequence.

Our result is as follows:

Theorem 1. *Let $N \geq 1$. If $|a_0| = |a_1| \neq 0$ and*

$$T = \begin{pmatrix} a_0 & \overline{a_1} & \cdots & \overline{a_N} \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{a_1} \\ a_N & \cdots & a_1 & a_0 \end{pmatrix} \geq 0,$$

then there exists a unique positive definite sequence f such that

$$f(i) = a_i \qquad \text{for any } i = 0, 1, ..., N.$$

Moreover, it holds

$$f(n) = f(0)\left(\frac{f(1)}{f(0)}\right)^n \ (\text{in particular, } |f(n)| = f(0)) \text{ for any } n \in \mathbb{N} \setminus \{0\}.$$

2 Proof of Theorem and Application

Let $T = \begin{pmatrix} 1 & \overline{\alpha} & \overline{\gamma} \\ \alpha & 1 & \overline{\beta} \\ \gamma & \beta & 1 \end{pmatrix}$ where α, β, γ are complex numbers and $|\alpha| = 1$. The following fact is known and is used in this paper.

(†) $\qquad T \geq 0$ if and only if $|\beta| \leq 1$ and $\gamma = \alpha\beta$.

The statement (†) for operators had been considered in [6], and we extend as follows:

Lemma 2. *Let u, v, w be bounded linear operators on a Hilbert space $\mathcal{H}$ and u isometric (that is, $u^*u = 1$). Then*

$$T = \begin{pmatrix} 1 & u^* & w^* \\ u & 1 & v^* \\ w & v & 1 \end{pmatrix} \geq 0 \textit{ if and only if } \|v\| \leq 1 \textit{ and } w = vu.$$

Proof. Assume $\|v\| \leq 1$ and $w = vu$. Since $(1 - uu^*)^2 = 1 - uu^*$, we have

$$\begin{aligned} T = &\begin{pmatrix} 1 \\ u \\ vu \end{pmatrix} \begin{pmatrix} 1 & u^* & u^*v^* \end{pmatrix} + \begin{pmatrix} 0 \\ 1 - uu^* \\ v(1 - uu^*) \end{pmatrix} \begin{pmatrix} 0 & 1 - uu^* & (1 - uu^*)v^* \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - vv^* \end{pmatrix} \geq 0. \end{aligned}$$

Conversely, Assume $T \geq 0$. Since $\begin{pmatrix} 1 & v^* \\ v & 1 \end{pmatrix}$ is positive, we have $\|v\| \leq 1$. For any vectors $x, y \in \mathcal{H}$, we have

$$\begin{aligned} 0 &\leq \left\langle T \begin{pmatrix} x \\ -ux \\ y \end{pmatrix}, \begin{pmatrix} x \\ -ux \\ y \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} w^*y \\ v^*y \\ wx - vux + y \end{pmatrix}, \begin{pmatrix} x \\ -ux \\ y \end{pmatrix} \right\rangle \\ &= \langle w^*y, x\rangle + \langle v^*y, -ux\rangle + \langle wx - vux + y, y\rangle \\ &= \langle y, wx - vux\rangle + \langle wx - vux + y, y\rangle. \end{aligned}$$

Set $y = -(wx - vux)$, then $-\|wx - vux\|^2 \geq 0$. This implies $w = vu$. $\square$

Proof of Theorem 1. By the assumption, we have $a_0 > 0$. We want to show that $a_i = a_0(a_i/a_0)^i$ for any $i = 1, 2, ..., N$.

For each $i = 2, 3, ..., N$, the matrix

$$\begin{pmatrix} 1 & \overline{(a_1/a_0)} & \frac{\overline{(a_i/a_0)}}{\overline{(a_{i-1}/a_0)}} \\ a_1/a_0 & 1 & \overline{(a_{i-1}/a_0)} \\ a_i/a_0 & a_{i-1}/a_0 & 1 \end{pmatrix} = \frac{1}{a_0} E_{3,i} T E_{3,i}^* \geq 0,$$

where $E_{3,i}$ is a $3 \times (N+1)$ matrix and its (a, b)-th component is

$$e_{a,b} = \begin{cases} 1 & ; \text{ if } (a, b) = (1, 1), (2, 2), (3, i+1) \\ 0 & ; \text{ otherwise} \end{cases}.$$

Since $|a_1/a_0| = 1$, we have $a_i/a_0 = (a_1/a_0)(a_{i-1}a_0)$ for any $i = 2, 3, ..., N$ by (†). This implies $a_i = a_0(a_1/a_0)^i$ for all $i = 1, 2, ..., N$. By setting $f(n) =$

$a_0(a_1/a_0)^n$ for any $n \in \mathbb{N}$ and continuing the above argument to the larger number than N, then f is a positive definite sequence with $f(i) = a_i$ for any $i = 0, 1, ..., N$ and

$$f(n) = f(0)\left(\frac{f(1)}{f(0)}\right)^n \text{ for any } n \in \mathbb{N} \setminus \{0\}.$$

We assume that there exists another positive definite sequence g with $g(i) = a_i$ for any $i = 0, 1, ..., N$. For $M > N$, we then have

$$\begin{pmatrix} g(0) & \overline{g(1)} & \cdots & \overline{g(M)} \\ g(1) & g(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{g(1)} \\ g(M) & \cdots & g(1) & g(0) \end{pmatrix} \geq 0.$$

Since $f(0) = g(0), f(1) = g(1)$ and $|g(0)| = |g(1)|$, we have

$$g(M) = g(0)\left(\frac{g(1)}{g(0)}\right)^M = f(0)\left(\frac{f(1)}{f(0)}\right)^M = f(M)$$

by the above argument. So, $f = g$. $\square$

Corollary 3. *Let f be a positive definite sequence. If there exists a positive integer K in which $f(0) = |f(K)|$, then*

$$f(nK) = f(0)\left(\frac{f(K)}{f(0)}\right)^n \text{ for any } n = 1, 2, ... \ .$$

Proof. We may assume that $f(0) > 0$. Define the $(n + 1) \times (nK + 1)$ matrix $F_{n,K}$ whose (a, b)-th component is

$$f_{a,b} = \begin{cases} 1 & ; \text{ if } (a, b) = (i + 1, iK + 1) \text{ for } i = 0, 1, ..., n \\ 0 & ; \text{ otherwise} \end{cases}.$$

Then we have

$$F_{n,K} \begin{pmatrix} f(0) & \overline{f(1)} & \cdots & \overline{f(nK)} \\ f(1) & f(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{f(1)} \\ f(nK) & \cdots & f(1) & f(0) \end{pmatrix} F_{n,K}^* \geq 0$$

and this matrix is equal to

$$\begin{pmatrix} f(0) & \overline{f(K)} & \cdots & \overline{f(nK)} \\ f(K) & f(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \overline{f(K)} \\ f(nK) & \cdots & f(K) & f(0) \end{pmatrix}.$$

Hence, by Theorem 1 we have

$$f(nK) = f(0)\left(\frac{f(K)}{f(0)}\right)^n.$$

$\square$

In the setting of Corollary 3 we have $f(0) = |f(nK)|$ for all $n \in \mathbb{N}$. In general, the sequence $\{|f(n)|\}$ is not necessarily constant. For instance, consider the function $f(n) = e^{\frac{2}{3}\pi n\sqrt{-1}}$. It is clear that f and $\overline{f}$ are positive definite sequences. Then, so is

$$g(n) = \frac{f(n) + \overline{f(n)}}{2} = \cos\left(\frac{2n\pi}{3}\right),$$

here we have

$$g(n) = \begin{cases} 1 & ;\ \text{if } n = 0, 3, 6, 9, ... \\ -\frac{1}{2} & ;\ \text{if } n = 1, 2, 4, 5, 7, 8, ... \end{cases}.$$

Let G be a group and e the unit of G. We say a complex-valued function φ on G is positive definite if for any positive integer N and for any $g_1, g_2, ..., g_N \in G$, the following $N \times N$ matrix

$$\begin{pmatrix} \varphi(g_1^{-1}g_1) & \varphi(g_2^{-1}g_1) & \cdots & \varphi(g_N^{-1}g_1) \\ \varphi(g_1^{-1}g_2) & \varphi(g_2^{-1}g_2) & \cdots & \varphi(g_N^{-1}g_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(g_1^{-1}g_N) & \varphi(g_2^{-1}g_N) & \cdots & \varphi(g_N^{-1}g_N) \end{pmatrix} \geq 0.$$

By definition, $\varphi(e) \geq 0$, $\varphi(g^{-1}) = \overline{\varphi(g)}$, and $|\varphi(g)| \leq \varphi(e)$ and for any $g \in G$.

Corollary 4. *Let φ be a positive definite function on G with $\varphi(e) \neq 0$ and K a subgroup of $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then*

$$H = \left\{ g \in G \,\middle|\, \frac{\varphi(g)}{\varphi(e)} \in K \right\}$$

is a subgroup of G and the function $\frac{1}{\varphi(e)}\varphi$ is multiplicative on H.

Proof. It is obvious that $e \in H$ and if $g \in H$, then $g^{-1} \in H$. Given $g, h \in H$, then by the assumption we have that the following matrix

$$\begin{pmatrix} \varphi(e) & \varphi(g^{-1}) & \varphi((gh)^{-1}) \\ \varphi(g) & \varphi(g^{-1}g) & \varphi((gh)^{-1}g) \\ \varphi(gh) & \varphi(g^{-1}(gh)) & \varphi((gh)^{-1}(gh)) \end{pmatrix} = \begin{pmatrix} \varphi(e) & \overline{\varphi(g)} & \overline{\varphi(gh)} \\ \varphi(g) & \varphi(e) & \overline{\varphi(h)} \\ \varphi(gh) & \varphi(h) & \varphi(e) \end{pmatrix} \geq 0.$$

By (†), we have

$$\frac{\varphi(gh)}{\varphi(e)} = \frac{\varphi(g)}{\varphi(e)} \frac{\varphi(h)}{\varphi(e)}.$$

It follows that $\varphi(gh)/\varphi(e) \in K$. That is, $gh \in H$. □

Let φ be a positive definite sequence, that is, a positive definite function on $\mathbb{Z}$. By Bochner's theorem (or Herglotz's theorem [5]), there exists a positive finite measure μ on $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ which is identified by $[0, 1)(\cong \mathbb{R}/\mathbb{Z})$ such that

$$\varphi(n) = \int_0^1 e^{2\pi\sqrt{-1}nx} d\mu(x) \quad \text{for all } n \in \mathbb{Z}.$$

It is known that

$$\mu(\{0\}) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n).$$

To see this, it suffices to show that

$$\mu(\{0\}) = 0 \Rightarrow \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n) = 0$$

by considering $\mu - \mu(\{0\})\delta_0$ instead of μ, where δ_0 is a Dirac measure at 0. Since $|\sin x| \leq |x|$ and $\frac{2x}{\pi} \leq \sin x$ for $x \in \left[0, \frac{\pi}{2}\right]$, we have

$$\begin{aligned} \left| \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n) \right| &= \left| \frac{1}{2N+1} \sum_{n=-N}^{N} \int_0^1 e^{2\pi\sqrt{-1}nx} d\mu(x) \right| \\ &= \left| \int_0^1 \frac{1}{2N+1} \frac{\sin(2N+1)\pi x}{\sin \pi x} d\mu(x) \right| \\ &\leq \int_{-\delta}^{\delta} \left| \frac{1}{2N+1} \frac{\sin(2N+1)\pi x}{\sin \pi x} \right| d\mu(x) \\ &\quad + \int_{\delta}^{1-\delta} \left| \frac{1}{2N+1} \frac{\sin(2N+1)\pi x}{\sin \pi x} \right| d\mu(x) \\ &\leq \frac{\pi}{2} \mu((-\delta, \delta)) + \frac{1}{(2N+1)\sin \pi\delta} \mu(\mathbb{T}). \end{aligned}$$

for any $\delta \in \left(0, \frac{1}{2}\right)$. Hence, $\limsup_{N\to\infty} \left| \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n) \right| \leq \frac{\pi}{2}\mu((-\delta,\delta))$. Since δ is arbitrary and $\mu(\{0\}) = 0$, then we have $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n) = 0$.

Proposition 5. *Let φ be a positive definite function on $\mathbb{Z}$. If $\lim_{n\to\infty} \varphi(n) = \varphi(0)$, then $\varphi(n) = \varphi(0)$ for all $n \in \mathbb{Z}$.*

Proof. Let $\varphi(n) = \int_0^1 e^{2\pi\sqrt{-1}nx} d\mu(x)$ $(n \in \mathbb{Z})$. Then, $\varphi(0) = \mu(\mathbb{T})$. Also, we have $\mu(\{0\}) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n) = \varphi(0)$. This means that μ is a non-negative scalar multiple of Dirac measure at 0 and so we have $\varphi(n) = \varphi(0)$ for all $n \in \mathbb{Z}$. $\square$

Corollary 6. *Let φ be a positive definite function on a group G and G is generated by $\{g_i \mid i \in I\}$. If*

$$\lim_{n\to\infty} \varphi(g_i^n) = \varphi(e) \quad \textit{for all } i \in I,$$

then $\varphi(g) = \varphi(e)$ for all $g \in G$.

Proof. We may assume that $\varphi(e) \neq 0$. By assumption and since $\lim_{n\to\infty} \varphi(g_i^n) = \varphi(e)$, we have $\varphi(g_i^n) = \varphi(e)$ for all n by Proposition 5. In particular $\varphi(g_i) = \varphi(e)$ for all $i \in I$. Set

$$H = \left\{ g \in G \;\middle|\; \frac{\varphi(g)}{\varphi(e)} \in \{1\} \right\}.$$

Using Corollary 4, we conclude that H is a subgroup of G. Since G is generated by $\{g_i \mid i \in I\}$ and $g_i \in H$ for any $i \in I$, we have $\varphi(g) = \varphi(e)$ for all $g \in G$. $\square$

Remark. Let φ be a positive definite function on the additive group $\mathbb{R}$. We assume that the sequence $\{\varphi(nx)\}_{n=1}^{\infty}$ converges to $\varphi(0)$ for any $x \in \mathbb{R}$. If φ is continuous, then there exists a finite positive measure μ on $\mathbb{R}$ such that $\varphi(x) = \int_{\mathbb{R}} e^{\sqrt{-1}tx} d\mu(t)$ $(x \in \mathbb{R})$ by Bochner's theorem. We can prove $\varphi(x) = \varphi(0)$ by using the fact

$$\mu(\{0\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(x) d\mu(x)$$

(see [3]:Appendices A.4). Without the assumption of the continuity of φ, we can also have $\varphi(x) = \varphi(0)$ by Corollary 6.

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