# EXPONENTIAL ATTRACTORS FOR SELF-REGULATING HOMEOSTASIS MODEL ON A SPHERE

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Abstract. This paper is devoted to studying a complete two-dimensional Daisyworld model on a sphere. The Daisyworld model which has been originally introduced by Andrew Watson and James Lovelock (1983) describes the process of planetary self-regulating homeostasis by a biota and its environment. After formulating our two-dimensional model, we construct global solutions, dynamical systems and exponential attractors. We also show some numerical results suggesting pattern formation of stripe segregation.

**1 Introduction** We are concerned with the initial-boundary value problem for a reaction-diffusion system

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + [(1-u-v)\Phi(u,v,w) - f]\, u & \text{in} \quad S\times(0,\infty), \\ \dfrac{\partial v}{\partial t} = d\Delta v + [(1-u-v)\Psi(u,v,w) - f]\, v & \text{in} \quad S\times(0,\infty), \\ \dfrac{\partial w}{\partial t} = D\Delta w + [1-g(u,v)]R(\omega) - \sigma w^4 & \text{in} \quad S\times(0,\infty), \\ u(\omega,0)=u_0(\omega), \ \ v(\omega,0)=v_0(\omega), \ \ w(\omega,0)=w_0(\omega) & \text{in} \quad S, \end{cases}$$

on a sphere $S \subset \mathbb{R}^3$. This is a tutorial mathematical model originally introduced by Watson-Lovelock [20] in order to investigate how the mechanism of global homeostasis works in Daisyworld which was ideally set as a biological and climatological system. Daisyworld is an imaginary planet that has only two types of daisies with contrasting brightness. They are expressly referred to as white and black daisy. On the planet, there are enough water and nutrients to sustain daisies, and thus the temperature is an only factor affecting the growth of daisies (for the details, see the review of Wood-Ackland-Dyke-Williams-Lenton [22]).

Unknown functions $u = u(\omega,t)$ and $v = v(\omega,t)$ denote a coverage rate of white and black daisy, respectively, at position $\omega \in S$ and time $t$. Therefore, $u \geq 0$, $v \geq 0$ and $u+v \leq 1$ at any $(\omega,t)$, and $1-u-v$ denotes a rate of uncovered ground. The third unknown function $w = w(\omega,t)$ denotes a surface temperature. We assume that $u$ and $v$ satisfy a diffusion equation on $S$ with diffusion rate $d > 0$. It is the same for $w$ with diffusion rate $D > 0$. So, $\Delta$ denotes a Laplace operator on the sphere $S$. It is natural to assume that $0 < d < D$. The function $g(u,v)$ stands for an averaged albedo of the surface that is given at each point as a function of $u$, $v$ in the form

$$\begin{aligned} g(u,v) &= a_w u + a_b v + a_g(1-u-v) \\ &= (a_w - a_g)u + (a_b - a_g)v + a_g, \end{aligned}$$

where $a_w$, $a_b$ and $a_g$ denote the proper albedo of white daisy, black daisy and bare ground, respectively. In general, we have $0 < a_b < a_g < a_w < 1$; as a consequence, it is always the case that

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$a_b \le g(u,v) \le a_w$. Furthermore, $\Phi(u,v,w)$ and $\Psi(u,v,w)$ denote a growth rate of white and black daisy, respectively. According to [20], they are set as

$$\Phi(u,v,w) = \{1 - \delta(\overline{w} - w - q[g(u,v) - a_w])^2\}_+,$$
$$\Psi(u,v,w) = \{1 - \delta(\overline{w} - w - q[g(u,v) - a_b])^2\}_+.$$

Here, $\overline{w}$ is a fixed optimal temperature for growing for both white daisy and black daisy. The term $q[g(u,v) - a_w]$ (resp. $q[g(u,v) - a_b]$) means some suitable adjustment on a local temperature to the global one (i.e., $w$) at each position where white daisy (resp. black daisy) grows, $q > 0$ being some coefficient. Since $g(u,v) \le a_w$ (resp. $g(u,v) \ge a_b$), we see that $w$ is always adjusted negatively (resp. positively) where white daisy (resp. black daisy) grows. The notation $\{w\}_+ = \max\{w, 0\}$ denotes a positive cutoff of the function $w$ for $-\infty < w < \infty$; consequently, $\{1 - \delta(\overline{w} - w)^2\}_+$ is a positive cutoff of the square function $1 - \delta(\overline{w} - w)^2$ for $-\infty < w < \infty$, $\delta > 0$ being some coefficient. Both white daisy and black daisy die at a rate $f > 0$. Finally, the term $[1 - g(u,v)]R(x)$ denotes an increasing rate of the global temperature which is determined by the averaged albedo $g(u,v)$ mentioned above and the incoming energy $R(\omega)$ from the sun which is a function of $\omega \in S$ hitting its maximum on the equator and vanishing at the two poles. And, the term $-\sigma w^4$ denotes a decaying rate of the temperature due to the Stefan-Boltzmann law, $\sigma > 0$ being the Stefan-Boltzmann constant of the surface.

A planetary biota modifies its environment and its environment regulates a biota by natural selection. Self-regulating homeostatic system is an idea that the feedback between a biota and its environment keeps the planetary surface environment stable and habitable for a biota. Daisyworld has been introduced by Lovelock [13] as a simple parable to verify a hypothesis that the Earth maintain self-regulating homeostasis (see Lovelock-Margulis [14] and Lenton [12]). In the original Daisyworld model due to Watson-Lovelock [20], the whole planet is regarded as a single point. The model is governed by rather simple rules: black daisies absorb more incoming energy, while white daisies reflect more. They showed that the competition of daisies controls the global albedo of the planet and regulate the global temperature to be more suitable for daisies. Their results suggested a possibility that the Daisyworld model is very valuable for understanding the mechanisms of self-regulating homeostasis of the Earth. The Daisyworld model was analyzed by several authors (e.g., [7, 16]), on the other hand, that inspired many modifications and extensions. Adams-Carr [2] and Adams-Carr-Lenton-White [3] extended the original model to one-dimensional one including variation of incoming solar energy and heat diffusion on the sphere. The one-dimensional model retains the temperature regulation and shows a stripe pattern that shows two types of daisies segregate. A two-dimensional extension of the Daisyworld model based on cellular automata was introduced by von Bloh-Block-Schellnhuber [19]. In [19], the equation for heat transfer on Daisyworld is governed by a simple energy balance equation and the spatial distribution of daisies are determined by discretized equations. Additional extensional models based on the two-dimensional one were presented (e.g., [1, 23]).

In this paper, we want to consider a complete two-dimensional version of the Daisyworld model adding diffusion terms of daisies on a sphere. After formulating our model as a reaction-diffusion equations on the sphere, we will analytically construct local solutions, global solutions, dynamical systems and exponential attractors. In the last section, we will show some numerical results suggesting two-dimensional pattern formation of the Turing type.

We denote by $S$ a sphere given by

$$S \equiv \{\omega = (x,y,z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 = \ell^2\}$$

with radius $\ell > 0$. And, $\Delta$ denotes a Laplace operator on $S$, namely, $\Delta$ is a Laplace-Beltrami operator on $S$ whose definition will be reviewed in the next section. Following [3], we assume that the solar energy incident on the surface is parallel to the latitude lines. The incoming energy $R(\omega)$ thus arrives symmetrically with respect to the equator, and it is given by

$$R(\omega) = R_0\sqrt{1-(z/\ell)^2}, \qquad \omega = (x, y, z) \in S, \tag{1.2}$$

with some coefficient $R_0 > 0$.

**2 Diffusion equations on $S$.** The theory of diffusion equations on Riemannian manifolds is already well known (see, e.g., [5]). It is however constructed in a very general context only using Riemannian metrics and without using any information in which Euclidean spaces the manifolds are embedded. On the other hand, in order to treat nonlinear diffusion equations like (1.1), the functional analytical approach has a great advantage over other ones.

By this reason, we want to review in this section the theory of diffusion equations on $S$ using the fact that $S$ is a special Riemannian manifold embedded in $\mathbb{R}^3$.

**2.1 Local coordinates of $S$.** We will use two polar coordinates in $\mathbb{R}^3$. First one is the usual one. Let $H_1 = \{(x, y, z) \in \mathbb{R}^3;\ y = 0,\ x \geq 0\}$. For $P = (x, y, z) \in \mathbb{R}^3 - H_1$, put

$$\begin{cases} x = r\sin\theta\cos\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\theta, \end{cases} \tag{2.1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, $\theta$ is the zenithal angle of $OP$ and $z$-axis and $\phi$ is the azimuthal angle of $OQ$ and $x$-axis, $Q = (x, y, 0)$ being the projected point of $P$ on the $xy$-plane. Therefore $(r, \theta, \phi)$ varies in $0 < r < \infty$, $0 < \theta < \pi$ and $0 < \phi < 2\pi$.

Second one is defined in $\mathbb{R}^3 - H_2$, where $H_2 = \{(x, y, z) \in \mathbb{R}^3;\ z = 0, x \leq 0\}$. For $P = (x, y, z) \in \mathbb{R}^3 - H_2$, put

$$\begin{cases} x = r\sin\vartheta\cos\varphi, \\ z = r\sin\vartheta\sin\varphi, \\ y = r\cos\vartheta, \end{cases} \tag{2.2}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, $\vartheta$ is the zenithal angle of $OP$ and $y$-axis and $\varphi$ is the azimuthal angle of $OQ$ and $x$-axis, $Q = (x, 0, z)$ being the projected point of $P$ on the $xz$-plane. Now $(r, \vartheta, \varphi)$ varies in $0 < r < \infty$, $0 < \vartheta < \pi$ and $-\pi < \varphi < \pi$.

These polar coordinates immediately provide a local coordinate system for $S$. Let $S_1 = S - \Gamma_1$, where $\Gamma_1 = S \cap H_1$. Then, fixing $r = \ell$, we get from (2.1) a homeomorphism $\Theta_1 : S_1 \to D_1$ with $D_1 = \{(\theta, \phi);\ 0 < \theta < \pi, 0 < \phi < 2\pi\}$. Similarly, setting $S_2 = S - \Gamma_2$, where $\Gamma_2 = S \cap H_2$, we get from (2.2) a homeomorphism $\Theta_2 : S_2 \to D_2$ with $D_2 = \{(\vartheta, \varphi);\ 0 < \vartheta < \pi, -\pi < \varphi < \pi\}$. By $\{(S_i, \Theta_i)\}_{i=1,2}$, $S$ becomes a differentiable manifold.

Let $\{\psi_i(\omega)\}_{i=1,2}$ be a partition of unity subordinate to $\{S_i, \Theta_i\}$, that is, $\psi_i(\omega)$ are smooth functions on $S$ such that $0 \leq \psi_i(\omega) \leq 1$, $\psi_1(\omega) + \psi_2(\omega) \equiv 1$ on $S$ and $\operatorname{supp}\psi_i \subset S_i$. We need also suitable rectangular domains $G_i \subset D_i$. For $i = 1, 2$, let $G_i$ be a rectangular domain such that

$$\Theta_i(\operatorname{supp}\psi_i) \subset G_i \subset \overline{G}_i \subset D_i.$$

We equip $S$ with the surface measure $d\omega$. For $1 \le p \le \infty$, $L_p(S)$ is the space of all measurable functions such that $|f(\omega)|^p$ is integrable on $S$. By the usual $L_p$-norm, $L_p(S)$ is a Banach space. When $p = 2$, $L_2(S)$ is a Hilbert space with the usual inner product. Of course, $f \in L_p(S)$ if and only if $\psi_i f \in L_p(S_i)$ for $i = 1,\ 2$; furthermore, $\psi_i f \in L_p(S_i)$ if and only if $(\psi_i f) \circ \Theta_i^{-1} \in L_p(G_i)$ with norm equivalence of $\|\psi_i f\|_{L_p(S_i)}$ and $\|(\psi_i f) \circ \Theta_i^{-1}\|_{L_p(G_i)}$.

**2.2 Laplace-Beltrami operator on $S$.** Let us denote by $\nabla_S$ the gradient operator acting to the differentiable functions on $S$.

In $S_1$, $\nabla_S u$ is described by the polar coordinate (2.1) in the form

$$\nabla_S u = \frac{1}{\ell}\left(\cos\theta\cos\phi\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{\sin\theta}\frac{\partial u}{\partial\phi},\ \cos\theta\sin\phi\frac{\partial u}{\partial\theta} + \frac{\cos\phi}{\sin\theta}\frac{\partial u}{\partial\phi},\ -\sin\theta\frac{\partial u}{\partial\theta}\right). \tag{2.3}$$

If $\omega \in S_1$, then the normal vector for $S$ at $\omega$ is given by $n_\omega = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Thereby, it is directly verified that $\nabla_S u(\omega) \cdot n_\omega = 0$, i.e., $\nabla_S u(\omega)$ is a tangential vector of $S$ at $\omega$.

It is the same for the description of $\nabla_S$ on $S_2$.

We can then give a definition of the first order Sobolev space $H^1(S)$ on $S$ using $\nabla_S$. In fact, $H^1(S)$ is the space of all functions $u \in L_2(S)$ for which $|\nabla_S u|$ also belong to $L_2(S)$. It is easy to see that $u \in H^1(S)$ if and only if $\psi_i u \in H^1(S)$ for $i = 1, 2$. Furthermore, in $S_1$ it follows from (2.3) that

$$|\nabla_S u|^2 = \frac{1}{\ell^2}\left[\left(\frac{\partial u}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial u}{\partial\phi}\right)^2\right].$$

Hence, $\psi_1 u \in H^1(S)$ if and only if $(\psi_1 u) \circ \Theta_1^{-1} \in H^1(G_1)$. It is the same for $\psi_2 u \in H^1(S)$. We equip $H^1(S)$ with the inner product

$$(u, v)_{H^1} = \int_S (\nabla_S u \cdot \nabla_S \overline{v} + u\overline{v})d\omega, \qquad u,\ v \in H^1(S).$$

Then, $H^1(S)$ becomes a Hilbert space. The norm $\|\psi_i u\|_{H^1}$ is equivalent to $\|(\psi_i u) \circ \Theta_i^{-1}\|_{H^1(G_i)}$ for $i = 1, 2$.

We are now led to define the Laplace-Beltrami operator $\Delta_S$ by

$$\Delta_S = \nabla_S \cdot \nabla_S. \tag{2.4}$$

In view of (2.3), in $S_1$ we observe that

$$\Delta_S u = \frac{1}{\ell^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right]. \tag{2.5}$$

In $S_2$, too, the similar description of $\Delta_S$ by means of (2.2) is verified.

**2.3 Realization of $\Delta_S$ in $L_2(S)$.** In order to formulate (1.1) in the space $L_2(S)$, we have to define $\Delta_S$ as a linear operator acting in $L_2(S)$. For this purpose, we consider the sesquilinear form $a(u, v) = (u, v)_{H^1}$, $u, v \in H^1(S)$. Trivially, $a(u, v)$ is continuous and coercive on $H^1(S)$. For each $u \in H^1(S)$, the mapping $v \mapsto a(u, v)$ is an anti-linear continuous functional on $H^1(S)$. Then, elements $u \in H^1(S)$ for which the mappings are continuous in the topology of $L_2(S)$ are picked up. By the Riesz theorem, for such a $u$, there exists a unique element $f \in L_2(S)$ such that $a(u,v) = (f,v)_{L_2}$ for all $v \in H^1(S)$. We then set $\Lambda u = f$, that is,

$$\begin{cases} \mathcal{D}(\Lambda) = \{u \in H^1(S);\ (u,v)_{H^1} = (f,v)_{L_2} \ \text{ for all } v \in H^1(S)\}, \\ \Lambda u = f. \end{cases}$$

It is immediate to see that $u \mapsto f = \Lambda u$ is a linear operator from $\mathcal{D}(\Lambda)$ into $L_2(S)$. Moreover, the theory of variation (see Dautray-Lions [6]) provides that $\mathcal{D}(\Lambda)$ is dense in $L_2(S)$ and $\Lambda$ is a positive definite self-adjoint operator of $L_2(S)$. Furthermore, $\mathcal{D}(\Lambda)$ is shown to coincide with the second order Sobolev space $H^2(S)$ which consists of functions $u \in L_2(S)$ such that $(\psi_i u) \circ \Theta_i^{-1} \in H^2(G_i)$ for $i = 1, 2$.

We here set $A = \Lambda - 1$ with $\mathcal{D}(A) = \mathcal{D}(\Lambda) = H^2(S)$. By definition, it holds for $u \in \mathcal{D}(A)$ that

$$(Au, v)_{L_2} = \int_S \nabla_S u \cdot \nabla_S \overline{v} d\omega \qquad \text{for all } v \in H^1(S).$$

Therefore, $A$ is a nonnegative self-adjoint operator of $L_2(S)$. And, $Au = 0$ implies $|\nabla_S u|^2 \equiv 0$ and hence $u \equiv \text{const}$. In the meantime, by integration by parts we verify that $Au = -\Delta_S u$ for $u \in \mathcal{D}(A)$. Hence, $A$ is a realization of $-\Delta_S$ in the space $L_2(S)$. Since $\mathcal{D}(A)$ is compactly embedded in $L_2(\Omega)$, $A$ can be decomposed of the form

$$Au = \sum_{k=0}^{\infty} \lambda_k (u, e_k)_{L_2} e_k, \tag{2.6}$$

where $\lambda_k$ are eigenvalues of $A$ and $e_k(\omega)$ are eigenfunctions of $A$ corresponding to $\lambda_k$, respectively. Clearly,

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \to \infty.$$

Meanwhile, $e_k(\omega)$ compose an orthonormal basis of $L_2(S)$. As noticed above, $e_0(\omega)$ must be a constant function on $S$, hence $e_0(\omega) \equiv \frac{1}{2\sqrt{\pi}\ell}$.

**2.4 Semigroup generated by $-A$.** Since $A$ is a nonnegative self-adjoint operator, $-A$ generates an analytic and contraction semigroup $e^{-tA}$, $0 \le t < \infty$, on $L_2(S)$. As the minimal eigenvalue $\lambda_0$ is zero, we have just

$$\|e^{-tA}\|_{\mathcal{L}(L_2(S))} = 1 \qquad \text{for any } 0 \le t < \infty. \tag{2.7}$$

For any initial function $u_0 \in L_2(S)$, $e^{-tA}$ gives a unique solution to the Cauchy problem of diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_S u & \text{in} \quad S \times (0,\infty), \\ u(\omega, 0) = u_0(\omega) & \text{in} \quad S, \end{cases} \tag{2.8}$$

on $S$. Indeed, $u(t) = e^{-tA}u_0$ is a unique solution to (2.8) in the function space:

$$u \in \mathcal{C}((0,\infty); H^2(S)) \cap \mathcal{C}([0,\infty); L_2(S)) \cap \mathcal{C}^1((0,\infty); L_2(S)).$$

From (2.6), $e^{-tA}u_0$ can be expressed by

$$e^{-tA}u_0 = \sum_{k=0}^{\infty} e^{-\lambda_k t}(u_0, e_k)_{L_2} e_k, \qquad u_0 \in L_2(S). \tag{2.9}$$

The formula (2.9) immediately provides various properties of the solution $u(t)$ to (2.8) as follows:

(1) If $u_0 \geq 0$, then $e^{-tA}u_0 \geq 0$ for any $0 < t < \infty$.

(2) It holds that $\int_S e^{-tA}u_0 d\omega = \int_S u_0 d\omega$ for any $0 < t < \infty$.

(3) Let $P_0 u_0 = (u_0, e_0)_{L_2} e_0$ be the projection from $L_2(S)$ onto the eigenspace of $\lambda_0 = 0$. Then,

$$\|e^{-tA} - P_0\|_{\mathcal{L}(L_2(S))} \leq e^{-\lambda_1 t} \qquad \text{for any } 0 \leq t < \infty. \tag{2.10}$$

(4) As an operator from $L_2(S)$ into $H^1(S)$, $e^{-tA}$ satisfies

$$\|\nabla_S e^{-tA}\|_{\mathcal{L}(L_2(S))} \leq \left(\lambda_1 + \frac{1}{et}\right)^{\frac{1}{2}} e^{-\lambda_1 t} \qquad \text{for any } 0 < t < \infty. \tag{2.11}$$

(5) As an operator from $L_2(S)$ into $H^2(S)$, $e^{-tA}$ satisfies

$$\|\Delta_S e^{-tA}\|_{\mathcal{L}(L_2(S))} \leq \left(\lambda_1 + \frac{1}{et}\right) e^{-\lambda_1 t} \qquad \text{for any } 0 < t < \infty. \tag{2.12}$$

The estimate (2.10) follows from

$$\|e^{-tA}u_0 - P_0 u_0\|_{L_2} = e^{-\lambda_1 t}\left(\sum_{k=1}^{\infty} e^{-2(\lambda_k - \lambda_1)t}|(u_0, e_k)_{L_2}|^2\right)^{\frac{1}{2}} \leq e^{-\lambda_1 t}\|u_0\|_{L_2}.$$

Similarly, the estimate (2.12) follows from

$$\begin{aligned}
\|\Delta_S e^{-tA}u_0\|_{L_2} = \|Ae^{-tA}u_0\|_{L_2} &= \left\|\sum_{k=1}^{\infty} \lambda_k e^{-\lambda_k t}(u_0, e_k)_{L_2} e_k\right\|_{L_2} \\
&= e^{-\lambda_1 t}\left\|\sum_{k=1}^{\infty} [t^{-1}(\lambda_k - \lambda_1)t + \lambda_1] e^{-(\lambda_k - \lambda_1)t}(u_0, e_k)_{L_2} e_k\right\|_{L_2} \\
&\leq e^{-\lambda_1 t}[t^{-1}e^{-1} + \lambda_1]\|u_0\|_{L_2}.
\end{aligned}$$

Finally, the estimate (2.11) is observed by

$$\begin{aligned}
\|\nabla_S e^{-tA}u_0\|_{L_2}^2 &= (Ae^{-tA}u_0, [e^{-tA} - P_0]u_0)_{L_2} \leq \|Ae^{-tA}u_0\|_{L_2}\|[e^{-tA} - P_0]u_0\|_{L_2} \\
&\leq \|Ae^{-tA}\|_{\mathcal{L}(L_2(S))}\|e^{-tA} - P_0\|_{\mathcal{L}(L_2(S))}\|u_0\|_{L_2}^2.
\end{aligned}$$

**3 Construction of Solutions.** In this section, we shall construct global solutions to the Cauchy problem (1.1) and a dynamical system generated by them. We begin with formulating (1.1) in an abstract form (cf. [11, 17, 24]).

**3.1 Abstract formulation.** We consider (1.1) in the product $L_2$-space

$$X = \left\{ U = \begin{pmatrix} u \\ v \\ w \end{pmatrix};\ u \in L_2(S),\ v \in L_2(S),\ w \in L_2(S) \right\},$$

$X$ being a complex Hilbert space by the usual inner product. Let $\mathcal{A}$ be a linear operator acting in $X$ which is given by

$$\mathcal{A} = \begin{pmatrix} dA & 0 & 0 \\ 0 & dA & 0 \\ 0 & 0 & DA \end{pmatrix},$$

where $A$ is the realization of $-\Delta_S$ in $L_2(S)$ introduced in Section 2. Then, $\mathcal{A}$ is a nonnegative self-adjoint operator of $X$ and generates an analytic semigroup $e^{-t\mathcal{A}}$ which is expressed by $\operatorname{diag}\{e^{-tdA}, e^{-tdA}, e^{-tDA}\}$ on $X$. It then follows from (2.7) that

$$\|e^{-t\mathcal{A}}\|_{\mathcal{L}(X)} \le 1 \qquad \text{for } 0 \le t < \infty. \tag{3.1}$$

Moreover, it is seen from (2.11) that

$$\|\nabla_S e^{-tdA}\|_{\mathcal{L}(L_2(S))} \le \left(\lambda_1 + \frac{1}{edt}\right)^{\frac{1}{2}} e^{-d\lambda_1 t} \qquad \text{for } 0 < t < \infty,$$

$$\|\nabla_S e^{-tDA}\|_{\mathcal{L}(L_2(S))} \le \left(\lambda_1 + \frac{1}{eDt}\right)^{\frac{1}{2}} e^{-D\lambda_1 t} \qquad \text{for } 0 < t < \infty.$$

As a consequence,

$$\|\nabla_S e^{-t\mathcal{A}}\|_{\mathcal{L}(X)} \le \left(\lambda_1 + \frac{1}{edt}\right)^{\frac{1}{2}} e^{-d\lambda_1 t} \qquad \text{for } 0 < t < \infty. \tag{3.2}$$

From the view point of modeling, we may expect that the solutions exist in the ranges of $u \ge 0$, $v \ge 0$, $u + v \le 1$ and $0 \le w \le (R_0/\sigma)^{\frac{1}{4}}$. On account of these range conditions, we introduce a nonlinear operator $\mathcal{F}$ of $X$ by

$$\mathcal{F}(U) = \begin{pmatrix} [\chi_1(1 - \operatorname{Re} u - \operatorname{Re} v)\Phi(\chi_1(\operatorname{Re} u), \chi_1(\operatorname{Re} v), \chi_2(\operatorname{Re} w)) - f]\chi_1(\operatorname{Re} u) \\ [\chi_1(1 - \operatorname{Re} u - \operatorname{Re} v)\Psi(\chi_1(\operatorname{Re} u), \chi_1(\operatorname{Re} v), \chi_2(\operatorname{Re} w)) - f]\chi_1(\operatorname{Re} v) \\ [1 - g(\chi_1(\operatorname{Re} u), \chi_1(\operatorname{Re} v))]R(\omega) - \sigma\chi_2(\operatorname{Re} w)^4 \end{pmatrix}.$$

Here, $\chi_1(\xi)$ and $\chi_2(\xi)$ are cutoff functions defined by

$$\chi_1(\xi) = \begin{cases} 0, & -\infty < \xi \le 0, \\ \xi, & 0 < \xi \le 1, \\ 1, & 1 < \xi < \infty, \end{cases} \qquad \chi_2(\xi) = \begin{cases} 0, & -\infty < \xi \le 0, \\ \xi & 0 < \xi \le (R_0/\sigma)^{\frac{1}{4}}, \\ (R_0/\sigma)^{\frac{1}{4}}, & (R_0/\sigma)^{\frac{1}{4}} < \xi < \infty, \end{cases}$$

respectively.

The problem (1.1) is then formulated as the Cauchy problem

$$\begin{cases} \dfrac{dU}{dt} + \mathcal{A}U = \mathcal{F}(U), & 0 < t < \infty, \\ U(0) = U_0, \end{cases} \tag{3.3}$$

in $X$, where $U(t) = {}^t(u(t), v(t), w(t))$ is an unknown function and $U_0$ is an initial value. As for the space of initial values, we set

$$K = \left\{ U_0 = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} \in X;\ u_0 \geq 0,\ v_0 \geq 0,\ u_0 + v_0 \leq 1, 0 \leq w_0 \leq \left(\frac{R_0}{\sigma}\right)^{\frac{1}{4}} \right\}. \tag{3.4}$$

It is clear that $\chi_1(\xi)$ and $\chi_2(\xi)$ are uniformly bounded and globally Lipschitz continuous functions for $-\infty < \xi < \infty$. Consequently, $\Phi(\chi_1(\mathrm{Re}\, u), \chi_1(\mathrm{Re}\, v), \chi_2(\mathrm{Re}\, w))$ and $\Psi(\chi_1(\mathrm{Re}\, u), \chi_1(\mathrm{Re}\, v), \chi_2(\mathrm{Re}\, w))$ are uniformly bounded and globally Lipschitz continuous functions for $(u, v, w) \in \mathbb{C}^3$. Therefore, it is easily verified that $\mathcal{F}$ is a bounded operator and satisfies the Lipschitz condition, i.e.,

$$\|\mathcal{F}(U)\|_X \leq C_1, \qquad U \in X, \tag{3.5}$$

$$\|\mathcal{F}(U) - \mathcal{F}(V)\|_X \leq C_2 \|U - V\|_X, \qquad U, V \in X, \tag{3.6}$$

with suitable constants $C_i > 0\, (i = 1, 2)$.

It is then possible to apply the general theory of abstract parabolic evolution equations, see [24, Theorem 4.4], to (3.3) to obtain that, for any $U_0 \in K$, there exists a unique local solution to (3.3) in the function space:

$$U \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(\mathcal{A})) \cap \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}^1((0, T_{U_0}]; X).$$

Here, the time $T_{U_0} > 0$ is determined by the norm $\|U_0\|_X$ alone.

**3.2 Global solutions.** We can verify that the local solution $U(t)$ constructed above takes its values in $K$.

**Proposition 3.1.** *The condition $U_0 \in K$ implies that $U(t) \in K$ for any $0 < t \leq T_{U_0}$.*

*Proof.* It is easy to verify that, if $U(t)$ is a local solution of (3.3), then its complex conjugate $\overline{U(t)}$ is also a local solution with the same initial condition. Therefore, $U(t) = \overline{U(t)}$ and $U(t)$ is real valued.

Firstly, let us see that $u(t) \geq 0$. For this purpose, we use a $\mathcal{C}^2$-cutoff function $H(u)$ given by

$$H(u) = \begin{cases} \frac{1}{2}u^2 + \frac{1}{2}u + \frac{1}{6}, & -\infty < u \leq -1, \\ -\frac{1}{6}u^3, & -1 \leq u < 0, \\ 0, & 0 \leq u < \infty. \end{cases}$$

Put $h(t) = \int_S H(u(\omega, t))d\omega$. Then, for $0 < t \leq T_{U_0}$,

$$\begin{aligned} \frac{dh_1}{dt}(t) = \int_S H'(u) \frac{\partial u}{\partial t}\, d\omega = d \int_S H'(u) \Delta_S u\, d\omega \\ + \int_S H'(u)[\chi_1(1 - u - v)\Phi(\chi_1(u), \chi_1(v), \chi_2(w)) - f]\chi_1(u)\, d\omega. \end{aligned}$$

Since

$$\int_S H'(u) \Delta_S u d\omega = -\int_S \nabla_S H'(u) \cdot \nabla_S u d\omega = -\int_S H''(u) |\nabla_S u|^2 d\omega \leq 0$$

and since $H'(u)\chi_1(u) \equiv 0$ for $-\infty < u < \infty$, it follows that $\frac{dh}{dt}(t) \leq 0$, i.e., $h(t) \leq h(0) = 0$ for any $0 < t \leq T_{U_0}$.

The same arguments for $v(t)$ conclude that $v(t) \geq 0$ for any $0 < t \leq T_{U_0}$.

Secondly, in order to see that $u(t) + v(t) \leq 1$, we notice that $z(t) = 1 - u(t) - v(t)$ is regarded as a solution to

$$\frac{\partial z}{\partial t} = d\Delta_S z - [\Phi(\chi_1(u), \chi_1(v), \chi_2(w)) + \Psi(\chi_1(u), \chi_1(v), \chi_2(w))]\chi_1(z) + f[\chi_1(u) + \chi_1(v)].$$

We can then repeat the same arguments as for $u(t)$ and $v(t)$ to conclude that $z(t) \geq 0$ for any $0 < t \leq T_{U_0}$.

Thirdly, let us observe that $0 \leq w \leq (R_0/\sigma)^{\frac{1}{4}}$ for any $0 < t \leq T_{U_0}$. The verification that $w(t) \geq 0$ is the same as for $u(t)$ and $v(t)$. Putting $w_1(t) = (R_0/\sigma)^{\frac{1}{4}} - w(t)$, we notice due to (1.2) that

$$\frac{\partial w_1}{\partial t} = D\Delta_S w_1 - \sigma[R_0/\sigma - \chi_2(w)^4] + R_0\left\{1 + [g(u,v) - 1]\sqrt{1 - (z/\ell)^2}\right\}.$$

Then, put $h_1(t) = \int_S H(w_1(\omega, t))d\omega$. Since $H'((R_0/\sigma)^{\frac{1}{4}} - w)[R_0/\sigma - \chi_2(w)^4] \equiv 0$ for $-\infty < w < \infty$, it follows that $\frac{dh_1}{dt}(t) \leq 0$, i.e., $h_1(t) \leq h_1(0) = 0$. Hence, $(R_0/\sigma)^{\frac{1}{4}} - w(t) \geq 0$ for any $0 < t \leq T_{U_0}$. □

This proposition shows that the norm $\|U(t)\|_X$ remains uniformly bounded on the interval $[0, T_{U_0}]$. This then means that one can always extend any local solution with a uniform time interval. Therefore, we obtain the following existence result.

**Theorem 3.1.** *For any $U_0 \in K$, (3.3) possesses a unique global solution $U(t)$ in the function space:*

$$U \in \mathcal{C}((0,\infty); \mathcal{D}(\mathcal{A})) \cap \mathcal{C}([0,\infty); X) \cap \mathcal{C}^1((0,\infty); X).$$

As verified by Proposition 3.1 $U(t)$ takes its values in $K$ for all $0 < t < \infty$. Thereby, $\chi_1(u(t)) = u(t)$, $\chi_1(v(t)) = v(t)$, $\chi_1(1 - u(t) - v(t)) = 1 - u(t) - v(t)$ and $\chi_2(w(t)) = w(t)$ for all $0 < t < \infty$. This in turn shows that the global solution $U(t)$ of (3.3) can be considered as a global solution to the original problem (1.1), too.

Let us finally verify global norm estimate and continuous dependence of solutions on initial values.

**Theorem 3.2.** *Let $U_0 \in K$ and let $U(t)$ be the global solution of (3.3). Then,*

$$\|\nabla_S U(t)\|_X \leq C_3\left[\left(1 + \frac{1}{t}\right)^{\frac{1}{2}} e^{-d\lambda_1 t}\|U_0\|_X + 1\right] \qquad \text{for} \quad 0 < t < \infty \tag{3.7}$$

*with some constant $C_3$.*

*Proof.* By Duhamel's formula, $U(t)$ can be written as

$$U(t) = e^{-t\mathcal{A}}U_0 + \int_0^t e^{-(t-s)\mathcal{A}}\mathcal{F}(U(s))ds.$$

Thereby,

$$\nabla_S U(t) = \nabla_S e^{-t\mathcal{A}}U_0 + \int_0^t \nabla_S e^{-(t-s)\mathcal{A}}\mathcal{F}(U(s))ds.$$

Due to (3.2) and (3.5), we have

$$\begin{aligned}\|\nabla_S U(t)\|_X &\le \left(\lambda_1 + \frac{1}{edt}\right)^{\frac{1}{2}} e^{-d\lambda_1 t}\|U_0\|_X + C_1 \int_0^t \left(\lambda_1 + \frac{1}{ed(t-s)}\right)^{\frac{1}{2}} e^{-d\lambda_1(t-s)} ds \\ &\le \left(\lambda_1 + \frac{1}{edt}\right)^{\frac{1}{2}} e^{-d\lambda_1 t}\|U_0\|_X + C_1 \int_0^\infty \left(\lambda_1 + \frac{1}{eds}\right)^{\frac{1}{2}} e^{-d\lambda_1 s} ds.\end{aligned}$$

Hence, (3.7) is verified. □

**Theorem 3.3.** *Let $U_0, V_0 \in K$ and let $U(t)$ and $V(t)$ be the global solutions of* (3.3) *with initial values $U_0$ and $V_0$, respectively. Then,*

$$\|U(t) - V(t)\|_X \le e^{C_2 t}\|U_0 - V_0\|_X \qquad \textit{for} \quad 0 \le t < \infty, \tag{3.8}$$

$$\|\nabla_S[U(t) - V(t)]\|_X \le C_4 \left[\left(1 + \frac{1}{t}\right)^{\frac{1}{2}} + te^{C_2 t}\right]\|U_0 - V_0\|_X \qquad \textit{for} \quad 0 < t < \infty \tag{3.9}$$

*with some constant $C_4$.*

*Proof.* By Duhamel's formula again, we have

$$U(t) - V(t) = e^{-t\mathcal{A}}[U_0 - V_0] + \int_0^t e^{-(t-s)\mathcal{A}}[\mathcal{F}(U(s)) - \mathcal{F}(V(s))]ds.$$

In view of (3.1) and (3.6),

$$\|U(t) - V(t)\|_X \le \|U_0 - V_0\|_X + C_2 \int_0^t \|U(s) - V(s)\|_X ds.$$

Hence, (3.8) is obtained.

Similarly, from

$$\nabla_S[U(t) - V(t)] = \nabla_S e^{-t\mathcal{A}}[U_0 - V_0] + \int_0^t \nabla_S e^{-(t-s)\mathcal{A}}[\mathcal{F}(U(s)) - \mathcal{F}(V(s))]ds,$$

it is estimated by (3.2), (3.6) and (3.8) that

$$\begin{aligned}&\|\nabla_S[U(t) - V(t)]\|_X \\ &\quad\le \left[\left(\lambda_1 + \frac{1}{edt}\right)^{\frac{1}{2}} e^{-d\lambda_1 t} + C_2 \int_0^t \left(\lambda_1 + \frac{1}{ed(t-s)}\right)^{\frac{1}{2}} e^{-d\lambda_1(t-s) + C_2 s}\right]\|U_0 - V_0\|_X \\ &\quad\le \left[\left(\lambda_1 + \frac{1}{edt}\right)^{\frac{1}{2}} + C_2 e^{C_2 t} \int_0^t \left(\lambda_1 + \frac{1}{ed(t-s)}\right)^{\frac{1}{2}} ds\right]\|U_0 - V_0\|_X.\end{aligned}$$

Hence, (3.9) has been verified. □

**4 Dynamical Systems** This section is devoted to constructing a dynamical system determined by (3.3) and showing existence of attractor sets. For this purpose, however, it suffices to simply follow the general procedure that is known for the Cauchy problems of semilinear abstract parabolic evolution equations, see [24, Section 6.5].

For $U_0 \in K$, let $U(t; U_0)$ denote the global solution of (3.3), and set

$$S(t)U_0 = U(t; U_0), \qquad 0 \le t < \infty.$$

Then, $S(t)$ is a nonlinear semigroup acting on $K$, i.e., $S(0) = I$ and $S(t+s) = S(t)S(s)$ for $0 \le s, t < \infty$. Furthermore, $S(t)$ is seen to be continuous in the sense that $(t, U_0) \mapsto S(t)U_0$ is continuous from $[0,\infty) \times K$ into $X$. Indeed, fix $(t, U_0) \in [0,\infty) \times K$. Due to (3.8),

$$\begin{aligned}\|S(t')U_0' - S(t)U_0\|_X &\le \|S(t')U_0' - S(t')U_0\|_X + \|S(t')U_0 - S(t)U_0\|_X \\ &\le e^{C_2 t'}\|U_0' - U_0\|_X + \|S(t')U_0 - S(t)U_0\|_X.\end{aligned}$$

Then, $(t', U_0') \to (t, U_0)$ implies $S(t')U_0' \to S(t)U_0$ in $X$. Hence, $S(t)$ defines a dynamical system in the space $X$ which is denoted by $(S(t), K, X)$ (cf. [4, 18]). The phase space $K$ is a bounded, closed subset of $X$.

As well known, the dissipative estimate (3.7) provides existence of the global attractor. Set a subset $B$ of $K$ by

$$B = K \cap \left\{U \in [H^1(S)]^3; \|\nabla U\|_X \le C_3 + 1\right\}.$$

Then, (3.7) means that there is a time $T$ such that $S(t)K \subset B$ for every $t \ge T$, i.e., $B$ is an absorbing set. In addition, $B$ is a compact set of $X$. Therefore, $B$ is a compact absorbing set of $(S(t), K, X)$. It is clear that $S(t)B \subset B$ for every $t \ge T$. So, we set again a subset of $K$ by

$$\mathcal{K} = \bigcup_{0 \le t \le T} S(t)B.$$

Then, $S(t)\mathcal{K} \subset \mathcal{K}$ for every $t > 0$, i.e., $\mathcal{K}$ is an invariant set. Therefore, $\mathcal{K}$ is not only compact and absorbing but also invariant. This means that the asymptotic behavior of trajectories of $(S(t), K, X)$ can be reduced to a sub dynamical system $(S(t), \mathcal{K}, X)$ in which the phase space $\mathcal{K}$ is a compact set of $X$.

By the usual arguments, it is seen that $\mathcal{B} = \bigcap_{0 \le t < \infty} S(t)\mathcal{K}$ becomes a global attractor of $(S(t), \mathcal{K}, X)$.

Furthermore, thanks to the estimate (3.9), we can construct the exponential attractors. Remember that a subset $\mathcal{M} \subset \mathcal{K}$ satisfying the following conditions is called an exponential attractor of $(S(t), \mathcal{K}, X)$:

1. $\mathcal{M}$ is a compact subset of $X$ with finite fractal dimension.
2. $\mathcal{M}$ includes the global attractor $\mathcal{B}$.
3. $\mathcal{M}$ is an invariant set, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t > 0$.
4. There exists an exponent $k > 0$ such that

$$h(S(t)\mathcal{K}, \mathcal{M}) \le C_5 e^{-kt}, \qquad 0 < t < \infty,$$

with a constant $C_5 > 0$.

Here, $h(K_1, K_2) = \sup_{F\in K_1} \inf_{G\in K_2} \|F - G\|_X$ is a semi-distance of two subsets $K_1$ and $K_2$ of $\mathcal{K}$.

As explained in [24, Section 6.4], the compact smoothing property

$$\|S(t^*)U_0 - S(t^*)V_0\|_{H^1(S)} \le C_6\|U_0 - V_0\|_X, \qquad U_0,\, V_0 \in \mathcal{K},$$

of $S(t^*)$ with any fixed time $t^* > 0$ provides existence of exponential attractors. But, in the present case, this property is nothing more than the estimates (3.8) and (3.9).

In this way, we have obtained the following theorem.

**Theorem 4.1.** *The dynamical system $(S(t), K, X)$ possesses exponential attractors.*

*Proof.* As explained above, we already know that there exists an exponential attractor $\mathcal{M}$ for $(S(t), \mathcal{K}, X)$. Then, as $S(T)K \subset B \subset \mathcal{K}$, it is readily verified that $\mathcal{M}$ is an exponential attractor for $(S(t), K, X)$, too. □

**5 Some Numerical Results** We shall conclude this paper with illustrating some numerical examples. Let us consider (1.1) in the sphere $S$ with $\ell = 1$. Numerical methods for partial differential equations on the spheres have been widely developed in the field of geodynamo simulations. For example, Yin-Yang grid by Kageyama-Sato [9], Cubed Sphere grid by Ronchi-Iacono-Paolucci [15], Half-Step-Shifted grid (e.g., [8]) and a method of applying l'Hospital's rule on the pole grids (e.g., [10]), see also the review of Williamson [21]. These numerical methods have in general a trade-off between computational cost and their accuracy.

We use the explicit Half-Step-Shifted grid scheme. As surveyed below, this scheme is based on the traditional finite difference methods with the spherical polar coordinate system. For spatial discretization, the $i$-th colatitude grid point $\theta_i$ and the $j$-th longitude grid point $\phi_j$ are defined by

$$\begin{aligned} \theta_i &= \left(i - \frac{1}{2}\right)\Delta\theta_i, && (i = 1, 2, \cdots, N),\\ \phi_j &= j\Delta\phi, && (j = 0, 1, \cdots, M), \end{aligned}$$

respectively, where $N$ and $M$ denote the numbers of grid points. And the $n$-th time step is defined by $t_n = n\Delta t$. We assume that $\Delta\theta_i$ is a non-uniform grid spacing which is smaller near the poles, while $\Delta\phi$ is a uniform one ($\Delta\phi = 2\pi/M$). This scheme is a simple idea which the horizontal grid lines are shifted by a distance of $\Delta\theta_i/2$ in order to remove coordinate singularity problems at the poles.

Let us denote the approximate values by $U^n_{i,j} \approx u(\theta_i, \phi_j, t_n), V^n_{i,j} \approx v(\theta_i, \phi_j, t_n)$ and $W^n_{i,j} \approx w(\theta_i, \phi_j, t_n)$, respectively. Then, (1.1) is discretized as follows:

$$\begin{aligned} \frac{U^{n+1}_{i,j} - U^n_{i,j}}{\Delta t} &= \left[\left(1 - U^n_{i,j} - V^n_{i,j}\right)\Phi(U^n_{i,j}, V^n_{i,j}, W^n_{i,j}) - f\right] U^n_{i,j}\\ &\quad + d\Bigg[\frac{1}{\sin\theta_i}\frac{1}{\Delta\theta_i}\left(\sin\theta_{i+\frac{1}{2}}\frac{U^n_{i+1,j} - U^n_{i,j}}{\Delta\theta_i} - \sin\theta_{i-\frac{1}{2}}\frac{U^n_{i,j} - U^n_{i-1,j}}{\Delta\theta_i}\right)\\ &\qquad\qquad + \frac{1}{\sin^2\theta_i}\left(\frac{U^n_{i,j+1} - 2U^n_{i,j} + U^n_{i,j-1}}{(\Delta\phi)^2}\right)\Bigg], \end{aligned}$$

$$
\begin{aligned}
\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta t} &= \left[\left(1-U_{i,j}^{n}-V_{i,j}^{n}\right)\Psi(U_{i,j}^{n},V_{i,j}^{n},W_{i,j}^{n})-f\right]V_{i,j}^{n} \\
&\quad + d\Bigg[\frac{1}{\sin\theta_i}\frac{1}{\Delta\theta_i}\left(\sin\theta_{i+\frac{1}{2}}\frac{V_{i+1,j}^{n}-V_{i,j}^{n}}{\Delta\theta_i}-\sin\theta_{i-\frac{1}{2}}\frac{V_{i,j}^{n}-V_{i-1,j}^{n}}{\Delta\theta_i}\right) \\
&\qquad\qquad + \frac{1}{\sin^2\theta_i}\left(\frac{V_{i,j+1}^{n}-2V_{i,j}^{n}+V_{i,j-1}^{n}}{(\Delta\phi)^2}\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\frac{W_{i,j}^{n+1}-W_{i,j}^{n}}{\Delta t} &= \left[1-g(U_{i,j}^{n},V_{i,j}^{n})\right]R(\theta_i)-\sigma\left(W_{i,j}^{n}\right)^4 \\
&\quad + D\Bigg[\frac{1}{\sin\theta_i}\frac{1}{\Delta\theta_i}\left(\sin\theta_{i+\frac{1}{2}}\frac{W_{i+1,j}^{n}-W_{i,j}^{n}}{\Delta\theta_i}-\sin\theta_{i-\frac{1}{2}}\frac{W_{i,j}^{n}-W_{i-1,j}^{n}}{\Delta\theta_i}\right) \\
&\qquad\qquad + \frac{1}{\sin^2\theta_i}\left(\frac{W_{i,j+1}^{n}-2W_{i,j}^{n}+W_{i,j-1}^{n}}{(\Delta\phi)^2}\right)\Bigg].
\end{aligned}
$$

Meanwhile, we set the parameters in (1.1) as: $d = 10^{-6}, D = 1.0, a_w = 0.75, a_g = 0.50, a_b = 0.25, \delta = 0.003265, f = 0.3, \overline{w} = 295.5, q = 40$ and $\sigma = 5.67 \times 10^{-8}$. The incoming energy $R(\theta)$ is taken as

$$
R(\theta) = \frac{4 \cdot 917}{\pi} L \sin\theta
$$

where $L = 0.85$ is the same as in Watson-Lovelock [20]. Initial functions $u_0(\theta,\phi)$, $v_0(\theta,\phi)$, $w_0(\theta,\phi)$ are constructed by slightly perturbing constant functions $\bar{u}_0(\theta,\phi) \equiv 0.321, \bar{v}_0(\theta,\phi) \equiv 0.291$ and $\bar{w}_0(\theta,\phi) \equiv 290.96$ for $(\theta,\phi) \in (0,\pi) \times [0,2\pi)$. In numerical computations, we apply the periodic boundary conditions at $j = 0$ and $j = M$ and the latitudinal boundary conditions:

$$
\begin{aligned}
&U_{0,j} = U_{1,\frac{M}{2}+j}, \qquad U_{N+1,j} = U_{N,\frac{M}{2}+j}, \qquad (j = 0,1,...,M/2)\,, \\
&U_{0,j} = U_{1,-\frac{M}{2}+j}, \qquad U_{N+1,j} = U_{N,-\frac{M}{2}+j}, \qquad (j = M/2+1,...,J)\,,
\end{aligned}
$$

at $i = 1$ and $i = N$. It is the same for $V_{i,j}$ and $W_{i,j}$.

The numerical solution to (1.1) stabilizes asymptotically. About $t = 600$, its evolution shows down evidently. This may mean that the solution is attracted by the global attractor. Fig.1 illustrates the graphs of $u(\theta,\phi,t)$, $v(\theta,\phi,t)$, $w(\theta,\phi,t)$ at $t = 600$.

Their graphs show a clear segregation strip pattern. The interface is given by zigzag curves which are almost parallel with the equator.

Similar results are obtained by another numerical method.

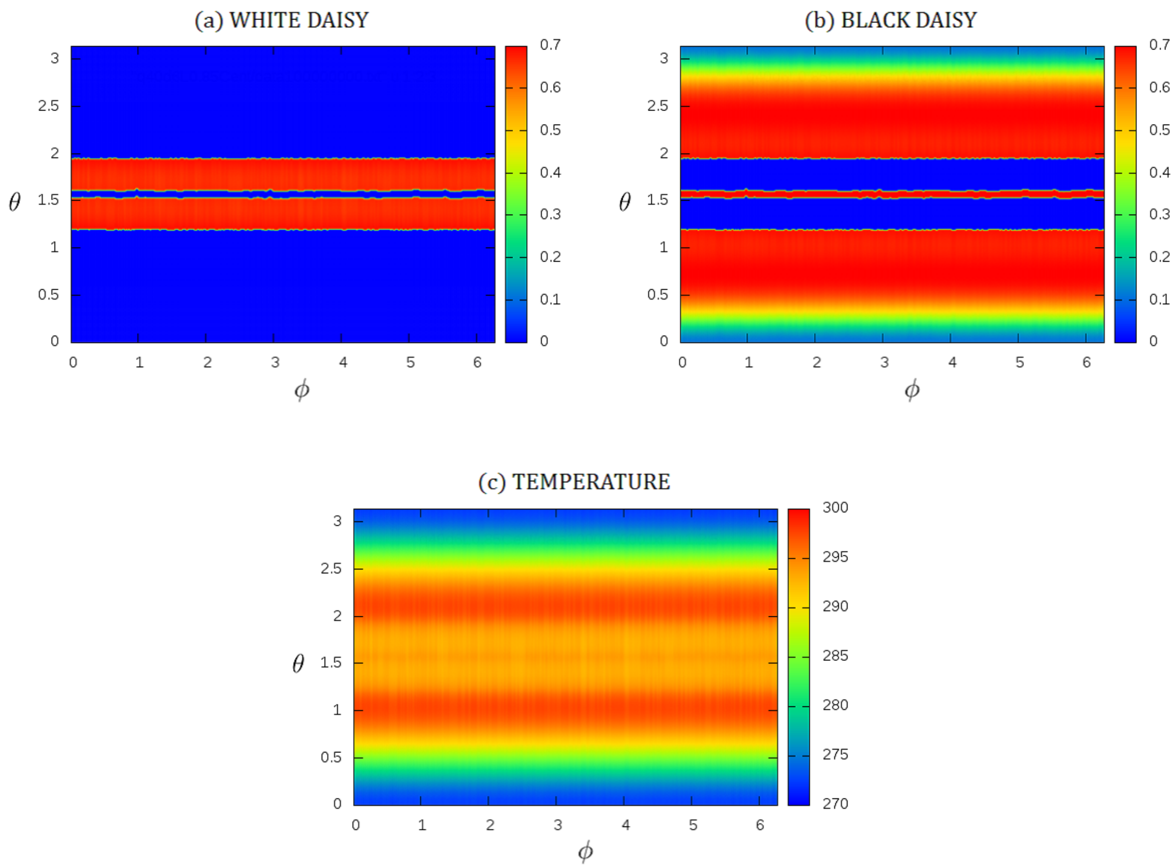


Fig. 1: (a) Graph of $u(\theta, \phi)$, (b) Graph of $v(\theta, \phi)$ and (c) Graph of $w(\theta, \phi)$ at time $t = 600$.

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