BIFURCATIONS WITH MULTI-DIMENSIONAL KERNEL IN A CHEMOTAXIS-GROWTH SYSTEM

Taka-aki Aoki and Koichi Osaki

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Abstract. We study the bifurcation problem for a chemotaxis-growth system with logistic growth in a two-dimensional rectangular domain. We apply the local bifurcation theorem by Ambrosetti and Prodi that does not require one-dimensional degeneration of the linearized operator around trivial solutions. We then obtain bifurcation solutions with two- and three-dimensional degeneration indicating spatially regular nesting patterns.

1 Introduction.

Budrene and Berg [2, 3] found that the chemotactic bacteria *E. coli* form remarkable macroscopic regular patterns in their colony, resulting from the interplay between diffusion, chemotaxis and growth. Mimura and Tsujikawa [12] studied the following chemotaxis-growth system to elucidate the mechanisms for pattern formation processes:

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u - \chi\nabla\cdot(u\nabla\rho) + f(u) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial \rho}{\partial t} = \Delta\rho - b\rho + cu & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x),\ \rho(x,0) = \rho_0(x) & \text{in } \Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$, and $\partial/\partial n$ denotes the derivative with respect to the outer normal of $\partial\Omega$. The function $u(x,t)$ is the population density of the chemotactic bacteria at position $x \in \Omega$ and time $t \in [0,\infty)$, and $\rho(x,t)$ is the concentration of chemical substance that is produced by the individuals. The function $f(u)$ denotes the growth of u, and several different forms have been proposed for $f(u)$ [7, 16]. We assume in this paper that $f(u)$ is a logistic saturating growth function,

$$f(u) = au(1-\mu u),$$

where a and μ are positive constants. The other coefficients b, c, d and χ are also positive constants. The advection term $-\chi\nabla\cdot(u\nabla\rho)$ corresponds to chemotaxis of bacteria, and the coefficient χ indicates the intensity of chemotaxis.

In this article, we consider a bifurcation problem for the stationary state of (E). In a two-dimensional rectangular domain, Kuto et al. [11] proved that one-mode bifurcations occurred for the uniform state $(u,\rho) = (1/\mu, c/(\mu b))$, that is, stripe and rectangle patterns occurred along destabilized x and y-directional double Fourier modes. Kuto et al. [11] also showed solutions for a hybrid mode bifurcation that formed hexagonal patterns. In the

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analytic proof, Kuto et al. [11] applied the classical local bifurcation theorem by Crandall and Rabinowitz [4], which requires one-dimensional degeneration of a linearized operator around the uniform state. In other words, the kernel of linearized operator is prohibited from containing any hybrid modes in the Crandall and Rabinowitz theorem [4]. Hexagonal pattern formation, however, does use two destabilized hybrid Fourier modes (see Section 4). Kuto et al. [11] therefore introduced a restricted functional space possessing $2\pi/3$-rotational symmetry, which is a closed one-dimensional subspace in the universal space, and applied the Crandall and Rabinowitz theorem in this subspace (see also [13]). An equivalent approach for hybrid mode bifurcation is the bifurcation theorem considered in [5], which is a similar bifurcation theorem that considers the symmetry of patterns.

In this article, we apply another type of bifurcation theorem introduced by Ambrosetti and Prodi [1], which admits multi-dimensional kernels of linearized operators. Indeed, we obtain another hexagonal pattern solution to (E) which does not have $2\pi/3$-rotational symmetry (see Figure 2). In addition, we study the bifurcation problem for (E) with three-dimensional kernel and obtain novel spatially regular nesting patterns. These patterns are not demonstrated in [11]. The advantage of no one-dimensional restrictions is, for instance, there is no need to know all of the symmetries in the hybrid mode in advance. Another advantage is the simpler the sufficient conditions for bifurcation are, the easier we can implement the algorithm for obtaining bifurcating solutions in a computer program.

We conclude this introduction by referring as other results as follows: In the one-dimensional chemotaxis-growth system (E), Kurata et al. [10] demonstrated that the instability of solutions occurred for the uniform state, and time-periodic solutions successively bifurcated from the nontrivial stationary solutions by utilizing the bifurcation software AUTO. Painter and Hillen [15] also showed that one-dimensional periodic solutions successively bifurcated and resulted in chaotic dynamics. If the logistic term is absent for (E) ($f(u) \equiv 0$), the chemotaxis-growth system (E) reduces to the celebrated Keller-Segel chemotaxis system [9], which admits the blow-up of solutions by overcrowding due to chemotaxis [6, 8, 18]. Meanwhile, for the two-dimensional chemotaxis-growth system (E), Osaki et al. [14] showed the existence of global-in-time solutions and a compact global attractor for the dynamical system generated by these solutions. For the case of more than three dimensions, Winkler [17] obtained the global-in-time existence of solutions for (E) under the quadratic suppression of $f(u)$ for sufficiently large μ. Zheng [19] also recently extended the valid region of μ for the global-in-time existence of solutions to the n-dimensional chemotaxis-growth system (E).

This paper is organized as follows. First, we provide a brief review of the multi-dimensional bifurcation theorem by Ambrosetti and Prodi [1] as a preliminary. In Section 3 we frame the chemotaxis-growth system (E) as a nonlinear bifurcation problem. In Sections 4 and 5 we study the bifurcation problems with two- and three-dimensional degeneration of the linearized operators, respectively.

2 A brief review of the multi-dimensional bifurcation theorem.

Let F be a nonlinear operator such that $F \in \mathcal{C}^\infty((\lambda_1, \lambda_2) \times X; Y)$. Here, X and Y are Banach spaces, and (λ_1, λ_2) is an interval in $\mathbb{R}$. We consider a bifurcation problem for a functional equation in the Banach space Y:

$$F(\lambda, u) = 0 \ \in Y. \tag{2.1}$$

Assume that the nonlinear equation (2.1) has a trivial solution $u = 0$ for arbitrary λ, i.e., $F(\lambda, 0) = 0$, $^\forall\lambda \in (\lambda_1, \lambda_2)$. We denote a bifurcation point by $\lambda = \lambda^*$. Then the linearized operator of $F(\lambda, u)$ around $(\lambda, u) = (\lambda^*, 0)$, $L = F_u(\lambda^*, 0) \in \mathcal{L}(X; Y)$, should degenerate, that is, L is not invertible, and then $V := \mathcal{K}(L) \neq \{0\}$. Let us denote $R := \mathcal{R}(L)$. Assume also that V has a topological complement W in X, and R is closed and also has a topological complement Z in Y:

$$X = V \oplus W, \quad Y = R \oplus Z.$$

The Taylor expansion of $F(\lambda, u)$ around $(\lambda, u) = (\lambda^*, 0)$ is expressed as

$$F(\lambda^* + \mu, u) = Lu + \mu M u + \frac{1}{2}\mathcal{B}[u, u] + \psi(\mu, u), \tag{2.2}$$

where $M := F_{u\lambda}(\lambda^*, 0)$, $\mathcal{B} := F_{uu}(\lambda^*, 0)$, and $\psi(\mu, u)$ is a smooth function such that $\psi(\mu, 0) \equiv 0$, $\psi_u(0, 0) = 0$, $\psi_{uu}(0, 0) = 0$, and $\psi_{\lambda u}(0, 0) = 0$.

By denoting the solution as $u = \mu(v+w)$, Ambrosetti and Prodi [1] derived a bifurcation equation with conjugate projections

$$P : Y \to Z, \quad Q : Y \to R.$$

By substituting $u = \mu(v + w)$ into the equation, we have

$$PM(v + w) + \frac{1}{2}P\mathcal{B}[v + w, v + w] + \mu P\tilde{\psi}(\mu, v, w) = 0, \tag{2.3}$$

$$\tilde{\Phi}(\mu, v, w) := Lw + \mu QM(v + w) + \frac{1}{2}\mu Q\mathcal{B}[v + w, v + w] + \mu^2 Q\tilde{\psi}(\mu, v, w) = 0. \tag{2.4}$$

Here, $\psi(\mu, \mu(v + w)) = \mu^3\tilde{\psi}(\mu, v, w)$ for a smooth function $\tilde{\psi}(\mu, v, w)$. Since $\tilde{\Phi}(0, v, 0) = 0$ for any $v \in V$ and $\tilde{\Phi}_w(0, v, 0) = L \neq 0$, the nonlinear equation $\tilde{\Phi}(\mu, v, w) = 0$ (which generally has an infinite number of dimensions) can be uniquely solved in w around the neighborhood $\Lambda \times \mathcal{V} \times \mathcal{W}$ of $(\mu, v, w) = (0, v^*, 0)$, where $v^* \in V$ is arbitrarily fixed in V. Then, the component w can be expressed uniquely as $w = \mu\,\gamma(\mu, v) \in \mathcal{W}$, $(\mu, v) \in \Lambda \times \mathcal{V}$, with a smooth function γ depending on v^*. Substituting this into the equation (2.3) (which is finite dimensional in a favorable case, e.g. L is a Fredholm operator), we obtain the bifurcation equation for $\Lambda \times \mathcal{V}$:

$$\begin{aligned} N(\mu, v) := PM(v + \mu\gamma(\mu, v)) + \frac{1}{2}P\mathcal{B}[v + \mu\gamma(\mu, v), v + \mu\gamma(\mu, v)] \\ + \mu P\tilde{\psi}(\mu, v, \mu\gamma(\mu, v)) = 0 \in Z, \end{aligned} \tag{2.5}$$

where $N(\mu, v)$ is smooth. We here note again that when the dimension of the subspace $Z \subset Y$ is finite, the bifurcation equation (2.5) consists of a finite number of equations.

The multi-dimensional bifurcation theorem introduced by Ambrosetti and Prodi [1] is as follows:

Theorem 2.1. [1, Theorem 5.1, p.102] Assume that two Banach spaces X and Y satisfy the conditions that $V = \mathcal{K}(L)$ has a topological complement in X, and $R = \mathcal{R}(L)$ is closed and has a topological complement in Y. Assume also that: for the nonlinear problem (2.5), there exists $v^* \in V$, $v^* \neq 0$, such that

(a) $N(0, v^*) = PMv^* + \frac{1}{2}P\mathcal{B}[v^*, v^*] = 0$;

(b) the linear operator $N_v(0, v^*) = S : V \to Z$, $Sv = PMv + P\mathcal{B}[v^*, v]$, is invertible.

Then, there exists a local branch of nontrivial solutions $(\lambda, u(\lambda))$ to (2.1) which bifurcates from $(\lambda^*, 0)$ such that

$$\lambda = \lambda^* + \mu, \quad u = \mu\,[v^* + \mu\tilde{v}(\mu)],$$

where $\tilde{v}(\mu)$ is a smooth function of μ. □

For the complete proof of Theorem 2.1 we refer to [1, p.102], but the above results are clear from the implicit function theorem. Indeed, for the bifurcation equation $N(\mu, v) = 0$, there exists a unique solution $(\mu, v(\mu))$ for small μ and $v(0) = v^*$. Then, by substituting $v = v(\mu)$ into $u = \mu(v + w)$ and $w = \mu\,\gamma(\mu, v)$ we obtain a nontrivial solution $u = \mu\,[\,v(\mu) + \mu\gamma(\,\mu, v(\mu)\,)\,]$ near $\mu = 0$. From the Taylor expansion of $v(\mu)$ around $\mu = 0$, we obtain the local solution $(\lambda, u(\lambda))$ to (2.1). □

3 Bifurcation equation of chemotaxis-growth model.

We return to the bifurcation problem for the following stationary system of (E):

$$\text{(SE)}\qquad \begin{cases} d\Delta u - \chi\nabla\cdot(u\nabla\rho) + au(1-\mu u) = 0 & \text{in } \Omega, \\ \Delta\rho - b\rho + cu = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega, \\ u \geq 0,\ \rho \geq 0 & \text{in } \Omega. \end{cases}$$

The spatial domain is specified as

$$\Omega = \left(0, \frac{\pi}{l}\right) \times \left(0, \frac{\pi}{\sqrt{3}l}\right). \tag{3.1}$$

Here $l > 0$ is a control parameter for bifurcation. The setting of Banach (Hilbert) spaces X and Y is

$$X = H_N^2(\Omega) \times H_N^2(\Omega), \quad Y = L^2(\Omega) \times L^2(\Omega)$$

with norms:

$$\|U\|_X := \sqrt{\|u\|_{H^2}^2 + \|\rho\|_{H^2}^2}, \quad \|U\|_Y := \sqrt{\|u\|_{L^2}^2 + \|\rho\|_{L^2}^2}, \quad U = {}^T[u\ \rho],$$

where $H_N^2(\Omega) = \left\{w \in H^2(\Omega);\ \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega\right\}$. Then, the inner product of Y is:

$$\langle U_1, U_2\rangle_Y := \langle u_1, u_2\rangle_{L^2} + \langle \rho_1, \rho_2\rangle_{L^2}, \quad U_1 = {}^T[u_1\ \rho_1],\ U_2 = {}^T[u_2\ \rho_2] \in Y.$$

We will show the existence of nontrivial solutions bifurcating from the positive trivial equilibrium to (SE):

$$U^* = \begin{bmatrix} u^* \\ \rho^* \end{bmatrix} := \begin{bmatrix} 1/\mu \\ c/(\mu b) \end{bmatrix}.$$

We set χ as a bifurcation parameter, and $F : (0, \infty) \times X \to Y$ by

$$F(\chi, U) := \begin{bmatrix} d\Delta u - \chi\nabla\cdot(u\nabla\rho) + au(1-\mu u) \\ \Delta\rho - b\rho + cu \end{bmatrix}. \tag{3.2}$$

Indeed, the nonlinear terms are L^2-valued functions in view of $\|\nabla\cdot(u\nabla\rho)\|_{L^2} \leq C\|u\nabla\rho\|_{H^1} \leq C\|u\|_{H^2}\|\rho\|_{H^2}$ and $\|u^2\|_{L^2} = \|u\|_{L^4}^2 \leq C\|u\|_{H^1}\|u\|_{L^2}$. Then, the nonlinear bifurcation problem for (SE) can be expressed as

$$F(\chi, U) = 0\ \in Y, \quad (\chi, U) \in (0, \infty) \times X.$$

The linearized operator

$$L := F_U\,(\chi^*, U^*) \in \mathcal{L}(X; Y)$$

around $U = U^*$ is calculated as

$$L\begin{bmatrix}h\\k\end{bmatrix} = \begin{bmatrix}d\Delta h - \frac{\chi^*}{\mu}\Delta k - ah\\ \Delta k + ch - bk\end{bmatrix} = \begin{bmatrix}d\Delta - a & -\frac{\chi^*}{\mu}\Delta\\ c & \Delta - b\end{bmatrix}\begin{bmatrix}h\\k\end{bmatrix}, \quad \begin{bmatrix}h\\k\end{bmatrix} \in X; \tag{3.3}$$

and we also obtain the second order derivatives of $F(\chi, U)$ as follows:

$$M\begin{bmatrix}h\\k\end{bmatrix} = F_{U,\chi}(\chi^*, U^*)\begin{bmatrix}h\\k\end{bmatrix} = \begin{bmatrix}-\frac{1}{\mu}\Delta k\\ 0\end{bmatrix},$$

$$\mathcal{B}\left(\begin{bmatrix}h\\k\end{bmatrix}, \begin{bmatrix}\tilde{h}\\\tilde{k}\end{bmatrix}\right) = F_{U,U}(\chi^*, U^*)\left(\begin{bmatrix}h\\k\end{bmatrix}, \begin{bmatrix}\tilde{h}\\\tilde{k}\end{bmatrix}\right) = \begin{bmatrix}-\chi^*[\nabla\cdot(h\nabla\tilde{k}) + \nabla\cdot(\tilde{h}\nabla k)] - 2a\mu h\tilde{h}\\ 0\end{bmatrix}$$

for ${}^T[h\ k]$, ${}^T[\tilde{h}\ \tilde{k}] \in X$. From the above we can set the bifurcation equation $N(\lambda, U) = 0$ for (SE) in the neighborhood of $(0, U^*) \in (-\varepsilon, \varepsilon) \times V$ with $\chi = \chi^* + \lambda$ and small ε.

We introduce double cosine functions for the usual orthogonal basis of $L^2(\Omega)$ under homogeneous Neumann boundary conditions:

$$\{\phi_m(x)\,\psi_n(y) \mid m, n \geq 0\}, \ \phi_m(x) = \cos(lmx), \ \psi_n(y) = \cos(\sqrt{3}lny).$$

Then, the orthogonal basis of Y is induced as:

$$\left\{{}^T[\,h_{mn}\,\phi_m(x)\psi_n(y) \ \ k_{mn}\,\phi_m(x)\psi_n(y)\,] \mid m, n \geq 0\right\}.$$

Proposition 3.1. The linearized operator $L = F_U(\chi^*, U^*)$ degenerates at $\chi^* = \chi(m, n)$, where $\chi(m, n)$ is defined as

$$\chi(m, n) := \frac{\mu}{c}\left[dl^2(m^2 + 3n^2) + \frac{ab}{l^2(m^2 + 3n^2)} + a + bd\right]. \tag{3.4}$$

Proof. Consider the linearized equation $L\,{}^T[k\ h] = 0$ with homogeneous Neumann boundary condition $\frac{\partial h}{\partial n} = \frac{\partial k}{\partial n} = 0$ on $\partial\Omega$. Substituting the two cosine Fourier series for $h(x, y)$ and $k(x, y)$:

$$\begin{bmatrix}h\\k\end{bmatrix} = \sum_{m,n=0}^{\infty}\begin{bmatrix}h_{mn}\\k_{mn}\end{bmatrix}\phi_m(x)\psi_n(y) \tag{3.5}$$

to the linearized equation, we have an equivalent equation for each Fourier coefficient ${}^T[h_{mn}\ k_{mn}]$ such that

$$\begin{bmatrix}-dl^2(m^2 + 3n^2) - a & \frac{\chi}{\mu}l^2(m^2 + 3n^2)\\ c & -l^2(m^2 + 3n^2) - b\end{bmatrix}\begin{bmatrix}h_{mn}\\k_{mn}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \quad m, n \in \mathbb{N}. \tag{3.6}$$

This indicates that there exists a nontrivial solution ${}^T[h_{mn}\ k_{mn}]$ to (3.6), if and only if the following characteristic equation holds:

$$\begin{vmatrix}-dl^2(m^2 + 3n^2) - a & \frac{\chi}{\mu}l^2(m^2 + 3n^2)\\ c & -l^2(m^2 + 3n^2) - b\end{vmatrix} \tag{3.7}$$
$$= [\,dl^2(m^2 + 3n^2) + a\,][\,l^2(m^2 + 3n^2) + b\,] - \chi\frac{c}{\mu}l^2(m^2 + 3n^2) = 0.$$

Solving this for χ, we have (3.4). □

Let V be the kernel of the linearized operator L: $V = \mathcal{K}(L)$. Then, for the composition of V we have:

Proposition 3.2. The kernel V is the linear span of the cosine Fourier basis

$$\Phi_{mn} \coloneqq \begin{bmatrix} 1 \\ \eta_{mn} \end{bmatrix} \phi_m(x)\psi_n(y)$$

of which modes (m,n) satisfy the characteristic equation (3.7), where

$$\eta_{mn} = \frac{c}{l^2(m^2+3n^2)+b}.$$

Proof. The proof is given in Kuto et al. [11, Theorem 5.1]. □

4 Two-dimensional kernel bifurcation of chemotaxis-growth model.

Kuto et al. [11] studied the bifurcation with two-dimensional kernel for (SE) by restricting the functional space to $2\pi/3$-rotational symmetry. The multiplicity occurred in the lowest Fourier modes $(m,n)=(2,0)$ and $(1,1)$, in the sense that $m^2+3n^2 = 2^2+3\cdot 0^2 = 1^2+3\cdot 1^2 = 4$, and there are not multiple solutions of (m,n) for $m^2+3n^2 \leq 3$.

In this section, we study the two-dimensional kernel bifurcation under the multiplicity of Fourier modes $(m,n) = (2,0)$ and $(1,1)$ without a one-dimensional kernel restriction. The kernel V is actually the linear span of the two Fourier bases:

$$V = \operatorname{span}\{\Phi_{20},\ \Phi_{11}\},$$

and hence $\dim V = 2$. Since R and W are isomorphic on $L|_W$, Z is the same linear span:

$$Z = \operatorname{span}\{\Phi_{20},\ \Phi_{11}\}.$$

The projection $P : Y \to Z$ is naturally introduced as:

$$P\,\Phi = \frac{\langle \Phi, \Phi_{20}\rangle_Y}{\|\Phi_{20}\|_Y^2}\,\Phi_{20} + \frac{\langle \Phi, \Phi_{11}\rangle_Y}{\|\Phi_{11}\|_Y^2}\,\Phi_{11} \in Z, \quad \Phi \in Y, \tag{4.1}$$

Where $\|\Phi_{20}\|_Y^2 = \frac{(1+\pi^2)\pi^2}{2\sqrt{3}l^2}$ and $\|\Phi_{11}\|_Y^2 = \frac{(1+\pi^2)\pi^2}{4\sqrt{3}l^2}$.

We extract $v^* \in V$ satisfying the sufficient conditions (a) and (b) in Theorem 2.1. By denoting

$$v^* = \alpha\,\Phi_{20} + \beta\,\Phi_{11} \coloneqq \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \in V; \quad \alpha,\ \beta \in \mathbb{R}, \tag{4.2}$$

we first determine α and β, as v^* satisfies the condition (a). The values Mv^* and $\mathcal{B}[v^*, v^*]$ are calculated as:

$$Mv^* = \begin{bmatrix} -\frac{1}{\mu}\Delta v_2^* \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4l^2\eta_{20}}{\mu}\left[\alpha\phi_2(x) + \beta\phi_1(x)\psi_1(y)\right] \\ 0 \end{bmatrix}, \tag{4.3}$$

$$\mathcal{B}[v^*, v^*] = \begin{bmatrix} -2\left[\chi^*\left(\nabla\cdot(v_1^*\nabla v_2^*)\right) + a\mu(v_1^*)^2\right] \\ 0 \end{bmatrix} = \begin{bmatrix} -\eta_{20}\chi^*\Delta\left(v_1^*\right)^2 - 2a\mu(v_1^*)^2 \\ 0 \end{bmatrix}. \tag{4.4}$$

By straightforward calculation we have

$$\frac{\langle Mv^*, \Phi_{20}\rangle_Y}{\|\Phi_{20}\|_Y^2} = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\alpha, \quad \frac{\langle Mv^*, \Phi_{11}\rangle_Y}{\|\Phi_{11}\|_Y^2} = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\beta.$$

We then obtain

$$PMv^* = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\left(\alpha\Phi_{20} + \beta\Phi_{11}\right). \tag{4.5}$$

Similarly, since

$$\frac{\langle \mathcal{B}[v^*, v^*], \Phi_{20}\rangle_Y}{\|\Phi_{20}\|_Y^2} = \frac{2\chi^* l^2\eta_{20} - a\mu}{2(1+\eta_{20}^2)}\beta^2, \quad \frac{\langle \mathcal{B}[v^*, v^*], \Phi_{11}\rangle_Y}{\|\Phi_{11}\|_Y^2} = \frac{2(2\chi^* l^2\eta_{20} - a\mu)}{1+\eta_{20}^2}\alpha\beta,$$

we have

$$P\mathcal{B}[v^*, v^*] = \frac{2\chi^* l^2\eta_{20} - a\mu}{2(1+\eta_{20}^2)}\left(\beta^2\Phi_{20} + 4\alpha\beta\Phi_{11}\right). \tag{4.6}$$

From the above, the condition (a) of Theorem 2.1 results in

$$\begin{aligned} PMv^* + \frac{1}{2}P\mathcal{B}[v^*, v^*] = {} & \frac{1}{4(1+\eta_{20}^2)\mu}\left[16l^2\eta_{20}\alpha + \mu(2\chi^* l^2\eta_{20} - a\mu)\beta^2\right]\Phi_{20} \\ & + \frac{\beta}{(1+\eta_{20}^2)\mu}\left[4l^2\eta_{20} + \mu(2\chi^* l^2\eta_{20} - a\mu)\alpha\right]\Phi_{11} = 0. \end{aligned} \tag{4.7}$$

As Φ_{20} and Φ_{11} are linearly independent in Y, we obtain the coefficients $(\alpha, \beta) \neq (0, 0)$ under the condition $2\chi^* l^2\eta_{20} - a\mu \neq 0$ where $(\alpha, \beta) = (A, -2A)$, $(A, 2A)$ with $A = -\frac{4l^2\eta_{20}}{\mu(2\chi^* l^2\eta_{20} - a\mu)}$. This shows that the v^* satisfying condition (a) are the following two candidates:

$$v^* = A\,(\Phi_{20} - 2\Phi_{11}), \quad A\,(\Phi_{20} + 2\Phi_{11}), \quad \text{where } A = -\frac{4l^2\eta_{20}}{\mu(2\chi^* l^2\eta_{20} - a\mu)}. \tag{4.8}$$

We display the profiles of these functions in Figure 1.

We here note that the latter result was first demonstrated in [11], and the former is newly derived in this paper, indeed, the former result does not have $2\pi/3$-rotational symmetry (see Figure 2).

Next, we consider the condition (b), that is, the invertibility of the operator $S : V \to Z$, $Sv = PMv + P\mathcal{B}[v^*, v]$, $v \in V$, with $v^* \in V$ fixed as in (4.8). Let us denote $v \in V$ as

$$v = \eta\,\Phi_{20} + \zeta\,\Phi_{11} \coloneqq \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}; \quad \eta,\ \zeta \in \mathbb{R}. \tag{4.9}$$

Then, we have

$$Mv = \begin{bmatrix} -\frac{1}{\mu}\Delta v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4l^2\eta_{20}}{\mu}\left[\eta\phi_2(x) + \zeta\phi_1(x)\psi_1(y)\right] \\ 0 \end{bmatrix}, \tag{4.10}$$

$$\begin{aligned} \mathcal{B}[v^*, v] &= \begin{bmatrix} -\chi^*\left[\nabla\cdot(v_1^*\nabla v_2) + \nabla\cdot(v_1\nabla v_2^*)\right] - 2a\mu v_1^* v_1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\eta_{20}\chi^*\Delta\left(v_1^* v_1\right) - 2a\mu v_1^*\, v_1 \\ 0 \end{bmatrix}. \end{aligned} \tag{4.11}$$

Since

$$\frac{\langle Mv, \Phi_{20}\rangle_Y}{\|\Phi_{20}\|_Y^2} = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\eta, \quad \frac{\langle Mv, \Phi_{20}\rangle_Y}{\|\Phi_{11}\|_Y^2} = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\zeta,$$

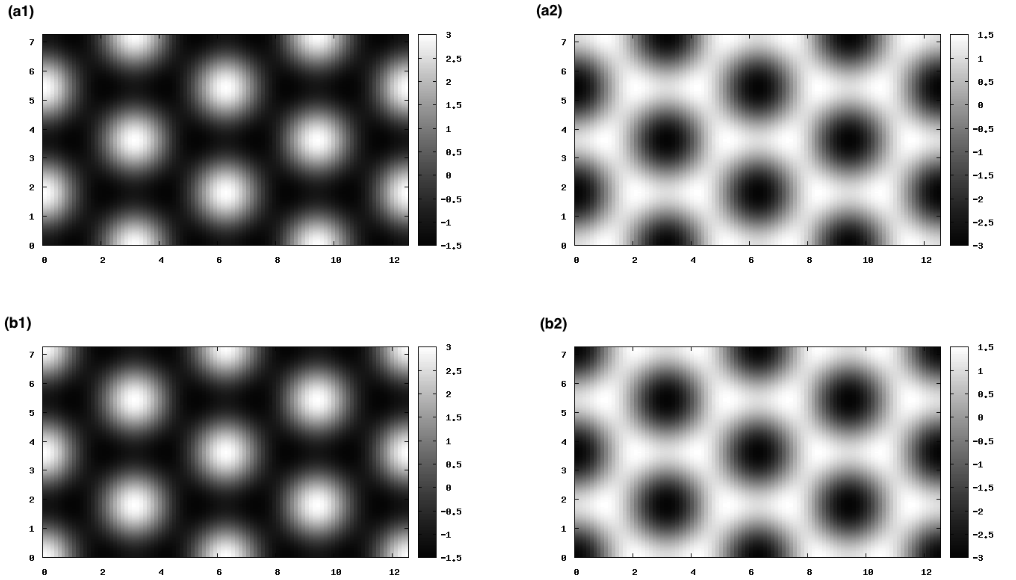

Figure 1: Plots of the functions belonging to the two-dimensional kernel $V = \mathrm{span}\{\Phi_{20}, \Phi_{11}\}$. The spatial domain is $\Omega = (0, 4\pi) \times (0, 4\sqrt{3}\pi)$. (a1) $v^* = A(\Phi_{20} - 2\Phi_{11})$ with $A = 1 > 0$. (a2) $v^* = A(\Phi_{20} - 2\Phi_{11})$ with $A = -1 < 0$. (b1) $v^* = A(\Phi_{20} + 2\Phi_{11})$ with $A = 1 > 0$. (b2) $v^* = A(\Phi_{20} + 2\Phi_{11})$ with $A = -1 < 0$.

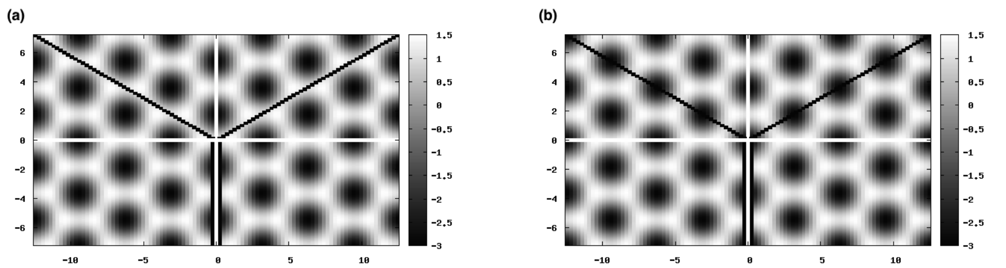

Figure 2: The spanning v^* in the spatial domain $(-4\pi, 4\pi) \times (-4\sqrt{3}\pi, 4\sqrt{3}\pi)$. The white horizontal and vertical lines represent the x and y axes, respectively. The black lines are auxiliary axes in the directions of $\pi/6$, $5\pi/6$, $3\pi/2$. (a) $v^* = A(\Phi_{20} - 2\Phi_{11})$ with $A = -1 < 0$, which does not have $2\pi/3$-rotational symmetry. And, (b) $v^* = A(\Phi_{20} + 2\Phi_{11})$ with $A = -1 < 0$, which have $2\pi/3$-rotational symmetry.

we then obtain

$$PMv = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\left(\eta\Phi_{20} + \zeta\Phi_{11}\right).$$

Similarly, since

$$\frac{\langle \mathcal{B}[v^*, v], \Phi_{20}\rangle_Y}{\|\Phi_{20}\|_Y^2} = \frac{2l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\zeta,$$
$$\frac{\langle \mathcal{B}[v^*, v], \Phi_{11}\rangle_Y}{\|\Phi_{11}\|_Y^2} = \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}(\eta-\zeta),$$

we have

$$P\mathcal{B}[v^*, v] = \frac{2l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\left[\zeta\Phi_{20} + 2(\eta-\zeta)\Phi_{11}\right].$$

From this, it follows that

$$Sv = \left(\frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\eta + \frac{2l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\zeta\right)\Phi_{20} + \frac{4l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\eta\,\Phi_{11} := [\Phi_{20}\ \Phi_{11}]\,\widetilde{S}\begin{bmatrix}\eta\\ \zeta\end{bmatrix}.$$

Here, $\widetilde{S}$ is the representation matrix of S:

$$\widetilde{S} = \frac{2l^2\eta_{20}}{\mu(1+\eta_{20}^2)}\begin{bmatrix}2 & 1\\ 2 & 0\end{bmatrix}.$$

Since

$$\det\widetilde{S} = -\frac{8l^4\eta_{20}^2}{\mu^2(1+\eta_{20}^2)^2} \neq 0,$$

the operator S is isomorphic.

We finally arrive at the main result of this section.

Theorem 4.1. Let $v^* \in V$ be the functions defined in (4.8), and $\chi^* = \chi(m,n)$. Then, under the conditions

$$2\chi^* l^2\eta_{20} - a\mu \neq 0,$$

there exists a local branch of nontrivial solutions $(\chi(\lambda), U(\lambda)) \in (0,\infty)\times X$ to (SE), with small parameter $\lambda \in (-\varepsilon, \varepsilon)$, which bifurcate from (χ^*, U^*) such that

$$\chi(\lambda) = \chi^* + \lambda, \quad U(\lambda) = U^* + \lambda[\,v^* + \lambda\tilde{v}(\lambda)\,],$$

where $\tilde{v}(\lambda)$ is a smooth function of λ.

5 Three-dimensional kernel bifurcation of chemotaxis-growth model.

In this section, we study the lowest dimension-three bifurcation along the Fourier modes $(m,n) = (1,3)$, $(4,2)$, $(5,1)$. Indeed, these are the triple solutions for $m^2+3n^2 = 28$, and there are no triple solutions for the case $m^2+3n^2 \leq 27$. Three-dimensional bifurcation is not analyzed in [11].

The kernel V is the linear span of the three Fourier bases:

$$V = \operatorname{span}\{\Phi_{13},\ \Phi_{42},\ \Phi_{51}\},$$

and hence $\dim V = 3$. Since R and W are isomorphic on $L|_W$, Z is the same linear span:

$$Z = \operatorname{span}\{\Phi_{13},\ \Phi_{42},\ \Phi_{51}\}.$$

The projection $P : Y \to Z$ is naturally introduced as:

$$P\,\Phi = \frac{\langle \Phi, \Phi_{13}\rangle_Y}{\|\Phi_{13}\|_Y^2}\,\Phi_{13} + \frac{\langle \Phi, \Phi_{42}\rangle_Y}{\|\Phi_{42}\|_Y^2}\,\Phi_{42} + \frac{\langle \Phi, \Phi_{51}\rangle_Y}{\|\Phi_{51}\|_Y^2}\,\Phi_{51} \in Z, \quad \Phi \in Y, \tag{5.1}$$

where $\|\Phi_{13}\|_Y^2 = \frac{(1+\eta_{13}^2)\pi^2}{4\sqrt{3}l^2}$, $\|\Phi_{42}\|_Y^2 = \frac{(1+\eta_{13}^2)\pi^2}{4\sqrt{3}l^2}$ and $\|\Phi_{51}\|_Y^2 = \frac{(1+\eta_{13}^2)\pi^2}{4\sqrt{3}l^2}$.

We set $v^* \in V$ so as to satisfy the sufficient conditions (a) and (b) in Theorem 2.1. By denoting

$$v^* = \alpha\,\Phi_{13} + \beta\,\Phi_{42} + \gamma\,\Phi_{51} \coloneqq \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \in V; \quad \alpha,\ \beta,\ \gamma \in \mathbb{R}, \tag{5.2}$$

we first determine α, β and γ, as v^* satisfies the condition (a). The values Mv^* and $\mathcal{B}[v^*, v^*]$ are calculated as:

$$Mv^* = \begin{bmatrix} -\frac{1}{\mu}\Delta v_2^* \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{28l^2\eta_{13}}{\mu}\left[\alpha\phi_1(x)\psi_3(y) + \beta\phi_4(x)\psi_2(y) + \gamma\phi_5(x)\psi_1(y)\right] \\ 0 \end{bmatrix}, \tag{5.3}$$

$$\mathcal{B}[v^*, v^*] = \begin{bmatrix} -2\left[\chi^*\left(\nabla\cdot(v_1^*\nabla v_2^*)\right) + a\mu(v_1^*)^2\right] \\ 0 \end{bmatrix} = \begin{bmatrix} -\eta_{13}\chi^*\Delta\,(v_1^*)^2 - 2a\mu(v_1^*)^2 \\ 0 \end{bmatrix}. \tag{5.4}$$

By straightforward calculation we have

$$\frac{\langle Mv^*, \Phi_{13}\rangle_Y}{\|\Phi_{13}\|_Y^2} = \frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\alpha, \quad \frac{\langle Mv^*, \Phi_{42}\rangle_Y}{\|\Phi_{42}\|_Y^2} = \frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\beta, \quad \frac{\langle Mv^*, \Phi_{51}\rangle_Y}{\|\Phi_{51}\|_Y^2} = \frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\gamma.$$

We then obtain

$$PMv^* = \frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\left(\alpha\,\Phi_{13} + \beta\,\Phi_{42} + \gamma\,\Phi_{51}\right). \tag{5.5}$$

Similarly, since

$$\begin{aligned} \frac{\langle \mathcal{B}[v^*, v^*], \Phi_{13}\rangle_Y}{\|\Phi_{13}\|_Y^2} &= \frac{14\chi^* l^2\eta_{13} - a\mu}{1+\eta_{13}^2}\,\beta\gamma, \\ \frac{\langle \mathcal{B}[v^*, v^*], \Phi_{42}\rangle_Y}{\|\Phi_{42}\|_Y^2} &= \frac{14\chi^* l^2\eta_{13} - a\mu}{1+\eta_{13}^2}\,\gamma\alpha, \\ \frac{\langle \mathcal{B}[v^*, v^*], \Phi_{51}\rangle_Y}{\|\Phi_{51}\|_Y^2} &= \frac{14\chi^* l^2\eta_{13} - a\mu}{1+\eta_{13}^2}\,\alpha\beta, \end{aligned}$$

we have

$$P\mathcal{B}[v^*, v^*] = \frac{14\chi^* l^2\eta_{13} - a\mu}{1+\eta_{13}^2}\left(\beta\gamma\,\Phi_{13} + \gamma\alpha\,\Phi_{42} + \alpha\beta\,\Phi_{51}\right). \tag{5.6}$$

From the above, the condition (a) of Theorem 2.1 gives

$$\begin{aligned} PMv^* + \frac{1}{2}P\mathcal{B}[v^*, v^*] = \frac{1}{2\mu(1+\eta_{13}^2)} \Big(&\left[56l^2\eta_{13}\alpha + \mu(14\chi^* l^2\eta_{13} - a\mu)\beta\gamma\right]\Phi_{13} \\ &+ \left[56l^2\eta_{13}\beta + \mu(14\chi^* l^2\eta_{13} - a\mu)\gamma\alpha\right]\Phi_{42} \\ &+ \left[56l^2\eta_{13}\gamma + \mu(14\chi^* l^2\eta_{13} - a\mu)\alpha\beta\right]\Phi_{51}\Big) = 0. \end{aligned} \tag{5.7}$$

As Φ_{13}, Φ_{42} and Φ_{51} are linearly independent in Y, we obtain the coefficients $(\alpha,\beta,\gamma)\neq(0,0,0)$ under the condition $14\chi^* l^2\eta_{13}-a\mu\neq 0$ that $(\alpha,\beta,\gamma)=(\widetilde{A},\widetilde{A},\widetilde{A})$, $(\widetilde{A},-\widetilde{A},-\widetilde{A})$, $(-\widetilde{A},\widetilde{A},-\widetilde{A})$, $(-\widetilde{A},-\widetilde{A},\widetilde{A})$ with $\widetilde{A}=-\frac{56l^2\eta_{13}}{\mu(14\chi^* l^2\eta_{13}-a\mu)}$. This shows that the v^* satisfying the condition (a) are the following four candidates:

$$
\begin{aligned}
(5.8)\quad v^* &= \widetilde{A}\,(\Phi_{13}+\Phi_{42}+\Phi_{51}), \quad \widetilde{A}\,(\Phi_{13}-\Phi_{42}-\Phi_{51}),\\
&\widetilde{A}\,(-\Phi_{13}+\Phi_{42}-\Phi_{51}), \quad \widetilde{A}\,(-\Phi_{13}-\Phi_{42}+\Phi_{51}), \quad \text{where } \widetilde{A}=-\frac{56l^2\eta_{13}}{\mu(14\chi^* l^2\eta_{13}-a\mu)}.
\end{aligned}
$$

We display the profiles of these functions in Figure 3. We here note that only the first $v^*=\widetilde{A}\,(\Phi_{13}+\Phi_{42}+\Phi_{51})$ has $2\pi/3$-rotational symmetry (see Figure 4).

Next, we consider the condition (b), that is, the invertibility of the operator $S:V\to Z$, $Sv=PMv+P\mathcal{B}[v^*,v]$, $v\in V$, with $v^*\in V$ fixed as in (5.8). Let us denote $v\in V$ as

$$
(5.9)\qquad v=\eta\,\Phi_{13}+\zeta\,\Phi_{42}+\xi\,\Phi_{51}:=\begin{bmatrix}v_1\\ v_2\end{bmatrix};\quad \eta,\ \zeta,\ \xi\in\mathbb{R}.
$$

Then, we have

$$
(5.10)\qquad Mv=\begin{bmatrix}-\frac{1}{\mu}\Delta v_2\\ 0\end{bmatrix}=\begin{bmatrix}\frac{28l^2\eta_{13}}{\mu}\left[\eta\,\phi_1(x)\psi_3(y)+\zeta\,\phi_4(x)\psi_2(y)+\xi\,\phi_5(x)\psi_1(y)\right]\\ 0\end{bmatrix},
$$

$$
\begin{aligned}
(5.11)\quad \mathcal{B}[v^*,v] &= \begin{bmatrix}-\chi^*\left[\nabla\cdot(v_1^*\nabla v_2)+\nabla\cdot(v_1\nabla v_2^*)\right]-2a\mu v_1^* v_1\\ 0\end{bmatrix}\\
&= \begin{bmatrix}-\eta_{13}\chi^*\Delta\,(v_1^* v_1)-2a\mu v_1^* v_1\\ 0\end{bmatrix}.
\end{aligned}
$$

Since

$$
\frac{\langle Mv,\Phi_{13}\rangle_Y}{\|\Phi_{13}\|_Y^2}=\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\eta,\quad \frac{\langle Mv,\Phi_{42}\rangle_Y}{\|\Phi_{42}\|_Y^2}=\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\zeta,\quad \frac{\langle Mv,\Phi_{51}\rangle_Y}{\|\Phi_{51}\|_Y^2}=\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\xi,
$$

we then obtain

$$
PMv=\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\left(\eta\Phi_{13}+\zeta\Phi_{42}+\xi\Phi_{51}\right).
$$

Similarly, since

$$
\begin{aligned}
\frac{\langle\mathcal{B}[v^*,v],\Phi_{13}\rangle_Y}{\|\Phi_{13}\|_Y^2} &= -\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\left(\zeta+\xi\right),\\
\frac{\langle\mathcal{B}[v^*,v],\Phi_{42}\rangle_Y}{\|\Phi_{42}\|_Y^2} &= -\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\left(\xi+\eta\right),\\
\frac{\langle\mathcal{B}[v^*,v],\Phi_{51}\rangle_Y}{\|\Phi_{51}\|_Y^2} &= -\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\left(\eta+\zeta\right),
\end{aligned}
$$

we have

$$
P\mathcal{B}[v^*,v]=-\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}[(\zeta+\xi)\Phi_{13}+(\xi+\eta)\Phi_{42}+(\eta+\zeta)\Phi_{51}].
$$

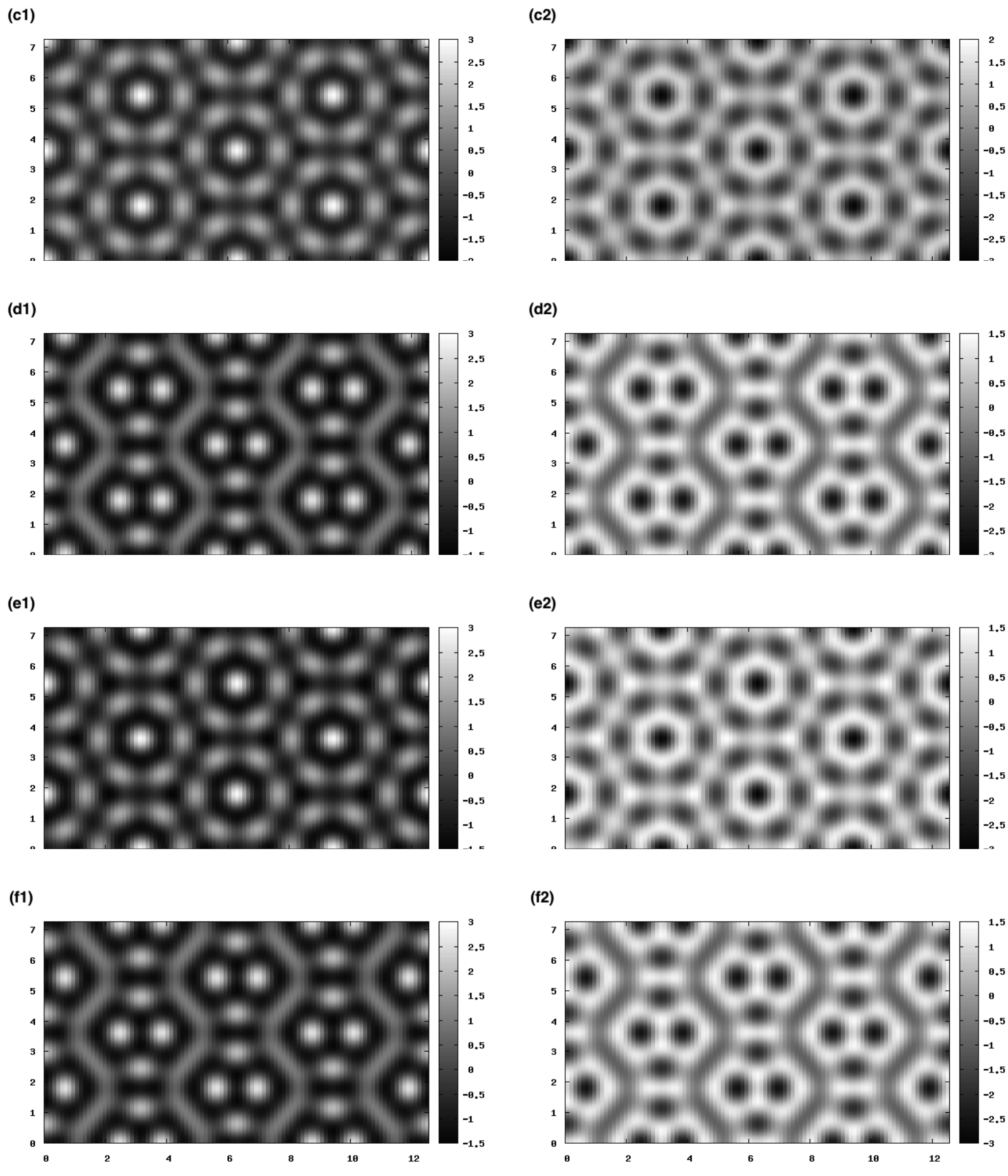

Figure 3: Plots of the functions for the three-dimensional kernel $V = \mathrm{span}\{\Phi_{13}, \Phi_{42}, \Phi_{51}\}$. The spatial domain is $\Omega = (0, 4\pi) \times (0, 4\sqrt{3}\pi)$. (c1) $v^* = \widetilde{A}(\Phi_{13}+\Phi_{42}+\Phi_{51})$ with $\widetilde{A} = 1 > 0$. (c2) $v^* = \widetilde{A}(\Phi_{13}+\Phi_{42}+\Phi_{51})$ with $\widetilde{A} = -1 < 0$. (d1) $v^* = \widetilde{A}(\Phi_{13}-\Phi_{42}-\Phi_{51})$ with $\widetilde{A} = 1 > 0$. (d2) $v^* = \widetilde{A}(\Phi_{13} - \Phi_{42} - \Phi_{51})$ with $\widetilde{A} = -1 < 0$. (e1) $v^* = \widetilde{A}(-\Phi_{13} + \Phi_{42} - \Phi_{51})$ with $\widetilde{A} = 1 > 0$. (e2) $v^* = \widetilde{A}(-\Phi_{13}+\Phi_{42}-\Phi_{51})$ with $\widetilde{A} = -1 < 0$. (f1) $v^* = \widetilde{A}(-\Phi_{13}-\Phi_{42}+\Phi_{51})$ with $\widetilde{A} = 1 > 0$. (f2) $v^* = \widetilde{A}(-\Phi_{13} - \Phi_{42} + \Phi_{51})$ with $\widetilde{A} = -1 < 0$.

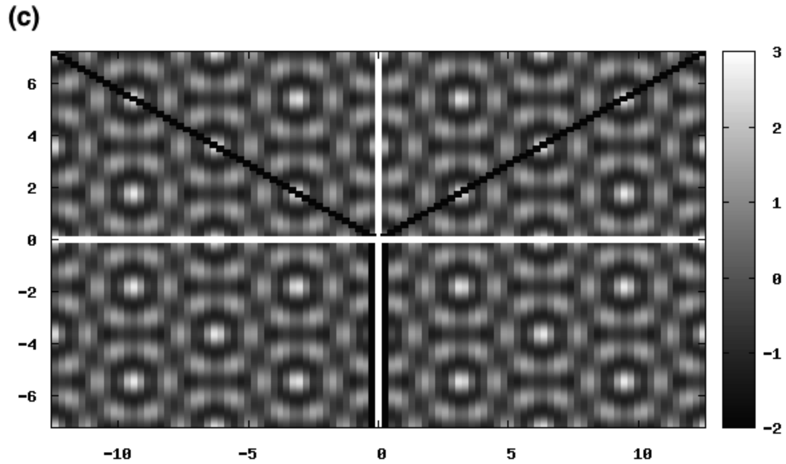

Figure 4: The spanning v^* for the spatial domain $(-4\pi, 4\pi) \times (-4\sqrt{3}\pi, 4\sqrt{3}\pi)$. The white horizontal and vertical lines represent the x and y axes, respectively. The black lines are the auxiliary axes in the directions of $\pi/6$, $5\pi/6$, $3\pi/2$. (c) $v^* = \widetilde{A}(\Phi_{13} + \Phi_{42} + \Phi_{51})$ with $\widetilde{A} = 1 > 0$, which has $2\pi/3$-rotational symmetry.

From this, it follows that

$$Sv = \widehat{S}\left([\eta - \zeta - \xi]\,\Phi_{13} + [-\eta + \zeta - \xi]\,\Phi_{42} + [-\eta - \zeta + \xi]\,\Phi_{51}\right)$$
$$:= [\Phi_{13}\ \Phi_{42}\ \Phi_{51}]\,\widetilde{S}\begin{bmatrix}\eta\\ \zeta\\ \xi\end{bmatrix}, \quad \widehat{S} = \frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}.$$

Here, $\widetilde{S}$ is the representation matrix of S:

$$\widetilde{S} = \begin{bmatrix}\widehat{S} & -\widehat{S} & -\widehat{S}\\ -\widehat{S} & \widehat{S} & -\widehat{S}\\ -\widehat{S} & -\widehat{S} & \widehat{S}\end{bmatrix}.$$

Because of

$$\det \widetilde{S} = -4\widehat{S}^{\,3} = -4\left(\frac{28l^2\eta_{13}}{\mu(1+\eta_{13}^2)}\right)^3 \neq 0,$$

the operator S is isomorphic.

We finally arrive at the main result of this section.

Theorem 5.1. Let $v^* \in V$ be the functions defined in (5.8). Then, under the conditions

$$14\chi^* l^2\eta_{13} - a\mu \neq 0,$$

there exists a local branch of nontrivial solutions $(\chi(\lambda), U(\lambda)) \in (0,\infty) \times X$ to (SE), with small parameter $\lambda \in (-\varepsilon, \varepsilon)$, which bifurcates from (χ^*, U^*) such that

$$\chi(\lambda) = \chi^* + \lambda, \quad U(\lambda) = U^* + \lambda[\, v^* + \lambda\tilde{v}(\lambda)\,],$$

where $\tilde{v}(\lambda)$ is a smooth function of λ.

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