ALMOST CONTRA-b-CONTINUOUS FUNCTIONS

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Abstract. In [1], the authors introduced and studied the notion of almost contra-b-continuity in topological spaces. In this paper, we investigate some more properties of this type of continuity.

1 Introduction Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, seperation axioms etc. by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of b-open [2] sets introduced by Andrijevic in 1996. This class is a subset of the class of semi-preopen sets [3], that is a subset of a topological space which is contained in the closure of the interior of its closure. Also, a class of b-open sets is a superset of the class of semi-open sets [17], that is a set which is contained in the closure of its interior, and the class of preopen sets [19], that is a set which is contained in the interior of its closure. Andrijevic studied several fundamental and interesting properties of b-open sets. In [1], the authors introduced and studied the notion of almost contra-b-continuity in topological spaces. In this paper, we investigate some more properties of this type of continuity.

2 Preliminaries Throughout the paper (X,τ) and (Y,σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), $Cl(A), Int(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X, respectively. A subset A of X is said to be regular open [26] (resp. semi-open [17], preopen [19], α-open [21], b-open [2]($=\gamma$-open [13])) if $A = Int(Cl(A))$ (resp. $A \subset Cl(Int(A))$, $A \subset Int(Cl(A))$, $A \subset Int(Cl(Int(A)))$, $A \subset Int(Cl(A)) \cup Cl(Int(A))$). The family of all α-open (resp. regular open, b-open) subsets of X is denoted by $\alpha O(X)$ (resp. $RO(X)$, $BO(X)$). The complement of semi-open (resp. regular open, preopen, b-open) is called semi-closed [7] (resp. regular closed, preclosed [19], b-closed [2]). The family of all regular closed sets (resp. b-closed sets) of (X,τ) is denoted by $RC(X)$ (resp. $BC(X)$). The intersection of all regular open sets containing A is called the r-kernal [9] of A and is denoted by $rKer(A)$. The intersection of all semi-closed (resp. preclosed, b-closed) sets containing A is called the semi-closure [6] (resp. pre-closure [19], b-closure [2]) of A and is denoted by $sCl(A)$ (resp. $pCl(A)$, $bCl(A)$). A subset A is b-closed if and only if $A= bCl(A)$. For each $x \in X$, the family of all b-open (resp. b-closed, semi-open, regular open, regular closed) sets containing x is denoted by $BO(X,x)$ (resp. $BC(X,x)$, $SO(X,x)$, $RO(X,x)$, $RC(X,x)$). The θ-semi-closure [16] of A, denoted by θ-$sCl(A)$, is defined to be the set of all $x \in X$ such that $A \cap Cl(U) \neq \varnothing$ for every $U \in SO(X,x)$. A subset A is called θ-semi-closed [16] if and only if $A = \theta$-$sCl(A)$.

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The complement of θ-semi-closed set is called θ-semi-open [16]. For a subset A of X, $sCl(A)$=$A \cup Int(Cl(A))$ [3], $pCl(A) = A \cup Cl(Int(A))$ [3] and $bCl(A) = sCl(A) \cap pCl(A)$ [2]. If A is open in a space X, then $sCl(A) = Int(Cl(A))$ [3]. It follows that, if A is open in a space X, then $bCl(A) = Int(Cl(A))$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be b-continuous [13] (resp. contra-b-continuous [20]) if $f^{-1}(V)$ is b-open (resp. b-closed) set in X for each open set V of Y. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly b-continuous [24] (or almost weakly b-continuous [1]) if for every $x \in X$ and every open set V of Y containing $f(x)$, there exists $U \in BO(X, x)$ such that $f(U) \subset Cl(V)$.

3 Almost contra-b-continuous functions

Definition 3.1 [1] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *almost contra-b-continuous* if $f^{-1}(V) \in BC(X)$ for each $V \in RO(Y)$ (cf. Remark 3.4 below).

It is clear that every contra-b-continuous function is almost contra-b-continuous but the converse is not true in general.

Example 3.2 Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \tau)$ is almost contra-b-continuous but not contra-b-continuous.

Theorem 3.3 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) *f is almost contra-b-continuous;*

(ii) *$f^{-1}(F) \in BO(X)$ for every $F \in RC(Y)$;*

(iii) *for each $x \in X$ and each $F \in RC(Y, f(x))$, there exists $U \in BO(X, x)$ such that $f(U) \subset F$;*

(iv) *$f^{-1}(Int(Cl(G))) \in BC(X)$ for every open subset G of (Y, σ);*

(v) *$f^{-1}(Cl(Int(F))) \in BO(X)$ for every closed subset F of (Y, σ);*

(vi) *$f(bCl(A)) \subset rKer(f(A))$ for every subset A of (X, τ);*

(vii) *$bCl(f^{-1}(B)) \subset f^{-1}(rKer(B))$ for every subset B of (Y, σ).*

Proof (i)⇔(ii): Let $F \in RC(Y)$. Then $Y \backslash F \in RO(Y)$. By (i), $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F) \in BC(X)$. We have $f^{-1}(F) \in BO(X)$. The proof of the reverse in similar.
(ii)⇒(iii): Let $F \in RC(Y, f(x))$. By (ii), $f^{-1}(F) \in BO(X)$ and $x \in f^{-1}(F)$. Take $U = f^{-1}(F)$, then $f(U) \subset F$.
(iii)⇒(ii): Let $F \in RC(Y)$ and $x \in f^{-1}(F)$. From (iii), there exists a b-open set U_x in X containing x such that $U_x \subset f^{-1}(F)$. We have $f^{-1}(F) = \bigcup\{U_x | x \in f^{-1}(F)\}$. Since any union of b-open sets is b-open, $f^{-1}(F)$ is b-open in X.
(i) ⇔(iv): Let G be an open subset of Y. Since $Int(Cl(G))$ is regular open, then by (i), it follows that, $f^{-1}(Int(Cl(G))) \in BC(X)$. The converse can be shown similarly.
(iii)⇒(vi): Let $A \subset X$ and let $x \in bCl(A)$ and $F \in RC(Y, f(x))$. By (iii), there exists $U \in BO(X, x)$ such that $f(U) \subset F$. Since $x \in bCl(A)$, we have $U \cap A \neq \emptyset$. Hence, $f(U) \cap f(A) \neq \emptyset$ and therefore $F \cap f(A) \neq \emptyset$. It follows from Proposition 24(i) of [9] that $f(x) \in rKer(f(A))$ and hence $f(bCl(A)) \subset rKer(f(A))$.
(vi)⇒(vii): Let $B \subset Y$. By (vii), $f(bCl(f^{-1}(B))) \subset rKer(f(f^{-1}(B))) \subset rKer(B)$. Hence $bCl(f^{-1}(B)) \subset f^{-1}(rKer(B))$.
(vii)⇒(i): Let $V \in RO(Y)$. Then by (vii), $bCl(f^{-1}(V)) \subset f^{-1}(rKer(V))$. Since $V \in RO(Y), rKer(V) = V$ and hence $bCl(f^{-1}(V)) \subset f^{-1}(V)$, which shows that $f^{-1}(V)$ is b-closed. Consequently, f is almost contra-b-continuous. □

Remark 3.4 (i) A function $f : (X, \tau) \to (Y, \sigma)$ is called: *almost contra-b-continuous at a point* $x \in X$, if for each regular closed subset V of (Y, σ) containing $f(x)$, there exists a b-open subset U of (X, τ) containing x such that $f(U) \subset V$.

(ii) By Theorem 3.3 and definitions, it is shown that a function $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous if and only if f is almost contra-b-continuous at each point of X.

Theorem 3.5 (i) *If $f : (X, \tau) \to (Y, \sigma)$ is weakly-b-continuous and (Y, σ) is regular, then f is b-continuous.*

(ii) *If $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous and (Y, σ) is regular, then f is b-continuous.*

(iii) *If $f : (X, \tau) \to (Y, \sigma)$ is contra-b-continuous and (Y, σ) is regular, then f is b-continuous.*

Proof. Clear. □

Sometimes, the concept of a b-open set (resp. b-closed set) of a topological space (X, τ) is called a γ-open set (resp. γ-closed set); and so the family $BO(X)$ (resp. $BC(X)$) is denoted by $\gamma O(X)$ (resp. $\gamma C(X)$).

Lemma 3.6 [13] *Let A and B be subsets of a topological space (X, τ).*

(i) *If $A \in \gamma O(X)$ and $B \in \alpha O(X)$, then $A \cap B \in \gamma O(B)$.*

(ii) *Let $A \subset B \subset X$, $A \in \gamma O(B)$ and $B \in \alpha O(X)$, then $A \in \gamma O(X)$.*

Theorem 3.7 *If $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous and $U \in \alpha O(X)$, then $f|U : (U, \tau|U) \to (Y, \sigma)$ is almost contra-b-continuous.*

Proof. Let V be a regular closed subset of Y. We have $(f|U)^{-1}(V) = f^{-1}(V) \cap U$. Since $f^{-1}(V)$ is b-open and U is α-open, it follows from the Lemma 3.6 (i) that $(f|U)^{-1}(V)$ is b-open in the relative topology of U. Thus, $f|U$ is almost contra-b-continuous. □

Theorem 3.8 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $x \in X$. If there exists $U \in BO(X, x)$ and $f|U : (U, \tau|U) \to (Y, \sigma)$ is almost contra-b-continuous at x, then f is almost contra-b-continuous at x.*

Proof. Suppose that $F \in RC(Y, f(x))$. Since $f|U$ is almost contra-b-continuous at x, there exists $V \in BO(U, x)$ such that $f(V) = (f|U)(V) \subset F$. Since $U \in \alpha O(X, x)$, it follows from Lemma 3.6 (ii) that $V \in BO(X, x)$. This shows that f is almost contra-b-continuous at x. □

Theorem 3.9 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $\Sigma = \{U_i : i \in I\}$ be a cover of X by α-open sets of (X, τ). If for each $i \in I$, $f|U_i : (U_i, \tau|U_i) \to (Y, \sigma)$ is almost contra-b-continuous, then $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous.*

Proof. Let $V \in RC(Y)$. Since $f|U_i$ is almost contra-b-continuous for each $i \in I$, $(f|U_i)^{-1}(V) \in BO(U_i)$, since $U_i \in \alpha O(X)$, by Lemma 3.6 (2), $(f|U_i)^{-1}(V) \in BO(X)$ for each $i \in U$. Then $f^{-1}(V) = \bigcup\{(f|U_i)^{-1}(V) \in BO(X) | i \in I\}$. This gives f is almost contra-b-continuous. □

Theorem 3.10 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and let $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ be the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is almost contra-b-continuous, then f is almost contra-b-continuous.*

Proof. Let $V \in RC(Y)$, then $X \times V = X \times Cl(Int(V)) = Cl(Int(X)) \times Cl(Int(V)) = Cl(Int(X \times V))$. Then $X \times V \in RC(X \times Y)$. Since g is almost contra-b-continuous, then $f^{-1}(V) = g^{-1}(X \times V) \in BO(X)$. Thus, f is almost contra-b-continuous. □

Definition 3.11 A function $f : (X, \tau) \to (Y, \sigma)$ is said to be
(i) *pre-b-open* if $f(U) \in BO(Y)$ for each $U \in BO(X)$,
(ii) *b-irresolute* [13] if for each $x \in X$ and each $V \in BO(Y, f(x))$, there exists $U \in BO(X, x)$ such that $f(U) \subset V$,
(iii) *θ-irresolute* [13] if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists $U \in SO(X, x)$ such that $f(Cl(U)) \subset Cl(V)$.

Theorem 3.12 *If $f : (X, \tau) \to (Y, \sigma)$ is a surjective pre-b-open and $g : (Y, \sigma) \to (Z, \gamma)$ is a function such that $g \circ f$: $(X, \tau) \to (Z, \gamma)$ is almost contra-b-continuous, then g is almost contra-b-continuous.*

Proof. Let V be any regular closed set in Z. Since $g \circ f$ is almost contra-b-continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is b-open. Since f is surjective pre-b-open, $f(f^{-1}(g^{-1}(V)))$ $= g^{-1}(V)$ is b-open. Therefore, g is almost contra-b-continuous. □

Theorem 3.13 (i) *If $f : (X, \tau) \to (Y, \sigma)$ is b-irresolute and $g : (Y, \sigma) \to (Z, \gamma)$ is almost contra-b-continuous, then $g \circ f$: $(X, \tau) \to (Z, \gamma)$ is almost contra-b-continuous.*
(ii) *If $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous and $g : (Y, \sigma) \to (Z, \gamma)$ is θ-irresolute, then $g \circ f$: $(X, \tau) \to (Z, \gamma)$ is almost contra-b-continuous.*

Proof. (i) Let $x \in X$ and $W \in SO(Z)$. Then there exists a set $U \in BO(X, x)$ such that $(g \circ f)(U) \subset Cl(W)$. Therefore, $g \circ f$ is almost contra-b-continuous.
(ii) Similar to (i). □

Definition 3.14 A filter base Λ is said to be *b-convergent* (resp. *rc-convergent* [12]) *to a point* $x \in X$ if for any $U \in BO(X, x)$ (resp. $U \in RC(X, x)$), there exists $B \in \Lambda$ such that $B \subset U$.

Theorem 3.15 *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous function, then for each point $x \in X$ and each filter base Λ in X b-converging to x, the filter base $f(\Lambda)$ is rc-convergent to $f(x)$.*

Proof. Let $x \in X$ and Λ be any filter base in X b-converging to x. Since f is almost contra-b-continuous, then for any $V \in RC(Y, f(x))$, there exists $U \in BO(X, x)$ such that $f(U) \subset V$. Since Λ is b-converging to x, there exists a $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and therefore the filter base $f(\Lambda)$ is rc-convergent to $f(x)$. □

4 Separation axioms and covering properties

Definition 4.1 A topological space (X, τ) is said to be
(i) $P_{\sum}$ [30] if for any open set V of (X, τ) and each $x \in V$, there exists $F \in RC(X, x)$ such that $x \in F \subset V$,
(ii) *weakly* $P_{\sum}$ [22] if for any $V \in RO(X, x)$, there exists $F \in RC(X, x)$ such that $x \in F \subset V$.

Theorem 4.2 *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous function and (Y, σ) is $P_{\sum}$, then f is b-continuous.*

Proof. Let V be any open set in Y. Since Y is $P_{\sum}$, there exists a subfamily $\mathcal{A}$ of $RC(Y)$ such that $V = \cup\{F : F \in \mathcal{A}\}$. Since f is almost contra-b-continuous, $f^{-1}(F)$ is b-open in X for each $F \in \mathcal{A}$ and $f^{-1}(V)$ is b-open in X. Therefore, f is b-continuous. $\square$

Theorem 4.3 *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous function and (Y, σ) is weakly $P_{\sum}$, then f is almost b-continuous.*

Proof. Similar to the proof of Theorem 4.2. $\square$

Definition 4.4 A topological space (X, τ) is said to be

(i) *weakly Hausdorff* [28] if each element of X is an intersection of regular closed sets,

(ii) *b-T_0* [5] if for each pair of distinct points in X, there exists a b-open set of (X, τ) containing one point but not the other,

(iii) *b-T_1* [5] if for each pair of distinct points x and y of X, there exist b-open sets U and V of (X, τ) containing x and y, respectively such that $y \notin U$ and $x \notin V$,

(iv) *b-T_2* [10] if for each pair of distinct points x and y of X, there exist b-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Theorem 4.5 *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous injection and (Y, σ) is weakly Hausdorff, then (X, τ) is b-T_1.*

Proof. Suppose that Y is weakly Hausdorff. For any two distinct points x and y in X, there exist $V, W \in RC(Y)$ such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is almost contra-b-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are b-open subsets of X such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that X is b-T_1. $\square$

Definition 4.6 A topological space (X, τ) is said to be

(i) *hyperconnected* [27] if every open set is dense,

(ii) *ultra b-connected* if every two non-void b-closed subsets of (X, τ) intersect,

(iii) *b-connected* [13] provided that X is not the union of two disjoint nonempty b-open sets.

Theorem 4.7 *If (X, τ) is ultra b-connected and $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous surjection, then (Y, σ) is hyperconnected.*

Proof. Assume that Y is not hyperconnected. Then there exists an open set V such that V is not dense in Y. Then there exist disjoint nonempty regular open subsets B_1 and B_2 in Y, namely $Int(Cl(V))$ and $Y \setminus Cl(V)$. Since f is almost contra-b-continuous surjection, $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ are disjoint nonempty b-closed subsets of X. By assumption, the ultra-b-connectedness of X implies that A_1 and A_2 must intersect. By contradiction, Y is hyperconnected. $\square$

Theorem 4.8 (i) [24] *If $f : (X, \tau) \to (Y, \sigma)$ is weakly b-continuous surjection and (X, τ) is b-connected, then (Y, σ) is connected.*

(ii) *If $f : (X, \tau) \to (Y, \sigma)$ is a almost contra-b-continuous surjection and (X, τ) is b-connected, then (Y, σ) is connected.*

(iii) [20] *If $f : (X, \tau) \to (Y, \sigma)$ is a contra-b-continuous surjection and (X, τ) is b-connected, then (Y, σ) is connected.*

Theorem 4.9 (i) [24] *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly b-continuous injection and (Y, σ) is Urysohn, then (X, τ) is b-T_2.*

(ii) *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous injection and (Y, σ) is Urysohn, then (X, τ) is b-T_2.*

(iii) [20] *If $f : (X, \tau) \to (Y, \sigma)$ is a contra-b-continuous injection and (Y, σ) is Urysohn, then (X, τ) is b-T_2.*

Definition 4.10 [15] A topological space (X, τ) is said to be *θ-irreducible* if every pair of nonempty regular closed sets of (X, τ) has a nonempty intersection.

Theorem 4.11 *If (X, τ) is b-connected and $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous surjection, then (Y, σ) is θ-irreducible.*

Proof. Similar to that proof of Theorem 4.7. □

Definition 4.12 [13] A topological space (X, τ) is said to be *b-normal* provided that every pair of nonempty disjoint closed sets can be separated by disjoint b-open sets.

Theorem 4.13 (i) *If (Y, σ) is normal and $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous closed injection, then (X, τ) is b-normal.*

(ii) [20] *If (Y, σ) is normal and $f : (X, \tau) \to (Y, \sigma)$ is a contra-b-continuous closed injection, then (X, τ) is b-normal.*

Proof. (i) Let F_1 and F_2 be disjoint nonempty closed sets of X. Since f is injective and closed, $f(F_1)$ and $f(F_1)$ are disjoint closed sets of Y. Since Y is normal, there exist open sets V_1 and V_2 of Y such that $f(F_1) \subset V_1$, $f(F_2) \subset V_2$ and $Cl(V_1) \cap Cl(V_2) = \emptyset$. Then, since $Cl(V_1), Cl(V_2) \in RC(Y)$ and f is almost contra-b-continuous, $f^{-1}(Cl(V_1)), f^{-1}(Cl(V_2)) \in BO(X)$. Since $F_1 \subset f^{-1}(V_1)$, $F_2 \subset f^{-1}(V_2)$ and $f^{-1}(Cl(V_1))$ and $f^{-1}(Cl(V_2))$ are disjoint, X is b-normal. □

Definition 4.14 A cover $\sum = \{U_i : i \in I\}$ of subsets of X is called a *b-cover* if U_i is b-open in (X, τ) for each $i \in I$.

Definition 4.15 A topological space (X, τ) is said to be

(i) *b-compact* [23] (resp. *S-closed* [29]) if every b-open (resp. regular closed) cover of X has a finite subcover,

(ii) *countably b-compact* [11] (resp. *countably S-closed* [8]) if every countable cover of X by b-open (resp. regular closed) sets has a finite subcover,

(iii) *b-Lindelöf* [11] (resp. *S-Lindelöf* [18]) if every b-open (resp. regular closed) cover of X has a countable subcover.

Definition 4.16 A topological space (X, τ) is said to be

(i) *b-closed compact* [11] (resp. *nearly compact* [25]) if every b-closed (resp. regular open) cover of X has a finite subcover,

(ii) *countably b-closed compact* [11] (resp. *nearly countably compact* [14]) if every countable cover of X by b-closed (resp. regular open) sets has a finite subcover,

(iii) *b-closed Lindelöf* [11] (resp. *nearly Lindelöf* [14]) if every b-closed (resp. regular open) cover of X has a countable subcover.

Theorem 4.17 *Let $f : (X, \tau) \to (Y, \sigma)$ be an almost contra-b-continuous surjection. Then the following statements hold.*

(i) *If (X, τ) is b-closed compact, then Y is nearly compact.*

(ii) *If (X, τ) is b-closed Lindelöf, then Y is nearly Lindelof.*

(iii) *If (X, τ) is countably b-closed compact, then Y is nearly countably compact.*

Proof. We prove only (i), the proofs of (ii) and (iii) being entirely analogous.
Let $\{V_i : i \in I\}$ be any regular open cover of Y. Since f is almost contra-b-continuous, $\{f^{-1}(V_i)$: $i \in I\}$ is a b-closed cover of X. Since X is b-closed compact, there exists a a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_i) : i \in I_0\}$. Therefore, we have $Y = \cup\{V_i : i \in I_0\}$ and Y is S-closed. □

Definition 4.18 [26] A topological space (X, τ) is said to be *mildly compact* (resp. *mildly contably compact, mildly Lindelöf*) if every clopen (resp. countable clopen, clopen) cover of X has a finite (resp. finite, countable) subcover.

Theorem 4.19 *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous and almost continuous surjection and (X, τ) is mildly compact* (resp. *mildly countably compact, mildly Lindelöf), then Y is nearly compact* (resp. *nearly countably compact, nearly Lindelöf*) *and S-closed* (resp. *countably S-closed, S-Lindelöf*).

Proof. Let $V \in RC(Y)$. Then since f is almost contra-b-continuous and almost continuous, $f^{-1}(V)$ is b-open and closed in X and hence $f^{-1}(V)$ is clopen (resp. open). Let $\{V_i : i \in I\}$ be any regular closed (resp. regular open) cover of Y. Then $\{f^{-1}(V_i) : i \in I\}$ is a clopen cover of X and since X is mildly compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_i) : i \in I_0\}$. Since f is surjective, we obtain $Y = \cup\{V_i : i \in I_0\}$. This shows that Y is S-closed (resp. nearly compact).
The other proofs can be obtained similarly. □

Definition 4.20 A topological space (X, τ) is said to be *s-Urysohn* [4] if for each pair of distinct points x and y in X, there exist $U \in SO(X, x)$ and $V \in SO(X, y)$ such that $Cl(U) \cap Cl(V) = \emptyset$.

Theorem 4.21 *If (Y, σ) is s-Urysohn and $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-b-continuous injection, then (X, τ) is b-T_2.*

Proof. It is similar to Proof of Theorem 4.5. □

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)); x \in X\} \subset X \times Y$ is called the *graph of* f and is denoted by $G(f)$.

Definition 4.22 A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be *regular b-closed* if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in BC(X, x)$ and $V \in RO(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Theorem 4.23 *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is regular b-closed if and only if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in BC(X, x)$ and $V \in RO(Y, y)$ such that $f(U) \cap V = \emptyset$.*

Proof. This is an immediate consequence of Definition 4.22. □

Theorem 4.24 *Let $f : (X, \tau) \to (Y, \sigma)$ have a regular b-closed graph $G(f)$. If f is injective, then (X, τ) is b-T_1.*

Proof. Let x and y be any two distinct points of X. Then, we have $(x, f(y)) \in (X \times Y)\backslash G(f)$. By definition of regular b-closed graph, there exist $U \in BC(X)$ and $V \in RO(Y)$ such that $(x, f(y)) \in U \times V$ and $f(U) \cap V = \emptyset$; hence $U \cap f^{-1}(V) = \emptyset$. Therefore, we have $y \notin U$. Thus, $y \in X\backslash U$ and $x \notin X\backslash U$. We obtain that $X\backslash U \in BO(X)$. This implies that X is b-T_1. □

Theorem 4.25 *If $f : (X, \tau) \to (Y, \sigma)$ is almost contra-b-continuous and (Y, σ) is T_2, then the graph $G(f)$ is regular b-closed.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$. Then $f(x) \neq y$. Snce Y is T_2, there exist open sets V and W containing $f(x)$ and y, respectively, such that $V \cap W = \emptyset$; hence $Int(Cl(V)) \cap Int(Cl(W)) = \emptyset$. Since f is almost contra-b-continuous, $f^{-1}(Int(Cl(V)))$ is b-closed containing x. Take $U = f^{-1}(Int(Cl(V)))$. Then $f(U) \subset Int(Cl(V))$. Therefore, $f(U) \cap Int(Cl(W)) = \emptyset$ and hence the graph $G(f)$ is regular b-closed. □

Theorem 4.26 *Let $f : (X, \tau) \to (Y, \sigma)$ have a regular b-closed graph $G(f)$. If f is surjective, then (Y, σ) is weakly Hausdorff.*

Proof. Let $y \in Y$. Since f is surjective, $f(x) = y$ for some $x \in X$ and $(x, a) \in (X \times Y)\backslash G(f)$ for any point $a \in Y$ such that $a \neq y$. For the points y and a, by definition of regular b-closed graph $G(f)$, there exists a b-closed set U_a of X and $F(a) \in RO(Y)$ such that $(x, a) \in U_a \times F(a)$ and $f(U_a) \cap F(a) = \emptyset$; hence $y \notin F(a)$. Then, $\{y\} \subset A$, where $A = \cap\{Y\backslash F(z) : z \neq y\}$. In order to prove $\{y\} \supset A$, let $w \in A$ and suppose that $w \notin \{y\}$. The, for any point z with $z \neq y$, we have that $w \in Y\backslash F(z)$. Since $w \neq y$, we can take $z = w$ and so $w \in F(w)$. This is a contradicition. Hence we show that $\{y\} = A$; and so $\{y\}$ is an intersection of regular closed sets $Y\backslash F(z)$, where $z \neq y$, that is (Y, σ) is weakly Hausdorff. □

5 Additional Properties

Theorem 5.1 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
(i) *f is almost contra-b-continuous;*
(ii) *$f^{-1}(V) \in BO(X)$ for each θ-semi-open set V of (Y, σ);*
(iii) *$f^{-1}(F) \in BC(X)$ for each θ-semi-closed set F of (Y, σ);*
(iv) *for each $x \in X$ and each $U \in SO(Y, f(x))$, there exists $V \in BO(X, x)$ such that $f(V) \subset Cl(U)$;*
(v) *$f^{-1}(U) \subset bInt(f^{-1}(Cl(U)))$ for every $U \in SO(Y)$;*
(vi) *$f(bCl(A)) \subset \theta\text{-}sCl(f(A))$ for every subset A of (X, τ);*
(vii) *$bCl(f^{-1}(B)) \subset f^{-1}(\theta\text{-}sCl(B))$ for every subset B of $(Y.\sigma)$;*
(viii) *$bCl(f^{-1}(V)) \subset f^{-1}(\theta\text{-}sCl(V))$ for every open subset V of (Y, σ);*
(ix) *$bCl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every open subset V of (Y, σ);*
(x) *$bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V))$ for every open subset V of (Y, σ).*

Proof. (i)⇒(ii): This follows from the fact that every θ-semi-open set is the union of regular closed sets.
(ii)⇔(iii): This is obvious.
(ii)⇒(iv): Let $x \in X$ and $U \in SO(Y, f(x))$. Since $Cl(U)$ is θ-semi-open in Y, there exists $V \in BO(X, x)$ such that $x \in V \subset f^{-1}(Cl(U))$ and hence $f(V) \subset Cl(U)$.
(iv)⇒(v): Let $U \in SO(Y)$ and $x \in f^{-1}(U)$. Then $f(x) \in U$. By (iv), there exists $V \in BO(X, x)$ such that $f(V) \subset Cl(U)$. It follows that $x \in V \subset f^{-1}(Cl(U))$. Hence, $x \in bInt(f^{-1}(Cl(U)))$.
(v)⇒(i): Let $F \in RC(Y)$. Since $F \in SO(Y)$, then by (v), $f^{-1}(F) \subset bInt(f^{-1}(Cl(F)))$ and consequently, $f^{-1}(F) \in BO(X)$. Hence, by Theorem 3.3, (i) holds.
(iv)⇒(vi): Let A be any subset of X. Suppose that $x \in bCl(A)$ and $G \in SO(Y, f(x))$. By (v), there exists $V \in BO(X, x)$ such that $f(V) \subset Cl(G)$. Since $x \in bCl(A)$, $V \cap A \neq \emptyset$ and hence $\emptyset \neq f(V) \cap f(A) \subset Cl(G) \cap f(A)$. Therefore, we obtain $f(x) \in \theta\text{-}sCl(f(A))$ and hence $f(bCl(A)) \subset \theta\text{-}sCl(f(A))$.

(vi)$\Rightarrow$(vii): Let B be any subset of Y. Then $f(bCl(f^{-1}(B))) \subset (\theta\text{-}sCl(f(f^{-1}(B))) \subset \theta\text{-}sCl(B)$ and hence $bCl(f^{-1}(B)) \subset f^{-1}(\theta\text{-}sCl(B))$.
(vii)$\Rightarrow$(viii): Obvious.
(viii)$\Rightarrow$(ix): Follows from the fact that $\theta\text{-}sCl(V) = sCl(V)$ for every open subset V of Y.
(ix)$\Rightarrow$(x): Obvious.
(x)$\Rightarrow$(i): Let $V \in RO(Y)$. By (x), $bCl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V))) = f^{-1}(V)$ and hence $f^{-1}(V) \in BC(X)$, which proves that f is almost contra-b-continuous. $\square$

Theorem 5.2 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*
(i) *f is almost contra-b-continuous;*
(ii) *$f^{-1}(Cl(V)) \in BO(X)$ for every $V \in SPO(Y)$;*
(iii) *$f^{-1}(Cl(V)) \in BO(X)$ for every $V \in SO(Y)$;*
(iv) *$f^{-1}(Int(Cl(V))) \in BC(X)$ for every $V \in PO(Y)$.*

Proof. (i)$\Rightarrow$(ii): Let V be any semi-preopen set of Y. It follows from Theorem 2.4 of [3] that $Cl(V)$ is regular closed. Then by Theorem 3.3 $f^{-1}(Cl(V)) \in BO(X)$.
(ii)$\Rightarrow$(iii): This is obvious since $SO(Y) \subset SPO(Y)$.
(iii)$\Rightarrow$(iv): Let $V \in PO(Y)$. Then $Y \setminus Int(Cl(V))$ is regular closed and hence it is semi-open. Then $X \setminus f^{-1}(Int(Cl(V)))=f^{-1}(Y \setminus Int(Cl(V))) = f^{-1}(Cl(Y \setminus Int(Cl(V)))) \in BO(X)$. Hence $f^{-1}(Int(Cl(V))) \in BC(X)$.
(iv)$\Rightarrow$(i): Let V be any regular open set of Y. Then $V \in PO(Y)$ and hence $f^{-1}(V) = f^{-1}(Int(Cl(V)))$ is b-closed in X. $\square$

Definition 5.3 [11] The *b-frontier* of a subset A of a topological space (X, τ), $bFr(A)$, is defined by $bFr(A) = bCl(A) \cap bCl(X \setminus A) = bCl(A) \cap (X \setminus bInt(A))$.

Theorem 5.4 *For a function $f : (X, \tau) \to (Y, \sigma)$, we introduce the following notations relating to f:*
$A_f := \{x \in X : f$ *is not almost contra-b-continuous at* $x\}$,
$B_f(x) := \cup\{bFr(f^{-1}(F_x)) : F_x \in RC(Y, f(x))\}$, *where* $x \in A_f$, *and*
$B_f := \bigcup\{B_f(x) | x \in A_f\}$.
Then, we have the following properties:
If $z \in A_f$, then $z \in B_f(z)$; and so $A_f \subset B_f$ holds in (X, τ).

Proof. Let $z \in A_f$. Namely, we suppose that f is not almost contra-b-continuous at $z \in X$. By Theorem 3.3, there exists a subset $F_z \in RC(Y, f(z))$ such that $f(U) \cap (Y \backslash F_z) \neq \emptyset$ for every $U \in BO(X, z)$. By the property (5) in Proposition 5 of [10], $z \in bCl(f^{-1}(Y \setminus F_z))$ holds: and so $z \in bCl(X \setminus f^{-1}(F_z))$. On the other hand, we obtain $z \in f^{-1}(F_z) \subset bCl(f^{-1}(F_z))$; and hence $z \in bFr(f^{-1}(F_z))$, $F_z \in RC(Y, f(z))$ and $z \in A_f$. Namely, if $z \in A_f$, then $z \in B_f(z)$ holds; and so we have $A_f \subset \bigcup\{B_f(z) | z \in A_f\} = B_f$ holds in (X, τ).

$\square$

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