TRANSFORMS ON OPERATOR MONOTONE FUNCTIONS

Masaru Nagisa

Received August 31, 2017 ; revised December 7, 2017

Abstract. Let f be an operator monotone function on $[0,\infty)$ with $f(t)\geq 0$ and $f(1)=1$. If $f(t)$ is neither the constant function 1 nor the identity function t, then

$$h(t)=\frac{(t-a)(t-b)}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(b))}\qquad t\geq 0$$

is also operator monotone on $[0,\infty)$, where $a,b\geq 0$ and

$$f^\sharp(t)=\frac{t}{f(t)}\qquad t\geq 0.$$

Moreover, we show some extensions of this statement.

1 Introduction and History We call a real continuous function $f(t)$ on an interval I operator monotone on I (in short, $f\in\mathbb{P}(I)$), if $A\leq B$ implies $f(A)\leq f(B)$ for any self-adjoint matrices A,B with their spectrum containd in I. In this paper, we consider only the case $I=[0,\infty)$ or $I=(0,\infty)$. We denote $f\in\mathbb{P}_+(I)$ if $f\in\mathbb{P}(I)$ satisfies $f(t)\geq 0$ for any $t\in I$.

Let $\mathbb{H}_+$ be the upper-half plain of $\mathbb{C}$, that is,

$$\mathbb{H}_+=\{z\in\mathbb{C}\mid \mathrm{Im}z>0\}=\{z\in\mathbb{C}\mid |z|>0,\ 0<\arg z<\pi\},$$

where $\mathrm{Im}z$ (resp. $\arg z$) means the imaginary part (resp. the argument) of z. As Loewner's theorem, it is known that f is operator monotone on I if and only if f has an analytic continuation to $\mathbb{H}_+$ that maps $\mathbb{H}_+$ into its closure $\overline{\mathbb{H}_+}$ and also has an analytic continuation to the lower half-plane $\mathbb{H}_-$, obtained by the reflection across I. (see [1],[3],[5]).

D. Petz [11] proved that an operator monotone function $f:[0,\infty)\longrightarrow[0,\infty)$ satisfying the functional equation

$$f(t)=tf(t^{-1})\qquad t\geq 0$$

2010 *Mathematics Subject Classification.* 15A45, 15A60, 47A63.

Key words and phrases. operator monotone functions, Pick functions, Petz-Hasegawa's functions.

is related to a Morozova-Chentsov function [9] which gives a monotone metric on the manifold of $n \times n$ density matrices. In the work [12], the concrete functions (Petz-Hasegawa's functions)

$$f_a(t) = a(1-a)\frac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} \qquad (-1 \le a \le 2)$$

appeared and their operator monotonicity was proved (see also [2]). V.E.S. Szabo introduced an interesting idea for checking their operator monotonicity in [13], but his idea was something strange. We use a similar idea in our argument. M. Uchiyama [14] proved the operator monotonicity of the following extended functions:

$$\frac{(t-a)(t-b)}{(t^p-a^p)(t^{1-p}-b^{1-t})} \in \mathbb{P}_+[0.\infty)$$

for $0 < p < 1$ and $a, b > 0$. The main result of this paper is as follows:

Theorem 1. *Let a and b be non-negative real numbers. If $f \in \mathbb{P}_+[0,\infty)$ and both f and $f^\sharp$ are not constant, then*

$$h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(b))} \in \mathbb{P}_+[0.\infty),$$

where

$$f^\sharp(t) = \frac{t}{f(t)} \qquad t \ge 0.$$

The proof of this statement was given in [7] by the author and his student, M. Kawasaki. We also made its revised version as [8]. But these manuscripts have been unpublished. F. Hansen [4] has inspired by [8] and given the different proof of the above statement in the case $a = b = 1$. The proof in this paper was based on the theory of Complex Analysis and we have considered this method useful. The same method was introduced for the proof of the operator monotonicity of Petz-Hasegawa's functions in [6]. Also the autor and S. Wada have extended the method and succeeded to justify Szabo's result in a sense [10]. The last section we will prove main theorem and give some applications.

2 Main result For $f \in \mathbb{P}[0,\infty)$, we have the following integral representation:

$$f(z) = f(0) + \beta z + \int_0^\infty \frac{\lambda z}{z+\lambda} dw(\lambda),$$

where $\beta \geq 0$ and
$$\int_0^\infty \frac{\lambda}{1+\lambda} dw(\lambda) < \infty$$
(see [1]). When $f(0) \geq 0$ (i.e., $f \in \mathbb{P}_+[0,\infty)$), it holds that $0 < \arg f(z) \leq \arg z$ whenever $0 < \arg z < \pi$.

For any $f \in \mathbb{P}_+[0,\infty)$ ($f \neq 0$), we define $f^\sharp$ as follows:
$$f^\sharp(t) = \frac{t}{f(t)} \qquad t \in [0,\infty).$$
Then it is well-known $f^\sharp \in \mathbb{P}_+[0,\infty)$.

Proposition 2. *Let f be an operator monotone function on $(0,\infty)$ and a be a positive real number.*

(1) *When $f(t)$ is not constant, we have*
$$g_1(t) = \frac{t-a}{f(t)-f(a)} \in \mathbb{P}_+[0,\infty).$$

(2) *When $f(t) \geq 0$ for $t \geq 0$, we have*
$$g_2(t) = \frac{f(t)(t-a)}{tf(t)-af(a)} \in \mathbb{P}_+[0,\infty).$$

Proof. (1) It has proved in [14]. We state the outline of the proof. For $f \in \mathbb{P}_+(0,\infty)$, we have
$$f(z) = \alpha + \beta z + \int_0^\infty (-\frac{1}{x+z} + \frac{x}{x^2+1}) d\nu(x) \quad (\alpha \in \mathbb{R}, \beta \geq 0)$$
for $z \in \mathbb{H}_+ \cup (0,\infty)$ by Loewner's theorem ([1], [3], [6]). Since
$$g_1(z) = \frac{z-a}{f(z)-f(a)} = \frac{1}{\beta + \int_0^\infty \frac{1}{(x+z)(x+a)} d\nu(x)},$$
we have
$$\begin{aligned}\mathrm{Im} g_1(z) &= \frac{-1}{|g_1(z)|^2} \mathrm{Im}(\beta + \int_0^\infty \frac{1}{(x+z)(x+a)} d\nu(x)) \\ &= \frac{-1}{|g_1(z)|^2} \int_0^\infty \frac{1}{x+a} \mathrm{Im} \frac{1}{x+z} d\nu(x) > 0\end{aligned}$$

for $z \in \mathbb{H}_+$. This implies $g_1 \in \mathbb{P}_+[0,\infty)$.

(2) Since $g_2([0,\infty)) \subset [0,\infty)$, it suffices to show that $g_2(\mathbb{H}_+) \subset \mathbb{H}_+$. By the calculation

$$\begin{aligned} g_2(z) &= \frac{zf(z) - af(a) + af(a) - f(z)a}{zf(z) - af(a)} = 1 - \frac{a(f(z) - f(a))}{zf(z) - af(a)} \\ &= 1 - \frac{a}{\dfrac{zf(z) - af(a)}{f(z) - f(a)}} = 1 - \frac{a}{z + f(a)g_1(z)}, \end{aligned}$$

we have

$$\mathrm{Im} g_2(z) = -\mathrm{Im}\frac{a}{z + f(a)g_1(z)} = \mathrm{Im}\frac{a(z + f(a)g_1(z))}{|z + f(a)g_1(a)|^2}.$$

When $z \in \mathbb{H}_+$, $\mathrm{Im} g_1(z) > 0$ by (1) and $\mathrm{Im} g_2(z) > 0$. So $g_2(t)$ belongs to $\mathbb{P}_+[0,\infty)$. □

Lemma 3. *Let $a \geq 0$ and $0 < p < 1$. If f is a non-constant operator monotone function on $(0,\infty)$, then we have*

$$f(t^p) - f(a^p) \neq 0 \text{ and } tf(t^p) - af(a^p) \neq 0$$

for any $t \in (-\infty, 0)$.

Proof. When f is operator monotone and not constant, we have $\mathrm{Im} f(z) > 0$ for any $z \in \mathbb{H}_+$ by the maximum principle for the harmonic function $\mathrm{Im} f$ on $\mathbb{H}_+$. For any $t = |t|e^{i\pi} \in (-\infty, 0)$, we have $t^p \in \mathbb{H}_+$ and $\mathrm{Im} f(t^p) > 0$. This implies

$$\mathrm{Im} f(t^p) \neq \mathrm{Im} f(a^p) = 0 \text{ and } \mathrm{Im} f(t^p) \neq \mathrm{Im}\frac{af(a^p)}{t} = 0,$$

that is, $f(t^p) - f(a^p) \neq 0$ and $tf(t^p) - af(a^p) \neq 0$. □

Lemma 4. *For any $z \in \mathbb{H}_+$ and a positive integer n $(n \geq 2)$, we have*

$$\arg z < \arg(z - l) < \frac{\pi + (n-1)\arg z}{n} \quad \text{if} \quad 0 < l \leq \frac{|z|}{n-1}.$$

Proof. It is clear that $\arg z < \arg(z - l)$ for $z \in \mathbb{H}_+$ and $l > 0$.

It suffices to show that, for $z = e^{i\theta}$ $(0 < \theta < \pi)$,

$$\arg(z - l) < \frac{\pi + (n-1)\theta}{n} \quad \text{if } 0 < l \leq \frac{1}{n-1}.$$

We set

$$w = \frac{\sin\theta}{\sin\frac{\pi+(n-1)\theta}{n}} e^{i(\pi+(n-1)\theta)/n}.$$

Then we have $\mathrm{Im} z = \mathrm{Im} w$ and

$$0 < z - w = \cos\theta - \frac{\sin\theta}{\sin\frac{\pi+(n-1)\theta}{n}} \cos\frac{\pi+(n-1)\theta}{n} = \frac{\sin\frac{\pi-\theta}{n}}{\sin\frac{\pi+(n-1)\theta}{n}}.$$

By the estimation

$$\inf\{z-w \mid 0<\theta<\pi\} = \inf\{\frac{\sin\frac{\pi-\theta}{n}}{\sin\frac{\pi+(n-1)\theta}{n}} \mid 0<\theta<\pi\}$$

$$= \inf\{\frac{\sin t}{\sin(\pi-(n-1)t)} \mid 0<t<\frac{\pi}{n}\} = \inf\{\frac{\sin t}{\sin(n-1)t} \mid 0<t<\frac{\pi}{n}\} = \frac{1}{n-1},$$

we can get the desired result. □

Now we can prove the following theorem and remark that Theorem 1 follows from this statement because $f(t)f^\sharp(t)/t = 1 \in \mathbb{P}_+[0,\infty)$:

Theorem 5. *Let n be a positive integer, $a, b, b_1, \ldots, b_n \geq 0$ and $f, g, g_1, \ldots, g_n$ be non-constant, non-negative operator monotone functions on $[0,\infty)$.*

(1) *If $\dfrac{f(t)g(t)}{t}$ is operator monotone on $[0,\infty)$, then the function*

$$h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(g(t)-g(b))}$$

is operator monotone on $[0,\infty)$ for any $a, b \geq 0$.

(2) *If $\dfrac{f(t)}{\prod_{i=1}^n g_i(t)}$ is operator monotone on $[0,\infty)$, then the function*

$$h(t) = \frac{(t-a)}{(f(t)-f(a))} \prod_{i=1}^n \frac{g_i(t)(t-b_i)}{tg_i(t)-b_i g_i(b_i)}$$

is operator monotone on $[0,\infty)$ for any $a, b \geq 0$.

Proof. (1) By $f, g \in \mathbb{P}_+[0,\infty)$ and Proposition 2 (1),

$$\frac{t-a}{f(t)-f(a)} \text{ and } \frac{t-b}{g(t)-g(b)}$$

are operator monotone on $[0,\infty)$. Therefore

$$h(z)=\frac{(z-a)(z-b)}{(f(z)-f(a))(g(z)-g(b))}$$

is holomorphic on $\mathbb{H}_+$, continuous on $\mathbb{H}_+\cup[0,\infty)$ and satisfies $h([0,\infty))\subset[0,\infty)$ and

$$\arg h(z)=\arg\frac{z-a}{f(z)-f(a)}+\arg\frac{z-b}{g(z)-g(b)}>0 \text{ for } z\in\mathbb{H}_+.$$

We assume that $f(z)$ and $g(z)$ are continuous on the closure $\overline{\mathbb{H}_+}$ of $\mathbb{H}_+$ and

$$f(t)-f(a)\neq 0 \text{ and } g(t)-g(b)\neq 0 \text{ for any } t\in(-\infty,0).$$

Then $h(z)$ is continuous on $\overline{\mathbb{H}_+}$.

In the case $z\in(-\infty,0)$, i.e., $|z|>0$ and $\arg z=\pi$, we have

$$\begin{aligned}
&\arg h(z)\\
=&\arg(z-a)-\arg(f(z)-f(a))+\arg(z-b)-\arg(g(z)-g(b))\\
\le&\pi-\arg f(z)+\pi-\arg g(z)\\
\le&2\pi-\arg z=\pi \qquad (\text{since } \arg f(z)+\arg g(z)-\arg z\ge 0).
\end{aligned}$$

So it holds $0\le\arg h(z)\le\pi$.

In the case that $z\in\mathbb{H}_+$ satisifying $|z|>\max\{a,b\}$, it holds that

$$\arg(z-a),\arg(z-b)<\frac{\pi+\arg z}{2}$$

by Lemma 4 (as $l=\max\{a,b\}$ and $n=2$). Since

$$\begin{aligned}
\arg h(z)&=\arg(z-a)-\arg(f(z)-f(a))+\arg(z-b)-\arg(g(z)-g(b))\\
&<\frac{\pi+\arg z}{2}-\arg f(z)+\frac{\pi+\arg z}{2}-\arg g(z)\\
&=\pi+\arg z-\arg f(z)-\arg g(z)\le\pi,
\end{aligned}$$

we have $0<\arg h(z)<\pi$.

For $r>0$, we define $H(r)=\{z\in\mathbb{C}\mid |z|\le r, \mathrm{Im}z\ge 0\}$. Whenever $r>l=\max\{a,b\}$, we can get

$$0\le\arg h(z)\le\pi$$

on the boundary of $H(r)$. Since $h(z)$ is holomorphic on the interior $H(r)^\circ$ of $H(r)$ and continuous on $H(r)$, $\mathrm{Im}h(z)$ is harmonic on $H(r)^\circ$ and continuous on

$H(r)$. Because $\mathrm{Im}h(z) \geq 0$ on the boundary of $H(r)$, we have $h(H(r)) \subset \overline{\mathbb{H}_+}$ by the minimum principle of harmonic functions. This implies

$$h(\overline{\mathbb{H}_+}) = h(\bigcup_{r>l} H(r)) \subset \bigcup_{r>l} h(H(r)) \subset \overline{\mathbb{H}_+},$$

and $h \in \mathbb{P}_+[0, \infty)$.

In general case, we define f_p and g_p as follows $(0 < p < 1)$:

$$f_p(t) = f(t^p) \text{ and } g_p(t) = t^{1-p} g(t^p).$$

It is clear $f_p \in \mathbb{P}_+[0, \infty)$. When $0 < \arg z < \pi$, we have

$$\begin{gathered} \arg g_p(z) \geq (1-p) \arg z > 0 \text{ and} \\ \arg g_p(z) = (1-p) \arg z + \arg g(z^p) \leq (1-p) \arg z + p \arg z < \pi. \end{gathered}$$

So $g_p \in \mathbb{P}_+[0, \infty)$. For any $t \in (-\infty, 0)$, we have $f_p(t) - f_p(a) \neq 0$ by Lemma 3 and $g_p(t) - g_p(a) \neq 0$ because $t = |t|e^{\pi i}$ and $(1-p)\pi < \arg g_p(t) \leq \pi$. So we have

$$h_p(z) = \frac{(z-a)(z-b)}{(f_p(z) - f_p(a))(g_p(z) - g_p(b))}$$

is holomorphic on $\mathbb{H}_+$ and continuous on $\overline{\mathbb{H}_+}$. Since $\frac{f(t)g(t)}{t} \in \mathbb{P}_+[0, \infty)$,

$$\frac{f_p(t)g_p(t)}{t} = \frac{f(t^p)t^{1-p}g(t^p)}{t} = \frac{f(t^p)g(t^p)}{t^p}$$

also belongs to $\mathbb{P}_+[0, \infty)$. By the above argument, we have $h_p \in \mathbb{P}_+[0, \infty)$.

Since

$$\begin{aligned} h_p(t) &= \frac{(t-a)(t-b)}{(f_p(t) - f_p(a))(g_p(t) - g_p(b))} \\ &= \frac{(t-a)(t-b)}{(f(t^p) - f(a^p))(t^{1-p}g(t^p) - b^{1-p}g(b^p))} \qquad \text{for } t \geq 0, \end{aligned}$$

we have

$$\lim_{p \to 1-0} h_p(t) = h(t).$$

So we can get the operator monotonicity of $h(t)$.

(2) We show this by the similar way as (1). By Proposition 2,

$$\frac{t-a}{f(t) - f(a)} \text{ and } \frac{g_i(t)(t-b_i)}{tg_i(t) - b_i g_i(b_i)} \quad (i = 1, 2, \ldots, n)$$

are operator monotone on $[0,\infty)$. So we have that

$$h(z) = \frac{z-a}{f(z)-f(a)} \prod_{i=1}^{n} \frac{g_i(z)(z-b_i)}{zg_i(z)-b_ig_i(b_i)}$$

is holomorphic on $\mathbb{H}_+$, continuous on $\mathbb{H}_+ \cup [0,\infty)$ and satisfies $h([0,\infty)) \subset [0,\infty)$ and

$$\arg h(z) = \arg \frac{z-a}{f(z)-f(a)} + \sum_{i=1}^{n} \arg \frac{g_i(z)(z-b_i)}{zg_i(z)-b_ig_i(b_i)} > 0$$

for $z \in \mathbb{H}_+$.

We assume that $f(z)$ and $g_i(z)$ $(i = 1, 2, \dots, n)$ are continuous on $\overline{\mathbb{H}_+}$ and

$$f(t) - f(a) \neq 0 \text{ and } tg_i(t) - b_ig_i(b_i) \neq 0 \text{ for any } t \in (-\infty, 0).$$

Then $h(z)$ is continuous on $\overline{\mathbb{H}_+}$.

In the case $z \in (-\infty, 0)$, i.e., $|z| > 0$ and $\arg z = \pi$, we have

$$\begin{aligned}
&\arg h(z) \\
=&\arg(z-a) + \sum_{i=1}^{n} \arg g_i(z)(z-b_i) - \arg(f(z)-f(a)) - \sum_{i=1}^{n} \arg(zg_i(z)-b_ig_i(b_i)) \\
\leq&\pi + \sum_{i=1}^{n} \arg g_i(z) + n\pi - \arg f(z) - n\pi \quad (\text{since } \arg(zg_i(z)-b_ig_i(b_i)) \geq \pi) \\
\leq&\pi \qquad (\text{since } \arg f(z) - \sum_{i=1}^{n} \arg g_i(z) \geq 0).
\end{aligned}$$

So it holds $0 \leq \arg h(z) \leq \pi$.

In the case $z \in \mathbb{H}_+$ satisifying $|z| > n \max\{a, b_1, b_2, \dots, b_n\}$, it holds that

$$\arg(z-a), \arg(z-b_i) < \frac{\pi + n \arg z}{n+1} \qquad (i = 1, 2, \dots, n)$$

by Lemma 4. We may assume that there exists a number k $(1 \leq k \leq n)$ such that

$$\arg(zg_i(z)) \leq \pi \quad (i \leq k), \qquad \arg(zg_i(z)) > \pi \quad (i > k).$$

Since

$$
\begin{aligned}
&\arg h(z) \\
=&\arg(z-a)+\sum_{i=1}^{n}\arg(z-b_i)+\sum_{i=1}^{n}\arg g_i(z) \\
&\qquad -\arg(f(z)-f(a))-\sum_{i=1}^{n}\arg(zg_i(z)-b_ig_i(b_i)) \\
\leq&\frac{\pi+n\arg z}{n+1}\times(n+1)+\sum_{i=1}^{n}\arg g_i(z) \\
&\qquad -\arg f(z)-\sum_{i=1}^{k}\arg zg_i(z)-(n-k)\pi \\
=&\pi+n\arg z+\sum_{i=k+1}^{n}\arg g_i(z)-\arg f(z)-k\arg z-(n-k)\pi \\
\leq&\pi+(n-k)\arg z-(n-k)\pi\leq\pi,
\end{aligned}
$$

we have $0 \leq \arg h(z) \leq \pi$.

This means that it holds

$$0 \leq \arg h(z) \leq \pi$$

if z belongs to the boundary of $H(r) = \{z \in \mathbb{C} \mid |z| \leq r, \mathrm{Im} z \geq 0\}$ for a sufficiently large r. Using the same argument in (1), we can prove the operator monotonicity of h.

In general case, we define functions, for p $(0 < p < 1)$, as follows:

$$f_p(t) = f(t^p), \quad g_{i,p}(t) = g_i(t^p) \quad (i = 1, 2, \dots, n).$$

Since $f, g_i \in \mathbb{P}_+[0, \infty)$,

$$0 < \arg f_p(z) < \pi, \quad 0 < \arg zg_{i,p}(z) < 2\pi$$

for $z \in \mathbb{H}_+$. This means that $f_p(z)$ and $g_{i,p}(z)$ are continuous on $\overline{\mathbb{H}_+}$ and

$$f_p(t) - f_p(a) \neq 0 \text{ and } tg_{i,p}(t) - b_i g_{i,p}(b_i) \neq 0 \text{ for any } t \in (-\infty, 0)$$

by Lemma 3. Since

$$\frac{f_p(t)}{\prod_{i=1}^{n} g_{i,p}(t)} = \frac{f(t^p)}{\prod_{i=1}^{n} g_i(t^p)} \qquad (0 < p < 1)$$

is operator monotone on $[0, \infty)$, we can get the operator monotonicity of

$$\begin{aligned} h_p(t) &= \frac{t-a}{f_p(t)-f_p(a)} \prod_{i=1}^{n} \frac{g_{i,p}(t)(t-b_i)}{tg_{i,p}(t)-b_i g_{i,p}(b_i)} \\ &= \frac{t-a}{f(t^p)-f(a^p)} \prod_{i=1}^{n} \frac{g_i(t^p)(t-b_i)}{tg_i(t^p)-b_i g_i(b_i^p)}. \end{aligned}$$

So we can see that

$$h(t) = \lim_{p \to 1-0} h_p(t)$$

is operator monotone on $[0, \infty)$. □

Remark 6. *Using Proposition 2 and Theorem 5, we can prove the operator monotonicity of the concrete functions in [12]. Since* t^a $(0 < a < 1)$ *and* $\log t$ *is operator monotone on* $(0, \infty)$*,*

$$\begin{aligned} f_a(t) &= a(1-a)\frac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} \qquad (-1 \le a \le 2) \\ &= \begin{cases} a(a-1)\dfrac{t^{-a}(t-1)^2}{(t^{-a}-1)(t \cdot t^{-a}-1)} & -1 \le a < 0 \\ \dfrac{t-1}{\log t} & a = 0, 1 \\ a(1-a)\dfrac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} & 0 < a < 1 \\ a(a-1)\dfrac{t^{a-1}(t-1)^2}{(t^{a-1}-1)(t \cdot t^{a-1}-1)} & 1 < a \le 2 \end{cases} \end{aligned}$$

becomes operator monotone.

We can also prove this remark and the first part of Example 10 using some formula stated in [10] as Theorem 1.2.

Corollary 7. *Let* $f \in \mathbb{P}_+(0, \infty)$ *and both* f *and* $f^\sharp$ *be not constant. For any* $a > 0$*, we define*

$$h_a(t) = \frac{(t-a)(t-a^{-1})}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(a^{-1}))} \qquad t \in (0, \infty).$$

Then we have

(1) h_a *is operator monotone on* $(0, \infty)$*.*

(2) $f(t) = t \cdot f(t^{-1})$ *implies* $h_a(t) = t \cdot h_a(t^{-1})$.

(3) $a = 1$ *and* $f(t^{-1}) = f(t)^{-1}$ *imply* $h_1(t) = t \cdot h_1(t^{-1})$.

Proof. We can directly prove (1) from Theorem 5. Because

$$\begin{aligned} t \cdot h_a(t^{-1}) &= \frac{t(t^{-1}-a)(t^{-1}-a^{-1})}{(f(t^{-1})-f(a))(f^\sharp(t^{-1})-f^\sharp(a^{-1}))} \\ &= \frac{(t-a)(t-a^{-1})}{t(f(t^{-1})-f(a))(f^\sharp(t^{-1})-f^\sharp(a^{-1}))}, \end{aligned}$$

we can compute

$$\begin{aligned} &t(f(t^{-1})-f(a))(f^\sharp(t^{-1})-f^\sharp(a^{-1})) - (f(t)-f(a))(f^\sharp(t)-f^\sharp(a^{-1})) \\ =&(f(t^{-1})-f(a))(1/f(t^{-1})-t/af(a^{-1})) - (f(t)-f(a))(t/f(t)-1/af(a^{-1})) \\ =&0 \end{aligned}$$

if it holds $f(t) = t \cdot f(t^{-1})$ or $a = 1$, $f(t^{-1}) = f(t)^{-1}$. So we have (2) and (3). □

The function h is called symmetric if it satisfies the following condition:

$$h(t) = th(t^{-1}), \qquad t \geq 0.$$

We can define a symmetric operator mean using a symmetric operator monotone function in the sense of Kubo-Ando ([5], [6]). Corollary 7 says that we can repeatedly construct a symmetric operator monotone function from a symmetric operator monotone function. We can give the following examples.

Example 8. *If we choose* t^p $(0 < p < 1)$ *as* $f(t)$ *in Corollary* 7(3),

$$h(t) = \frac{(t-1)^2}{(t^p-1)(t^{1-p}-1)}.$$

If we choose $t^p + t^{1-p}$ $(0 < p < 1)$ *as* $f(t)$ *in Corollary* 7(2),

$$\begin{aligned} h(t) &= \frac{t-a}{t^p+t^{1-p}-a^p-a^{1-p}} \times \frac{t-a^{-1}}{\dfrac{1}{t^{p-1}+t^{-p}} - \dfrac{1}{a^p+a^{1-p}}} \qquad (a > 0) \\ &= \frac{\sqrt{t}(\cosh(\log t)-\cosh(\log a))}{\cosh(\log\sqrt{t}) - \cosh(\log\sqrt{t} + \log(t^p+t^{1-p}) - \log(a^p+a^{1-p}))}. \end{aligned}$$

These functions, $h \in \mathbb{P}_+[0,\infty)$*, are symmetric.*

3 Extension of Theorem 4 Let m and n be positive integers and $f_1, f_2, \ldots, f_m$, $g_1, g_2, \ldots, g_n$ be non-constant, non-negative operator monotone functions on $[0, \infty)$. We assume that the function

$$F(t) = \frac{\prod_{i=1}^m f_i(t)}{t^{m-1} \prod_{j=1}^n g_j(t)}$$

is operator monotone on $[0, \infty)$. For non-negative numbers $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$, we define the function $h(t)$ as follows:

$$h(t) = \prod_{i=1}^m \frac{t - a_i}{f_i(t) - f_i(a_i)} \prod_{j=1}^n \frac{g_j(t)(t - b_j)}{tg_j(t) - b_j g_j(b_j)} \qquad (t \geq 0).$$

Then it follows from Proposition 2 that $h(z)$ is holomorphic on $\mathbb{H}_+$, $h([0, \infty)) \subset [0, \infty)$ and $\arg h(z) > 0$ for any $z \in \mathbb{H}_+$.

Theorem 9. *In the above setting, we have the following:*

(1) *When f_i and g_j $(1 \leq i \leq m,\ 1 \leq j \leq n)$ are continuous on $\overline{\mathbb{H}_+}$ and*

$$f_i(t) - f_i(a_i) \neq 0, \quad tg_j(t) - b_j g_j(b_j) \neq 0, \quad t \in (-\infty, 0),$$

$h(t)$ is operator monotone on $[0, \infty)$.

(2) *When there exists a positive number α such that $\alpha \arg z \leq \arg F(z)$ for all $z \in \mathbb{H}_+$,$h(t)$ is operator monotone on $[0, \infty)$.*

Proof. (1) Using the same argument of proof of Theorem 5 (1), it suffices to show that $0 \leq \arg h(z) \leq \pi$ for $z \in \mathbb{R}$ or $z \in \mathbb{H}_+$ whose absolutely value is sufficiently large.

In the case $z \in (-\infty, 0)$, i.e., $|z| > 0$ and $\arg z = \pi$, we have

$$\begin{aligned}
&\arg h(z) \\
=&\sum_{i=1}^m \arg(z - a_i) + \sum_{j=1}^n \arg(g_j(z)(z - b_j)) \\
&\qquad - \sum_{i=1}^n \arg(f_i(z) - f_i(a_i)) - \sum_{j=1}^n (zg_j(z) - b_j g_j(b_j)) \\
\leq& m\pi + n\pi + \sum_{j=1}^n \arg g_j(z) - \sum_{i=1}^m \arg f_i(z) - n\pi \\
=&\pi - \arg \frac{\prod_{i=1}^n f_i(z)}{z^{m-1} \prod_{j=1}^n g_j(z)} \leq \pi.
\end{aligned}$$

So it holds $0 \leq \arg h(z) \leq \pi$.

In the case that $z \in \mathbb{H}_+$ satisifies

$$|z| > (m+n-1)\max\{a_i, b_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Then it holds that

$$\arg(z-a_i), \arg(z-b_j) < \frac{\pi + (m+n-1)\arg z}{m+n}$$

by Lemma 4. We may assume that there exists k $(1 \leq k \leq n)$ such that

$$\arg(zg_j(z)) \leq \pi \quad (j \leq k), \qquad \arg(zg_j(z)) > \pi \quad (j > k).$$

Since

$$\begin{aligned}
&\arg h(z) \\
\leq& \frac{\pi + (m+n-1)\arg z}{m+n} \times m + \frac{\pi + (m+n-1)\arg z}{m+n} \times n + \sum_{j=1}^{n} \arg g_j(z) \\
&\quad - \sum_{i=1}^{m} \arg f_i(z) - \sum_{j=1}^{k} \arg zg_j(z) - (n-k)\pi \\
=& \pi + (m+n-k-1)\arg z + \sum_{j=k+1}^{n} \arg g_j(z) - \sum_{i=1}^{m} \arg f_i(z) - (n-k)\pi \\
\leq& \pi + (n-k)(\arg z - \pi) - \arg \frac{\prod_{i=1}^{m} f_i(z)}{z^{m-1}\prod_{j=1}^{m} g_j(z)} \\
\leq& \pi - \arg F(z) \leq \pi,
\end{aligned}$$

we have $0 \leq \arg h(z) \leq \pi$. So $h(t)$ is operator monotone on $[0,\infty)$.

(2) We choose a positive number p as follows:

$$\frac{m-1}{\alpha + m - 1} < p < 1.$$

We define functions $f_{i,p}, g_{j,p}$ as follows:

$$f_{i,p}(z) = f_i(z^p), \quad g_{j,p}(z) = g_j(z^p) \qquad (z \in \mathbb{H}_+).$$

Since $f_i, g_j \in \mathbb{P}_+[0,\infty)$, $f_{i,p}, g_{j,p}$ are continuous on $\overline{\mathbb{H}_+}$ and satisfy the condition

$$f_{i,p}(t) - f_{i,p}(a_i) \neq 0, \quad tg_{j,p}(t) - b_j g_{j,p}(b_j) \neq 0, \quad t \in (-\infty, 0)$$

by Lemma 3. We put

$$F_p(t) = \frac{\prod_{i=1}^{m} f_{i,p}(t)}{t^{m-1}\prod_{j=1}^{n} g_{j,p}(t)} = F(t^p)t^{-(m-1)(1-p)}.$$

Then F_p is holomorphic on $\mathbb{H}_+$ and satisfies $F_p((0,\infty)) \subset (0,\infty)$. For any $z \in \mathbb{H}_+$, we have

$$\arg F_p(z) = \arg F(z^p) - (m-1)(1-p)\arg z \le \arg F(z^p) \le \pi$$

and

$$\begin{aligned}\arg F_p(z) &\ge \alpha \arg z^p - (m-1)(1-p)\arg z \\ &= (\alpha p - (m-1)(1-p))\arg z \\ &= ((\alpha + m - 1)p - (m-1))\arg z > 0.\end{aligned}$$

So we can see $F_p \in \mathbb{P}_+[0,\infty)$. By (1), we can show that

$$h_p(t) = \prod_{i=1}^m \frac{(t-a_i)}{f_{i,p}(t) - f_{i,p}(a_i)} \prod_{j=1}^n \frac{g_{j,p}(t)(t-b_j)}{tg_{j,p}(t) - b_j g_{j,p}(b_j)}$$

is operator monotone on $[0,\infty)$. When p tends to 1, $h_p(t)$ also tends to $h(t)$. Hence $h(t)$ is operator monotone on $[0,\infty)$. $\square$

Example 10. *Let $0 < p_i \le 1$ $(i = 1, 2, \ldots, m)$ and $0 \le q_j \le 1$ $(j = 1, 2, \ldots, n)$. We put*

$$f_i(t) = t^{p_i}, \quad g_j(t) = t^{q_j} \qquad (t \ge 0).$$

By the calculation

$$F(t) = \frac{\prod_{i=1}^m f_i(t)}{t^{m-1}\prod_{j=1}^n g_j(t)} = t^{\sum_{i=1}^m p_i - \sum_{j=1}^n q_j - (m-1)},$$

we have, for real numbers a_i, $b_j \ge 0$,

$$h(t) = t^{\sum_{j=1}^n q_j} \frac{(t-a_1)\cdots(t-a_m)(t-b_1)\cdots(t-b_n)}{(t^{p_1} - a_1^{p_1})\cdots(t^{p_m} - a_m^{p_m})(t^{1+q_1} - b_1^{1+q_1})\cdots(t^{1+q_n} - b_n^{1+q_n})}$$

is operator monotone on $[0,\infty)$ by Theorem 9 if it holds

$$0 \le \sum_{i=1}^m p_i - \sum_{j=1}^n q_j - (m-1) \le 1,$$

i.e., $F(t)$ is operator monotone on $[0,\infty)$.

When $\sum_{i=1}^m p_i = \sum_{j=1}^n q_j + (m-1)$, we can see that

$$h(t) = \frac{t^{\sum_{j=1}^n q_j}(t-1)^{m+n}}{\prod_{i=1}^m (t^{p_i} - 1)\prod_{j=1}^n (t^{1+q_j} - 1)}$$

is operator monotone on $[0, \infty)$ *and symmetric.*

We can easily check that, if h_1, $h_2 \in \mathbb{P}_+[0, \infty)$ *are symmetric, then the functions*

$$\begin{aligned} f(t) &= h_1(t)^{1/p} h_2(t)^{1-1/p} \qquad (p > 1) \\ g(t) &= \frac{t}{h_1(t)} \end{aligned}$$

are also operator monotone on $[0, \infty)$ *and symmetric.*

Combining these facts, for r_i, s_i $(i = 1, 2, \ldots, n)$ *with*

$$\begin{gathered} 0 < r_1, \ldots, r_c \leq 1, \quad 1 \leq r_{c+1}, \ldots, r_n \leq 2 \\ 0 < s_1, \ldots, s_d \leq 1, \quad 1 \leq s_{d+1}, \ldots, s_n \leq 2 \\ \sum_{i=1}^{c} r_i = \sum_{i=c+1}^{n} r_i - 1, \quad \sum_{i=1}^{d} s_i = \sum_{i=d+1}^{n} s_j - 1, \end{gathered}$$

we can see that the function

$$h(t) = \sqrt{t^\gamma \prod_{i=1}^{n} \frac{r_i(t^{s_i} - 1)}{s_i(t^{r_i} - 1)}}$$

is operator monotone on $[0, \infty)$ *and symmetric with* $h(1) = 1$*, where* $\gamma = 1 - c + d + \sum_{i=1}^{c} r_i - \sum_{i=1}^{d} s_i$.

By such a way, we can also construct from a symmetric operator monotone function to new one.

Ackowledgemant. The author is grateful to the refree for his careful comments.

References

[1] R. Bhatia, *Matrix Analysis* , Springer-Verlag, New York, Berlin , Heiderberg 1996.

[2] L. Cai and F. Hansen, Metric-adjusted skew information: Convexity and restricted forms of superadditivity, Lett. Math. Phys. 93(2010) 1–13.

[3] W. F. Donoughue, Jr., *Monotone Matrix Functions and Analytic Continuation*, Springer-Verlag, Berlin , Heiderberg, New York 1974.

[4] F. Hansen, WYD-like skew information measures, J. Stat. Phys. 151(2013), 974–979.

[5] F. Hiai, Matrix Analysis: Matrix monotone functions, matrix means, and majorization, Interdecip. Inform. Sci. 16 (2010) 139–248.

[6] F. Hiai, D. Petz, *Introduction to Matrix Analysis and Applications*, Universitext, Springer, 2014.

[7] M. Kawasaki, M. Nagisa, Some Operator monotone functions related to Petz-Hasegawa's functions, Technical Reports Math. Sci. Chiba Univ. 2012.

[8] M. Kawasaki, M. Nagisa, Transforms on operator monotone functions, arXiv:1206.5452v1.

[9] E.A. Morozova, N.N. Chentsov, Markov invariant geometry on state manifolds, Itogi Nauki i Techniki 36 (1990), 69–102, Translated in J. Soviet Math. 56(1991) 2648–2669.

[10] M. Nagisa, S. Wada, Operator monotonicity of some functions, Linear Algebra Appl. 486, (2015), 389–408.

[11] D. Petz, Monotone metric on matrix spaces, Linear Algebra Appl. 244 (1996) 81–96.

[12] D. Petz, H. Hasegawa, On the Riemannian metric of α-entropies of density matrices, Lett. Math. Phys. 38 (1996) 221–225.

[13] V.E.S. Szabo, A class of matrix monotone functions, Linear Algebra Appl. 420 (2007) 79–85.

[14] M. Uchiyama, Majorization and some operator monotone functions, Linear Algebra Appl. 432(2010) 1867–1872.

Communicated by *Hiroyuki Osaka*

Masaru NAGISA
Graduate School of Science
Chiba University
Chiba 263-8522
Japan
e-mail address: nagisa@math.s.chiba-u.ac.jp