MARTINGALE BESOV SPACES AND MARTINGALE TRIEBEL-LIZORKIN SPACES

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Abstract. In this paper, we give a definition of martingale Besov spaces and martingale Triebel-Lizorkin spaces for general filtrations. We investigate several fundamental properties of these spaces.

1 Introduction The theory of Besov spaces and Triebel-Lizorkin spaces provides us a unified approach to various important function spaces such as L_p-spaces, Hardy spaces, BMO spaces, Lipschitz spaces and Sobolev spaces. From such diversity, Besov spaces and Triebel-Lizorkin spaces are useful in various mathematical branches.

In martingale theory, Chao and Peng [5] gave a definition of Besov spaces and Triebel-Lizorkin spaces for p-adic martingales and pointed out some fundamental properties of these spaces. They used martingale Besov spaces for characterization of Schatten-von Neumann properties of commutators. For general filtrations, Weisz [17] proved duality theorems among martingale Hardy spaces of q-variations, including the duality between martingale Hardy spaces and martingale BMO spaces of q-variations. We note that these spaces coincide with martingale Triebel-Lizorkin spaces when the smoothness parameter equals to 0, and that Weisz's duality theorem is an early general result on martingale Triebel-Lizorkin spaces.

In this paper, we give a definition of martingale Besov spaces and martingale Triebel-Lizorkin spaces for general filtrations. We give proofs for several fundamental properties of these spaces such as duality, complex interpolation and norm equivalence in a general framework. We also study some embeddings under additional assumptions on filtrations. It relates to recent progress of the theory of fractional integral of martingales ([4], [7], [8], [11], [14]). In fact, we apply our results to the boundedness of fractional integrals of martingales and obtain some improvement.

The organization of this paper is as follows. In the next section, we give the definition of martingale Besov-Triebel-Lizorkin spaces for general filtrations and describe our results. In Section 3, we prove some basic properties of martingale Besov-Triebel-Lizorkin spaces. In Section 4, we show a duality between martingale Besov-Triebel-Lizorkin spaces. In Section 5, we study complex interpolation of martingale Besov-Triebel-Lizorkin spaces. In Section 6, we show a norm equivalence in terms of mean oscillations. In Section 7, we prove some embedding theorem under additional assumptions on filtrations. Finally in Section 8, we give an application of our results to the boundedness of fractional integral of martingales.

2 Notations, definitions and results Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{\mathcal{F}_n\}_{n\geq 0}$ be a filtration, that is, nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n, respectively. For simplicity, we use the convention $E_{-1} = 0$.

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We say a sequence of measurable functions $f=(f_n)_{n\geq 0}$ is adapted if f_n is $\mathcal{F}_n$-measurable for every $n\geq 0$.

We denote by $\mathcal{V}$ the set of all adapted sequence of functions $v=(v_n)_{n\geq 0}$ satisfying that $v_0=1$ and that there exist constants $\delta_2\geq\delta_1>1$ such that

$$\delta_1 v_{n-1}\leq v_n\leq \delta_2 v_{n-1}\quad \text{for all}\quad n\geq 1. \tag{2.1}$$

By $v_0=1$ and (2.1), if $(v_n)_{n\geq 0}\in\mathcal{V}$, then

$$\delta_1^n\leq v_n\leq \delta_2^n\quad \text{for all}\quad n\geq 0 \tag{2.2}$$

for some $\delta_2\geq\delta_1>1$. For $(v_n)_{n\geq 0}\in\mathcal{V}$, we use the convention $v_{-1}=v_0$.

Let $(f_n)_{n\geq 0}$ be a sequence of integrable functions. We say $(f_n)_{n\geq 0}$ is a martingale relative to $\{\mathcal{F}_n\}_{n\geq 0}$ if it is adapted and satisfies $E_n[f_m]=f_n$ for every $n\leq m$. For a martingale $f=(f_n)_{n\geq 0}$, let $d_nf=f_n-f_{n-1}$ with convention $f_{-1}=0$. We denote by $\mathcal{M}$ the set of all martingales.

For $p\in[1,\infty)$, let $\mathcal{M}_p$ be the set of all L_p-bounded martingales. It is known that, if $p\in(1,\infty)$, then any L_p-bounded martingale converges in L_p. Moreover, if $f\in L_p$, $p\in[1,\infty)$, then $(f_n)_{n\geq 0}$ with $f_n=E_nf$ is in $\mathcal{M}_p$ and converges to f in L_p (see for example [10]). For this reason a function $f\in L_1$ and the corresponding martingale $(f_n)_{n\geq 0}$ with $f_n=E_nf$ will be denoted by the same symbol f. Note also that $\|f\|_{L_p}=\sup_{n\geq 0}\|E_nf\|_{L_p}$.

We now introduce martingale Besov spaces and martingale Triebel-Lizorkin spaces. Our definition is a generalization of Chao and Peng's one in [5].

Definition 2.1. Let $p\in(0,\infty]$, $q\in(0,\infty]$, $s\in\mathbb{R}$ and $v=(v_n)_{n\geq 0}\in\mathcal{V}$. For $f=(f_n)_{n\geq 0}\in\mathcal{M}$, define $\|f\|_{B^s_{pq}}=\|f\|_{B^s_{pq}(v)}$ and $\|f\|_{F^s_{pq}}=\|f\|_{F^s_{pq}(v)}$ by

$$\|f\|_{B^s_{pq}}=\left(\sum_{n=0}^{\infty}\|v_{n-1}^s d_nf\|_{L_p}^q\right)^{1/q}\quad\text{and}\quad \|f\|_{F^s_{pq}}=\left\|\left(\sum_{n=0}^{\infty}|v_{n-1}^s d_nf|^q\right)^{1/q}\right\|_{L_p} \tag{2.3}$$

respectively if $p<\infty$ and $q<\infty$ with convention $v_{-1}=v_0$ and $f_{-1}=0$.

If $p<\infty$ and $q=\infty$, then define

$$\|f\|_{B^s_{p\infty}}=\sup_{n\geq 0}\|v_{n-1}^s d_nf\|_{L_p}\quad\text{and}\quad \|f\|_{F^s_{p\infty}}=\left\|\sup_{n\geq 0}|v_{n-1}^s d_nf|\right\|_{L_p},$$

and if $p=\infty$ and $q<\infty$, then define

$$\|f\|_{B^s_{\infty q}}=\left(\sum_{n=0}^{\infty}\|v_{n-1}^s d_nf\|_{L_\infty}^q\right)^{1/q}\quad\text{and}\quad \|f\|_{F^s_{\infty q}}=\sup_{n\geq 0}\left\|E_n\left[\sum_{k=n}^{\infty}|v_{k-1}^s d_kf|^q\right]^{1/q}\right\|_{L_\infty},$$

and if $p=q=\infty$, then define

$$\|f\|_{B^s_{\infty\infty}}=\sup_{n\geq 0}\|v_{n-1}^s d_nf\|_{L_\infty}\quad\text{and}\quad \|f\|_{F^s_{\infty\infty}}=\left\|\sup_{n\geq 0}|v_{n-1}^s d_nf|\right\|_{L_\infty}$$

respectively with the same convention as in (2.3).

Then, the spaces $B^s_{pq}=B^s_{pq}(v)$ and $F^s_{pq}=F^s_{pq}(v)$ are defined by

$$B^s_{pq}=\{f\in\mathcal{M}:\|f\|_{B^s_{pq}}<\infty\}\quad\text{and}\quad F^s_{pq}=\{f\in\mathcal{M}:\|f\|_{F^s_{pq}}<\infty\}$$

respectively.

$\|f\|_{B^s_{pq}}$ and $\|f\|_{F^s_{pq}}$ are quasi-norms on B^s_{pq} and F^s_{pq} respectively. We call $B^s_{pq} = B^s_{pq}(v)$ a martingale Besov space associated to v and call $F^s_{pq} = F^s_{pq}(v)$ a martingale Triebel-Lizorkin space associated to v.

Remark 2.1. For $f = (f_n)_{n\geq 0} \in \mathcal{M}$, the square functions $S_n(f)$, where $n \geq 0$, and $S(f)$ are defined by

$$S_n(f) = \left(\sum_{k=0}^{n} |d_k f|^2\right)^{1/2} \quad \text{and} \quad S(f) = \left(\sum_{n=0}^{\infty} |d_n f|^2\right)^{1/2}$$

with convention $f_{-1} = 0$. Then, for $p \in (0,\infty)$, the martingale Hardy spaces H^S_p is defined by

$$H^S_p = \{f \in \mathcal{M} : \|S(f)\|_p < \infty\}.$$

The space F^0_{p2} coincides with H^S_p for $p \in (0,\infty)$. Moreover, if $p > 1$, then $F^0_{p2} = H^S_p \sim L_p$. Furthermore, martingale space BMO^{S-}_2 is defined by

$$\mathrm{BMO}^{S-}_2 = \{f \in \mathcal{M} : \|f\|_{\mathrm{BMO}^{S-}_2} < \infty\},$$

where

$$\|f\|_{\mathrm{BMO}^{S-}_2} = \sup_{n\geq 0} \|E_n[S(f)^2 - S_{n-1}(f)^2]^{1/2}\|_\infty$$

with convention $S_{-1}(f) = 0$. The space $F^0_{\infty 2}$ coincides with BMO^{S-}_2. For the theory of martingale Hardy spaces and martingale BMO spaces, we refer to [6], [10] and [16].

For $v = (v_n)_{n\geq 0} \in \mathcal{V}$, define $u = (u_n)_{n\geq 0}$ by $u_n = v_n^{-1}$ for $n \geq 0$. For $\alpha \in \mathbb{R}$ and $f = (f_n)_{n\geq 0} \in \mathcal{M}$, define a martingale $I^u_\alpha f = ((I^u_\alpha f)_n)_{n\geq 0}$ by

$$(I^u_\alpha f)_n = \sum_{k=0}^{n} u^\alpha_{k-1} d_k f$$

with convention $u_{-1} = u_0$, $f_{-1} = 0$ and $(I^u_\alpha f)_{-1} = 0$.

Our first result is a lifting property of I^u_α. It is a direct consequence of the definition, but for its importance, we give a proof.

Theorem 2.1. *Let $v = (v_n)_{n\geq 0} \in \mathcal{V}$. Define $u = (u_n)_{n\geq 0}$ by $u_n = v_n^{-1}$ for $n \geq 0$. Let $\alpha \in \mathbb{R}$. Then, I^u_α is an isometric isomorphism from B^s_{pq} to $B^{s+\alpha}_{pq}$ and F^s_{pq} to $F^{s+\alpha}_{pq}$ respectively for $p \in (0,\infty]$, $q \in (0,\infty]$ and $s \in \mathbb{R}$.*

Proof. Since $d_n(I^u_\alpha f) = u^\alpha_{n-1} d_n f$ for $n \geq 0$, it is clear that I^u_α is a bijection from $\mathcal{M}$ to $\mathcal{M}$ with the inverse map $I^u_{-\alpha}$. Moreover, we have

$$v^{s+\alpha}_{n-1} d_n(I^u_\alpha f) = v^s_{n-1} d_n f \quad \text{for all} \quad n \geq 0. \tag{2.4}$$

By (2.4), we have

$$\|I^u_\alpha f\|_{B^{s+\alpha}_{pq}} = \|f\|_{B^s_{pq}} \quad \text{and} \quad \|I^u_\alpha f\|_{F^{s+\alpha}_{pq}} = \|f\|_{F^s_{pq}}.$$

This is the desired conclusion. □

Our next result is a duality between martingale Besov-Triebel-Lizorkin spaces. For $p \in [1,\infty]$, we denote by p' the conjugate exponent of p, that is,

$$p' = \begin{cases} p/(p-1) & \text{if} \quad 1 < p < \infty, \\ 1 & \text{if} \quad p = \infty, \\ \infty & \text{if} \quad p = 1. \end{cases}$$

We use the notation A^s_{pq} to denote either B^s_{pq} or F^s_{pq} for short.

Theorem 2.2. *Let $v = (v_n)_{n\geq 0} \in \mathcal{V}$, $s \in \mathbb{R}$, $p \in [1,\infty)$ and $q \in [1,\infty)$. Denote by p' and q' the conjugate exponents of p and q respectively. Let $(A^s_{pq})'$ denote the topological dual space of A^s_{pq}. Then, $(A^s_{pq})'$ is isomorphic to $A^{-s}_{p'q'}$ under the pairing $(g,f) \mapsto \sum_{n=0}^\infty E[d_n\overline{g}d_nf]$ with convention $g_{-1} = f_{-1} = 0$. More precisely, there exists a positive constant C depending only on p and q such that the following (1) and (2) hold:*

(1) If $g \in A^{-s}_{p'q'}$, then the infinite sum $\sum_{n=0}^\infty E[d_n\overline{g}d_nf]$ converges for every $f \in A^s_{pq}$. Moreover,

$$\left|\sum_{n=0}^\infty E[d_n\overline{g}d_nf]\right| \leq C\|g\|_{A^{-s}_{p'q'}}\|f\|_{A^s_{pq}} \quad (f \in A^s_{pq}).$$

(2) Conversely, for each $\Phi \in (A^s_{pq})'$, there exists $h \in A^{-s}_{p'q'}$ such that

$$\Phi(f) = \sum_{n=0}^\infty E[d_n\overline{h}d_nf] \quad (f \in A^s_{pq})$$

and that $\|h\|_{A^{-s}_{p'q'}} \leq C\|\Phi\|_{(A^s_{pq})'}$.

The proof of Theorem 2.2 is given in Section 4.

Remark 2.2. The duality of the case $s = 0$ and $A = F$ was proved in [17, Theorem 14 and 17].

Further, we investigate the complex interpolation between martingale Besov-Triebel-Lizorkin spaces. We recall the definition of the first Calderón's complex interpolation functor.

Let $S = \{z \in \mathbb{C} : 0 \leq \mathrm{Re} z \leq 1\}$ and $S_0 = \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$. Let (A_0, A_1) be a compatible couple of Banach spaces. We denote by $\mathcal{F}(A_0, A_1)$ the set of all $(A_0 + A_1)$-valued bounded continuous functions F on S which is holomorphic in S_0 and moreover, $t \mapsto F(j + it)$ $(j = 0, 1)$ is a function from $\mathbb{R}$ into A_j satisfying $\|F(j + it)\|_{A_j} \to 0$ as $|t| \to \infty$. As is shown in [2, Lemma 4.1.1], the space $\mathcal{F}(A_0, A_1)$ equipped with the norm

$$\|F\|_{\mathcal{F}(A_0,A_1)} = \max\left(\sup_{t\in\mathbb{R}}\|F(it)\|_{A_0}, \sup_{t\in\mathbb{R}}\|F(1+it)\|_{A_1}\right)$$

is a Banach space.

Definition 2.2. Let (A_0, A_1) be a compatible couple of Banach spaces. For $\theta \in [0, 1]$, define $[A_0, A_1]_\theta$ by

$$[A_0, A_1]_\theta = \{f \in A_0 + A_1 : f = F(\theta) \quad \text{for some} \quad F \in \mathcal{F}(A_0, A_1)\}$$

equipped with the norm

$$\|f\|_{[A_0,A_1]_\theta} = \inf_{F(\theta)=f}\|F\|_{\mathcal{F}(A_0,A_1)}.$$

We now state our result on complex interpolation of martingale Besov-Triebel-Lizorkin spaces.

Theorem 2.3. *Let $v \in \mathcal{V}$, $\theta \in (0, 1)$, $s_0, s_1 \in \mathbb{R}$ and $p_0, p_1, q_0, q_1 \in [1,\infty]$ with $\min(q_0, q_1) < \infty$. Define s, p and q by*

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{2.5}$$

with convention $1/\infty = 0$. Then, the following (i) and (ii) hold.

(i) $[B^{s_0}_{p_0q_0}, B^{s_1}_{p_1q_1}]_\theta = B^s_{pq}$ *with equivalence of norms.*

(ii) Assume that $1 < p_0, p_1 < \infty$*. Then,* $[F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1}]_\theta = F^s_{pq}$ *with equivalence of norms.*

The proof of Theorem 2.3 is given in Section 5.

Remark 2.3. In the theory of Besov-Triebel-Lizorkin spaces on Euclidean spaces, the complex interpolation is investigated for $p_0, p_1, q_0, q_1 \in (0,\infty]$ by using the framework of distribution valued analytic functions ([15, Section 2.4.4]) and by using isomorphisms to sequence spaces ([9, Theorem 9.1]). Since these methods are not known for martingales of general filtrations, we restrict ourselves to the case where $p_0, p_1, q_0, q_1 \in [1,\infty]$.

In the next section, we will show that if $s \in (0,\infty)$, $p \in [1,\infty]$ and $q \in (0,\infty]$, then $B^s_{pq} \subset L_p$ and $F^s_{pq} \subset L_p$. Further, in Section 6, we prove the following norm equivalence in terms of mean oscillations.

Theorem 2.4. *Let* $v \in \mathcal{V}, s \in (0,\infty)$*,* $p \in [1,\infty]$ *and* $q \in (0,\infty]$*. Let* $f \in L_p$ *and identify* f *with the corresponding martingale* $(f_n)_{n\geq 0} = (E_n f)_{n\geq 0}$*. Then, the following norm equivalence holds:*

$$\|f\|_{B^s_{pq}} \sim \left\| (\|v^s_{n-1} E_n|f - f_{n-1}|\|_{L_p})_{n\geq 0} \right\|_{\ell_q}. \tag{2.6}$$

Moreover, if $1 < p < \infty$ *and* $q \geq 1$*, then*

$$\|f\|_{F^s_{pq}} \sim \left\| \|(v^s_{n-1} E_n|f - f_{n-1}|)_{n\geq 0}\|_{\ell_q} \right\|_{L_p}. \tag{2.7}$$

Note that we do not need any assumption on $\{\mathcal{F}_n\}_{n\geq 0}$ in Theorems 2.1, 2.2, 2.3 and 2.4.

To study embeddings, we need some assumptions on $\{\mathcal{F}_n\}_{n\geq 0}$. $B \in \mathcal{F}_n$ is called an atom (more precisely a $(\mathcal{F}_n, P)$-atom), if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $P(A) = P(B)$ or $P(A) = 0$. Below, we assume that

$$\text{every } \sigma\text{-algebra } \mathcal{F}_n \text{ is generated by countable atoms.} \tag{2.8}$$

We denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. We define $\mathcal{F}_n$-measurable functions b_n and v_n by

$$b_n = \sum_{B\in A(\mathcal{F}_n)} P(B)\chi_B, \quad v_n = b_n^{-1}. \tag{2.9}$$

We also assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, that is, there exists $R \geq 2$ such that

$$E_n f \leq R E_{n-1} f \text{ for all } n \geq 1 \text{ and non-negative integrable function } f. \tag{2.10}$$

Further, for the sake of simplicity, we assume that

$$\text{If } B \in A(\mathcal{F}_{n-1}),\ B' \in A(\mathcal{F}_n) \text{ and } B' \subset B, \text{ then } P(B') < P(B) \text{ for every } n \geq 1. \tag{2.11}$$

$$\mathcal{F}_0 = \{\emptyset, \Omega\}. \tag{2.12}$$

If (2.8), (2.10), (2.11) and (2.12) hold, then, by [11, Lemma 3.3],

$$\left(1 + \frac{1}{R}\right) b_n \leq b_{n-1} \leq R b_n$$

for every $n \geq 1$. Hence, we obtain that the sequence $v = (v_n)_{n\geq 0}$ defined in (2.9) belongs to $\mathcal{V}$.

As for embeddings, we show the following two theorems. For quasi-normed space X and Y, we denote by $X \hookrightarrow Y$ if the identity map from X is a continuous map into Y.

Theorem 2.5. *Suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. Furthermore, assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular with* (2.11) *and* (2.12)*. Let $v = (v_n)_{n\geq 0}$ be the sequence of functions defined in* (2.9)*. Let $s \in \mathbb{R}$, $q \in (0,\infty)$ and $p_0, p_1 \in (0,\infty)$ with $p_0 < p_1$. Let $\alpha = 1/p_0 - 1/p_1$. Then,*

$$B^{s+\alpha}_{p_0 q} \hookrightarrow B^s_{p_1 q} \quad \text{and} \quad F^{s+\alpha}_{p_0\infty} \hookrightarrow F^s_{p_1 q}. \tag{2.13}$$

Theorem 2.6. *Suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. Furthermore, assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular with* (2.11) *and* (2.12)*. Let $v = (v_n)_{n\geq 0}$ be the sequence of functions defined in* (2.9)*. Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. Let A^s_{pq} denote either B^s_{pq} or F^s_{pq}. If $s > 1/p$, then*

$$A^s_{pq} \hookrightarrow B^{s-1/p}_{\infty\infty}. \tag{2.14}$$

The proofs of Theorems 2.5 and 2.6 are given in Section 7.

We apply our results to the boundedness of fractional integral for martingales. To explain this application, we recall the definition of fractional integrals for martingales.

Definition 2.3. Let $\alpha \in \mathbb{R}$. Suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. Let b_n be the function defined in (2.9). For a martingale $(f_n)_{n\geq 0}$, define a martingale $I_\alpha f = ((I_\alpha f)_n)_{n\geq 0}$ by

$$(I_\alpha f)_n = \sum_{k=0}^{n} b^\alpha_{k-1} d_k f$$

with convention $b_{-1} = b_0$ and $f_{-1} = 0$. If $\alpha > 0$, then we call $I_\alpha f$ the fractional integral of f of order α.

Further, we recall the definition of martingale Lipschitz spaces ([16, page 7]). For $s > 0$ and $f \in L_1$, let

$$\|f\|_{\Lambda_1^-(s)} = \sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{P(B)^{1+s}} \int_B |f(\omega) - (E_{n-1}f)(\omega)|\, dP(\omega)$$

with convention $E_{-1}f = 0$. We do not assume $E_0 f = 0$, different from [16]. Then define

$$\Lambda_1^-(s) = \{f \in L_1 : \|f\|_{\Lambda_1^-(s)} < \infty\}. \tag{2.15}$$

We regard $\Lambda_1^-(s)$ as martingale spaces by the identification $f \in L_1$ with the corresponding martingale $(E_n f)_{n\geq 0}$.

We now state the application of our results. For two quasi-normed spaces X and Y, we denote by $B(X,Y)$ the set of all bounded linear maps from X to Y.

Theorem 2.7. *Suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. Furthermore, assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular with* (2.11) *and* (2.12)*. Let $v = (v_n)_{n\geq 0}$ be the sequence of functions defined in* (2.9)*. Let $s \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty)$ and $\alpha \in (0,\infty)$. If $\alpha < 1/p$, then define p_1 by $1/p_1 = 1/p - \alpha$. Then, the following boundedness holds for the fractional integral I_α:*

$$I_\alpha \in B(F^s_{p\infty}, F^s_{p_1 q}) \quad \text{if} \quad \alpha < 1/p, \tag{2.16}$$

$$I_\alpha \in B(F^s_{pq}, F^s_{\infty q}) \quad \text{if} \quad \alpha = 1/p \quad \text{and} \quad q \geq 1, \tag{2.17}$$

$$I_\alpha \in B(F^s_{p\infty}, B^{s+\alpha-1/p}_{\infty\infty}) \quad \text{if} \quad \alpha > 1/p. \tag{2.18}$$

Theorem 2.7 is an extension of the following known fact shown in [4], [11] and [14]. Indeed, we can obtain it as a corollary of Theorem 2.7.

Corollary 2.8. *Under the assumptions in Theorem 2.7, the following boundedness holds for the fractional integral* I_α*:*

$$I_\alpha \in B(H_p^S, H_{p_1}^S) \quad \text{if} \quad \alpha < 1/p, \tag{2.19}$$

$$I_\alpha \in B(H_p^S, \mathrm{BMO}_2^{S-}) \quad \text{if} \quad \alpha = 1/p, \tag{2.20}$$

$$I_\alpha \in B(H_p^S, \Lambda_1^-(\alpha - 1/p)) \quad \text{if} \quad \alpha > 1/p. \tag{2.21}$$

In Section 8, we give proofs of Theorem 2.7 and Corollary 2.8.

3 Some basic properties In this section, we show several basic properties of martingale Besov spaces and martingale Triebel-Lizorkin spaces.

Proposition 3.1. *Let* $v \in \mathcal{V}, s \in \mathbb{R}$, $p \in [1, \infty]$ *and* $q \in (0, \infty]$. *Then* B_{pq}^s *and* F_{pq}^s *are quasi-Banach spaces.*

Proof. Let A_{pq}^s denote either B_{pq}^s or F_{pq}^s. Let $(f^{(N)})_{N\geq 1}$ be a Cauchy sequence in A_{pq}^s. By (2.2), the sequence $(d_n f^{(N)})_{N\geq 1}$ is a Cauchy sequence in L_p for every $n \geq 0$. Let $g_n \in L_p$ be the limit function of the sequence $(d_n f^{(N)})_{N\geq 1}$. Noting that $p \geq 1$, we have $E_{n-1} g_n = 0$ for all $n \geq 1$. Therefore, the sequence $f = (f_n)_{n\geq 0}$ defined by $f_n = \sum_{k=0}^n g_k$ for $n \geq 0$ is a martingale. Hence, by a standard argument, we have that $(f^{(N)})_{N\geq 1}$ converges to f in A_{pq}^s. We obtain the desired conclusion. □

Proposition 3.2. *Let* $v \in \mathcal{V}$, $s \in \mathbb{R}$, $p \in (0, \infty]$ *and* $q, q_1, q_2 \in (0, \infty]$.

(1) If $p < \infty$ *and* $q_1 \leq q_2$*, then*

$$B_{pq_1}^s \hookrightarrow B_{pq_2}^s \quad \text{and} \quad F_{pq_1}^s \hookrightarrow F_{pq_2}^s. \tag{3.1}$$

(2) For each $s \in \mathbb{R}$, $p \in (0, \infty]$ *and* $q \in (0, \infty]$,

$$B_{p\min(p,q)}^s \hookrightarrow F_{pq}^s \hookrightarrow B_{p\max(p,q)}^s. \tag{3.2}$$

Proof. (3.1) is a consequence of the known fact $\|(a_n)_{n\geq 0}\|_{\ell_{q_2}} \leq \|(a_n)_{n\geq 0}\|_{\ell_{q_1}}$ for any sequence $(a_n)_{n\geq 0}$.

To show (3.2), we first note that

$$B_{pp}^s = F_{pp}^s, \tag{3.3}$$

which is derived from the definition. Furthermore, we recall the following fact for any sequence of measurable functions $(g_n)_{n\geq 0}$, which is proved by the use of Minkowski's inequality:

$$\left\| (\|g_n\|_{L_p})_{n\geq 0} \right\|_{\ell_q} \leq \left\| \|(g_n)_{n\geq 0}\|_{\ell_q} \right\|_{L_p} \quad \text{if} \quad p \leq q, \tag{3.4}$$

$$\left\| (\|g_n\|_{L_p})_{n\geq 0} \right\|_{\ell_q} \geq \left\| \|(g_n)_{n\geq 0}\|_{\ell_q} \right\|_{L_p} \quad \text{if} \quad p \geq q. \tag{3.5}$$

We now show (3.2) in case $p < \infty$. If $p \leq q$, then, using (3.3), (3.1) and (3.4), we have (3.2) as follows:

$$B_{p\min(p,q)}^s = B_{pp}^s = F_{pp}^s \hookrightarrow F_{pq}^s \hookrightarrow B_{pq}^s = B_{p\max(p,q)}^s.$$

Similarly, if $p \geq q$, then we have (3.2) as follows:

$$B^s_{p\min(p,q)} = B^s_{pq} \hookrightarrow F^s_{pq} \hookrightarrow F^s_{pp} = B^s_{pp} = B^s_{p\max(p,q)}.$$

Thus, we obtain (3.2) in case $p < \infty$.

If $p = \infty$ and $q < \infty$, then we have $\|f\|_{F^s_{\infty q}} \leq \|f\|_{B^s_{\infty q}}$ by the following inequality:

$$\sum_{k=n}^{\infty} |v^s_{k-1} d_k f|^q \leq \sum_{n=0}^{\infty} \|v^s_{n-1} d_n f\|^q_{L_\infty} = \|f\|^q_{B^s_{\infty q}}.$$

We also have $\|f\|_{B^s_{\infty\infty}} \leq \|f\|_{F^s_{\infty q}}$ for $q < \infty$ by the following inequality:

$$|v^s_{n-1} d_n f|^q = E_n[|v^s_{n-1} d_n f|^q] \leq E_n\left[\sum_{k=n}^{\infty} |v^s_{k-1} d_k f|^q\right] \leq \|f\|^q_{F^s_{\infty q}}.$$

The proof is completed. □

Concerning Theorem 2.4, we show the following proposition.

Proposition 3.3. *Let $v \in \mathcal{V}$, $s > 0$, $p \in [1,\infty]$, and $q \in (0,\infty]$. Then,*

$$B^s_{pq} \hookrightarrow L_p \quad \text{and} \quad F^s_{pq} \hookrightarrow L_p \tag{3.6}$$

under the identification of $(f_n)_{n\geq 0} \in A^s_{pq}$ with its limit function, where A^s_{pq} denote either B^s_{pq} or F^s_{pq}.

Proof. By Proposition 3.2, we only have to show that

$$B^s_{p\infty} \hookrightarrow L_p.$$

Let $f = (f_n)_{n\geq 0} \in B^s_{p\infty}$. By (2.2), we have

$$\sum_{n=1}^{\infty} \|d_n f\|_{L_p} \leq \sum_{n=1}^{\infty} \delta_1^{-s(n-1)} \|v^s_{n-1} d_n f\|_{L_p} \leq \sum_{n=1}^{\infty} \delta_1^{-s(n-1)} \|f\|_{B^s_{p\infty}} < \infty.$$

Thus, $(f_n)_{n\geq 0} = (\sum_{k=0}^n d_k f)_{n\geq 0}$ converges in L_p. Denote the limit function by the same symbol f. Then we have $E_n f = f_n$ and $\|f\|_{L_p} \leq 2(1-\delta_1^{-s})^{-1}\|f\|_{B^s_{p\infty}}$. The proof is completed. □

4 Proof of Theorem 2.2.

In this section, we prove Theorem 2.2. To do this, we need two lemmas.

Lemma 4.1. *Let $p \in [1,\infty]$ and $q \in [1,\infty]$. Let $(f_n)_{n\geq 0}$ be a sequence of integrable functions. If $1 \leq q \leq p < \infty$ or $1 < p \leq q \leq \infty$, then, there exists a constant $C_{p,q}$ depending only on p and q such that*

$$\left\| \|(E_n f_n)_{n\geq 0}\|_{\ell_q} \right\|_{L_p} \leq C_{p,q} \left\| \|(f_n)_{n\geq 0}\|_{\ell_q} \right\|_{L_p}.$$

For the proof of Lemma 4.1, we refer to [1, Theorem 3.1].

Remark 4.1. Since $\|E_n f_n\|_{L_p} \leq \|f_n\|_{L_p}$ by Jensen's inequality for E_n, it is clear that

$$\left\| (\|E_n f_n\|_{L_p})_{n\geq 0} \right\|_{\ell_q} \leq \left\| (\|f_n\|_{L_p})_{n\geq 0} \right\|_{\ell_q}$$

for $p \in [1,\infty]$ and $q \in (0,\infty]$.

Lemma 4.2. *Let $q \in [1,\infty)$. Denote by q' the conjugate exponent of q. Then, there exists a positive constant C depending only on q such that the following (1) and (2) hold:*

(1) If $g \in F^0_{\infty q'}$, then the infinite sum $\sum_{n=0}^\infty E[d_n\overline{g} d_n f]$ converges for every $f \in F^0_{1q}$. Moreover,
$$\left|\sum_{n=0}^\infty E[d_n\overline{g} d_n f]\right| \le C\|g\|_{F^0_{\infty q'}}\|f\|_{F^0_{1q}} \quad (f \in F^0_{1q}).$$

(2) Conversely, for each $\Phi \in (F^0_{1q})'$, there exists $g \in F^0_{\infty q'}$ such that
$$\Phi(f) = \sum_{n=0}^\infty E[d_n\overline{g} d_n f] \quad (f \in F^0_{1q})$$
and that $\|g\|_{F^0_{\infty q'}} \le C\|\Phi\|_{(F^0_{1q})'}$.

For the proof of Lemma 4.2, we refer to [17, Theorem 17 and Corollary 10].

Remark 4.2. We remark on the difference between our convention and the one in [17]. In [17, Corollary 10], it was shown the duality between $H_1^{S_q} = \{(f_n)_{n\ge0} \in F^0_{1q} : f_0 = 0\}$ and $\mathcal{BMO}^-_{q'} = \{(f_n)_{n\ge0} \in F^0_{\infty q'} : f_0 = 0\}$. For this difference, we note that
$$f \in F^0_{1q} \quad \text{if and only if} \quad f - f_0 \in H_1^{S_q} \quad \text{and} \quad f_0 \in L_1,$$
$$f \in F^0_{\infty q'} \quad \text{if and only if} \quad f - f_0 \in \mathcal{BMO}^-_{q'} \quad \text{and} \quad f_0 \in L_\infty$$
with
$$\|f\|_{F^0_{1q}} \sim \|f - f_0\|_{F^0_{1q}} + \|f_0\|_{L_1}, \quad \|f\|_{F^0_{\infty q'}} \sim \|f - f_0\|_{F^0_{\infty q'}} + \|f_0\|_{L_\infty},$$
where $f = (f_n)_{n\ge0}$ and $f - f_0 = (f_n - f_0)_{n\ge0}$.

Proof of Theorem 2.2. The proof below is a modification of the one given in [17, Theorems 14-17], but, to include the Besov space case, we give a proof.

We first prove the case where $p \in (1,\infty)$. Let $g \in A^{-s}_{p'q'}$ and $f \in A^s_{pq}$. If $A^s_{pq} = F^s_{pq}$, then using Hölder's inequality, we have

$$\begin{aligned}(4.1)\qquad \sum_{n=0}^\infty E\left[|d_n g d_n f|\right] &= E\left[\sum_{n=0}^\infty |v^{-s}_{n-1} d_n g v^s_{n-1} d_n f|\right] \\ &\le E\left[\|(v^{-s}_{n-1} d_n g)_{n\ge0}\|_{\ell_{q'}}\|(v^s_{n-1} d_n f)_{n\ge0}\|_{\ell_q}\right] \\ &\le \|g\|_{F^{-s}_{p'q'}}\|f\|_{F^s_{pq}}.\end{aligned}$$

If $A^s_{pq} = B^s_{pq}$, then similarly we have
$$\begin{aligned}\sum_{n=0}^\infty E\left[|d_n g d_n f|\right] &= \sum_{n=0}^\infty E\left[|v^{-s}_{n-1} d_n g v^s_{n-1} d_n f|\right] \\ &\le \sum_{n=0}^\infty \|v^{-s}_{n-1} d_n g\|_{L_{p'}}\|v^s_{n-1} d_n f\|_{L_p} \\ &\le \|g\|_{B^{-s}_{p'q'}}\|f\|_{B^s_{pq}}.\end{aligned}$$

Therefore, we have obtained (1) in case $p \in (1,\infty)$.

We next show (2) in case $p \in (1,\infty)$. Define A_{pq} by

$$A_{pq} = \begin{cases} \ell_q(L_p) & \text{if} \quad A^s_{pq} = B^s_{pq}, \\ L_p(\ell_q) & \text{if} \quad A^s_{pq} = F^s_{pq}. \end{cases}$$

Let $\Phi \in (A^s_{pq})'$ and let $u_n = v_n^{-1}$ for $n \geq 0$. By Theorem 2.1, the functional $f \mapsto \Phi \circ I^u_s(f)$ on A^0_{pq} is bounded. We denote by $i : A^0_{pq} \to A_{pq}$ the isometric embedding defined by $i(f) = (d_n f)_{n\geq 0}$ $(f \in A^0_{pq})$. Using Hahn-Banach's theorem, we take $\Psi \in (A_{pq})'$ such that $\|\Psi\|_{(A_{pq})'} = \|\Phi\|_{(A^s_{pq})'}$ and that $\Psi \circ i = \Phi \circ I^u_s$ on A^0_{pq}. Furthermore, using the fact that $(A_{pq})'$ is isometric to $A_{p'q'}$, we take $g = (g_n)_{n\geq 0} \in A_{p'q'}$ such that

$$\|g\|_{A_{p'q'}} = \|\Phi\|_{(A^s_{pq})'} \quad \text{and that} \quad \Phi(f) = \sum_{n=0}^{\infty} E[\overline{g_n} v^s_{n-1} d_n f] \quad \text{for} \quad f \in A^s_{pq}.$$

Then define $h = (h_n)_{n\geq 0}$ by

$$h_n = \sum_{k=0}^{n} v^s_{k-1}(E_k g_k - E_{k-1} g_k)$$

with convention $v_{-1} = v_0$ and $E_{-1} g_0 = 0$. It is clear that $h = (h_n)_{n\geq 0}$ is a martingale. If $A^s_{pq} = B^s_{pq}$, then by Remark 4.1,

$$\begin{aligned} \left\| (\|v^{-s}_{n-1} d_n h\|_{L_{p'}})_{n\geq 0} \right\|_{\ell_{q'}} &= \left\| (\|E_n g_n - E_{n-1} g_n\|_{L_{p'}})_{n\geq 0} \right\|_{\ell_{q'}} \\ &\leq \left\| (\|E_n g_n\|_{L_{p'}})_{n\geq 0} \right\|_{\ell_{q'}} + \left\| (\|E_{n-1} g_n\|_{L_{p'}})_{n\geq 0} \right\|_{\ell_{q'}} \\ &\leq 2 \left\| (\|g_n\|_{L_{p'}})_{n\geq 0} \right\|_{\ell_{q'}} = 2\|\Phi\|_{(B^s_{pq})'}, \end{aligned}$$

that is, we have $h \in B^{-s}_{p'q'}$ with $\|h\|_{B^{-s}_{p'q'}} \leq 2\|\Phi\|_{(B^s_{pq})'}$.

Similarly, if $A^s_{pq} = F^s_{pq}$, then by Lemma 4.1,

$$\begin{aligned} \left\| \|(v^{-s}_{n-1} d_n h)_{n\geq 0}\|_{\ell_{q'}} \right\|_{L_{p'}} &= \left\| \|(E_n g_n - E_{n-1} g_n)_{n\geq 0}\|_{\ell_{q'}} \right\|_{L_{p'}} \\ &\leq \left\| \|(E_n g_n)_{n\geq 0}\|_{\ell_{q'}} \right\|_{L_{p'}} + \left\| \|(E_{n-1} g_n)_{n\geq 0}\|_{\ell_{q'}} \right\|_{L_{p'}} \\ &\leq 2C_{p',q'} \left\| \|(g_n)_{n\geq 0}\|_{\ell_{q'}} \right\|_{L_{p'}} = 2C_{p',q'}\|\Phi\|_{(F^s_{pq})'}, \end{aligned}$$

that is, we have $h \in F^{-s}_{p'q'}$ with $\|h\|_{F^{-s}_{p'q'}} \leq 2C_{p',q'}\|\Phi\|_{(F^s_{pq})'}$.

Let $f \in A^s_{pq}$. Then, by the formal self-adjointness of E_n, we have

$$\begin{aligned} \sum_{n=0}^{\infty} E[d_n \overline{h} d_n f] &= \sum_{n=0}^{\infty} (E[E_n(v^s_{n-1}\overline{g_n}) d_n f] - E[E_{n-1}(v^s_{n-1}\overline{g_n}) d_n f]) \\ &= \sum_{n=0}^{\infty} (E[\overline{g_n} v^s_{n-1} E_n(d_n f)] - E[\overline{g_n} v^s_{n-1} E_{n-1}(d_n f)]) \\ &= \sum_{n=0}^{\infty} E[\overline{g_n} v^s_{n-1} d_n f] = \Phi(f). \end{aligned}$$

Hence, we have the desired conclusion for the case where $p \in (1,\infty)$.

For the case where $A^s_{pq} = B^s_{pq}$ with $p = 1$, we can obtain the desired conclusion by the same way as in the case where $p \in (1,\infty)$.

We now give a proof for the case where $A^s_{pq} = F^s_{pq}$ with $p = 1$. Let $g \in F^{-s}_{\infty q'}$ and let $u = (v_n^{-1})_{n\geq 0}$. Then, by (1) of Lemma 4.2 and Theorem 2.1, we obtain that the infinite sum

$$\sum_{n=0}^{\infty} E[d_n\overline{g}d_nf] = \sum_{n=0}^{\infty} E[d_n(I^u_s\overline{g})d_n(I^u_{-s}f)]$$

converges and that

$$\left|\sum_{n=0}^{\infty} E[d_n\overline{g}d_nf]\right| \leq C\|I^u_s g\|_{F^0_{\infty q'}}\|I^u_{-s}f\|_{F^0_{1q}} = C\|g\|_{F^{-s}_{\infty q'}}\|f\|_{F^s_{1q}}$$

for $f \in F^s_{1q}$.

We next show (2). Let $\Phi \in (F^s_{1q})'$. By Theorem 2.1, $\Phi \circ I^u_s$ belongs to $(F^0_{1q})'$. Using (2) of Lemma 4.2, we take $g \in F^0_{\infty q'}$ such that

$$\Phi \circ I^u_s(\tilde{f}) = \sum_{n=0}^{\infty} E[d_n\overline{g}d_n\tilde{f}] \quad (\tilde{f} \in F^0_{1q}) \tag{4.2}$$

and that $\|g\|_{F^0_{\infty q'}} \leq C\|\Phi \circ I^u_s\|_{(F^0_{1q})'} = C\|\Phi\|_{(F^s_{1q})'}$. Let $f \in F^s_{1q}$. We put $h = I^u_{-s}g$ and $\tilde{f} = I^u_{-s}f$ in (4.2). Then, we have $\|h\|_{F^{-s}_{\infty q'}} \leq C\|\Phi\|_{(F^s_{1q})'}$ and

$$\Phi(f) = \sum_{n=0}^{\infty} E[d_n(I^u_s\overline{h})d_n(I^u_{-s}f)] = \sum_{n=0}^{\infty} E[d_n\overline{h}d_nf].$$

Therefore, we have the desired conclusion for the case where $A^s_{pq} = F^s_{pq}$ with $p = 1$. The proof is completed.

□

5 Proof of Theorem 2.3. In this section, we give a proof of Theorem 2.3. For the proof, we need some lemmas.

For $0 \leq x \leq 1$ and $t \in \mathbb{R}$, let $\mu_0(z,t)$, $\mu_1(z,t)$ be the Poisson kernel on $S = \{0 \leq \mathrm{Re} z \leq 1\}$, that is,

$$\mu_j(x+iy,t) = \frac{e^{-\pi(t-y)}\sin\pi x}{\sin^2\pi x + (\cos\pi x - e^{ij\pi-\pi(t-y)})^2}, \quad j = 0,1.$$

Lemma 5.1. *Let (A_0,A_1) be a compatible couple of Banach spaces. Let $f \in \mathcal{F}(A_0,A_1)$. Then, for $0 < \theta < 1$,*

$$\|f(\theta)\|_{[A_0,A_1]_\theta} \leq \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty}\|f(it)\|_{A_0}\mu_0(\theta,t)dt\right)^{1-\theta}\left(\frac{1}{\theta}\int_{-\infty}^{\infty}\|f(1+it)\|_{A_1}\mu_1(\theta,t)dt\right)^{\theta}.$$

For the proof of Lemma 5.1, see [2, Lemma 4.3.2].

Lemma 5.2. *Let f be a non-negative bounded measurable function on Ω. Let $a,b \in \mathbb{R}$ and let $\rho(z) = az+b$, $z \in \mathbb{C}$. Suppose that either* $\operatorname{ess\,inf} f > 0$ *or both a and b are positive. Then, the map $F : S \to L_\infty$ defined by $F(z) = f^{\rho(z)}$ is holomorphic on S_0.*

Proof. We first give the proof for the case where $\operatorname{ess\,inf} f > 0$. Since $f^{\rho(z)} = f^b(f^a)^z$, we only have to prove in case where $\rho(z) = z$. Let $z \in S_0$ and let $h \in \mathbb{C} \setminus \{0\}$ such that $z + h \in S_0$. By the fundamental theorem of calculus, we have

$$\left\| \frac{f^{z+h} - f^z}{h} - f^z \log f \right\|_{L_\infty} = \left\| f^z (\log f)^2 \frac{1}{h} \int_0^h \left(\int_0^t f^s \, ds \right) dt \right\|_{L_\infty} \leq C|h|,$$

where $C = (1 + \|f\|_{L_\infty}^2)\{\log(\|f^{-1}\|_{L_\infty} + \|f\|_{L_\infty})\}^2$. Therefore, we have $F'(z) = f^z \log f$ in L_∞.

We next give the proof for the case where both a and b are positive. Since $f^{\rho(z)} = f^{a(z+b/a)}$, we only have to show in case where $\rho(z) = z + c$ with $c > 0$. Then, as above, we have

$$\begin{aligned} \left\| \frac{f^{z+c+h} - f^{z+c}}{h} - f^{z+c} \log f \right\|_{L_\infty} &= \left\| (f^{c/2} \log f)^2 \frac{f^z}{h} \int_0^h \left(\int_0^t f^s \, ds \right) dt \right\|_{L_\infty} \\ &\leq \sup_{0 \leq x \leq \|f\|_{L_\infty}} (x^{c/2} \log x)^2 (1 + \|f\|_{L_\infty}^2)|h|. \end{aligned}$$

We have the desired conclusion. □

Lemma 5.3. *Let $(c_n)_{n=1}^\infty$ be a sequence of positive numbers and $\alpha > 0$. Then,*

$$\sum_{n=1}^\infty c_n \left(\sum_{k=1}^n c_k \right)^{\alpha - 1} \leq \frac{1}{\min(\alpha, 1)} \left(\sum_{n=1}^\infty c_n \right)^\alpha.$$

For the proof of Lemma 5.3, see [15, Section 2.4.6] and [13, Lemma 2.17].

In the next lemma, we give a dense subspace of B_{pq}^s and F_{pq}^s. Let $\mathcal{M}_b$ be the set of all martingales $(f_n)_{n\geq 0}$ which satisfies $\sup_{n\geq 0} \|f_n\|_{L_\infty} < \infty$.

Then define

$$\mathcal{T} = \{(f_n)_{n\geq 0} \in \mathcal{M}_b : \text{there exists } N \geq 0 \text{ such that } f_n = f_N \text{ for all } n \geq N\}.$$

Lemma 5.4. *Let $v \in \mathcal{V}$. Let $p \in [1, \infty]$, $q \in [1, \infty)$ and $s \in \mathbb{R}$. Then, $\mathcal{T}$ is dense in B_{pq}^s. Moreover, if $p < \infty$, then $\mathcal{T}$ is also dense in F_{pq}^s.*

Proof. We first show that $\mathcal{T}$ is dense in F_{pq}^s if $p < \infty$. Let $f = (f_n)_{n\geq 0} \in F_{pq}^s$. For $N \geq 0$, let $f^N = (f_{n\wedge N})_{n\geq 0}$ where $n \wedge N = \min(n, N)$. Then,

$$\left(\sum_{n=0}^\infty |v_{n-1}^s d_n(f - f^N)|^q \right)^{1/q} = \left(\sum_{n=N+1}^\infty |v_{n-1}^s d_n f|^q \right)^{1/q}. \tag{5.1}$$

By Lebesgue's convergence theorem, we have

$$\lim_{N\to\infty} \|f - f^N\|_{F_{pq}^s} = 0.$$

Therefore, to obtain the conclusion, we only have to show that each f^N is approximated by some sequences in $\mathcal{T}$.

For $R > 0$, let $g(N,R) = (E_n[f_N \chi_{\{|f_N| \le R\}}])_{n\ge 0}$. It is clear that $g(N,R) \in \mathcal{T}$. Noting that $v_n \le \delta_2^n$ for some $\delta_2 > 1$, we have

$$\begin{aligned}
(5.2) \qquad \left(\sum_{n=0}^{\infty} |v_{n-1}^s d_n(f^N - g(N,R))|^q \right)^{1/q} &= \left(\sum_{n=0}^{N} |v_{n-1}^s d_n(f^N - g(N,R))|^q \right)^{1/q} \\
&\le \delta_2^{Ns^+} \left(\sum_{n=0}^{N} |d_n(f^N - g(N,R))|^q \right)^{1/q} \\
&\le \delta_2^{Ns^+} \sum_{n=0}^{N} |d_n(f^N - g(N,R))| \\
&\le 2\delta_2^{Ns^+} \sum_{n=0}^{N} E_n |f_N \chi_{\{|f_N|>R\}}|
\end{aligned}$$

where $s^+ = \max(s,0)$. By (5.2), we have

$$\begin{aligned}
\|f^N - g(N,R)\|_{F_{pq}^s} &\le 2\delta_2^{Ns^+} \sum_{n=0}^{N} \big\| E_n |f_N \chi_{\{|f_N|>R\}}| \big\|_{L_p} \\
&\le 2\delta_2^{Ns^+} \sum_{n=0}^{N} \|f_N \chi_{\{|f_N|>R\}}\|_{L_p}.
\end{aligned}$$

Since $f_N \in L_p$, we have

$$\lim_{R\to\infty} \|f^N - g(N,R)\|_{F_{pq}^s} = 0.$$

Therefore, we have that $\mathcal{T}$ is dense in F_{pq}^s.

We next show that $\mathcal{T}$ is dense in B_{pq}^s. As in (5.1), we have

$$(5.3) \qquad \left(\sum_{n=0}^{\infty} \|v_{n-1}^s d_n(f - f^N)\|_{L_p}^q \right)^{1/q} = \left(\sum_{n=N+1}^{\infty} \|v_{n-1}^s d_n f\|_{L_p}^q \right)^{1/q}.$$

Note that (5.3) holds even if $p = \infty$. Then we have

$$\lim_{N\to\infty} \|f - f^N\|_{B_{pq}^s} = 0.$$

Similarly, as in (5.2), we have

$$\left(\sum_{n=0}^{\infty} \|v_{n-1}^s d_n(f^N - g(N,R))\|_{L_p}^q \right)^{1/q} \le 2\delta_2^{Ns^+} \sum_{n=0}^{N} \|f_N \chi_{\{|f_N|>R\}}\|_{L_p}.$$

Hence, we obtain

$$\lim_{R\to\infty} \|f^N - g(N,R)\|_{B_{pq}^s} = 0.$$

Therefore, we have the desired conclusion. □

The following is the key lemma for the proof of Theorem 2.3.

Lemma 5.5. *Let $v \in \mathcal{V}$, $\theta \in (0,1)$, $s_0, s_1 \in \mathbb{R}$ and $p_0, p_1, q_0, q_1 \in [1,\infty]$. Define s, p and q by (2.5) with convention $1/\infty = 0$. Then, there exists a positive constant C_1 depending only on p_0, p_1, q_0, q_1 and θ such that the following (i) and (ii) hold.*

(i) For each $f \in \mathcal{T}$, there exists $H \in \mathcal{F}(B^{s_0}_{p_0q_0}, B^{s_1}_{p_1q_1})$ such that

$$\|H\|_{\mathcal{F}(B^{s_0}_{p_0q_0}, B^{s_1}_{p_1q_1})} \le C_1 \|f\|_{B^s_{pq}}, \quad H(\theta) = f \tag{5.4}$$

and that

$$H(z) \in \mathcal{T} \quad \text{for all} \quad z \in S \quad \text{and} \quad \sup_{n\ge 0} \sup_{z\in S} \|d_n H(z)\|_{L_\infty} < \infty. \tag{5.5}$$

(ii) Assume that $1 < p_0, p_1 < \infty$. Then, for each $f \in \mathcal{T}$, there exists $H \in \mathcal{F}(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})$ such that

$$\|H\|_{\mathcal{F}(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})} \le C_1 \|f\|_{F^s_{pq}}, \quad H(\theta) = f \tag{5.6}$$

and that (5.5).

Proof. We first show (i). To do this, we introduce functions ρ_1, ρ_2 and ρ_3 defined on $\mathbb{C}$ by

$$\begin{aligned}
\rho_1(z) &= \left(s\frac{p}{p_0} - s_0 \right)(1-z) + \left(s\frac{p}{p_1} - s_1 \right) z, \\
\rho_2(z) &= \frac{p}{p_0}(1-z) + \frac{p}{p_1} z, \\
\rho_3(z) &= \left(\frac{q}{q_0} - \frac{p}{p_0} \right)(1-z) + \left(\frac{q}{q_1} - \frac{p}{p_1} \right) z,
\end{aligned}$$

with convention $1/\infty = 0$ and $\infty/\infty = 1$.

Furthermore, define $\mathrm{sgn} : \mathbb{C} \to \mathbb{C}$ by

$$\mathrm{sgn}(z) = \begin{cases} z/|z| & \text{if} \quad z \ne 0, \\ 0 & \text{if} \quad z = 0. \end{cases}$$

Let $f \in \mathcal{T}$ such that $\|f\|_{B^s_{pq}} = 1$. For $n \ge 0$, $z \in S$ and $\omega \in \Omega$, define $g_n(z,\omega)$ by

$$g_n(z,\omega) = v_{n-1}(\omega)^{\rho_1(z)} |d_n f(\omega)|^{\rho_2(z)} \mathrm{sgn}(d_n f(\omega)) \|v^s_{n-1} d_n f\|^{\rho_3(z)}_{L_p}.$$

Then, define $H(z) = (H_n(z))_{n\ge 0}$ by

$$h_n(z) = g_n(z) - E_{n-1}[g_n(z)], \quad H_n(z) = \sum_{k=0}^n h_k(z) \tag{5.7}$$

with convention $E_{-1}[g_0(z)] = 0$. $H(z)$ is a martingale for every $z \in S$. Noting that $\rho_1(\theta) = \rho_3(\theta) = 0$ and $\rho_2(\theta) = 1$, we have $g_n(\theta,\omega) = d_n f(\omega)$ and then have

$$H(\theta) = f. \tag{5.8}$$

By Lemma 5.2, we obtain that g_n is an L_∞-valued holomorphic function on S_0. Moreover, since $f \in \mathcal{T}$ and $\mathrm{Re}\rho_j$ $(j = 1,2,3)$ is bounded on S, we have

$$H(z) \in \mathcal{T} \quad \text{for all} \quad z \in S. \tag{5.9}$$

Thus, H is a $(B^{s_0}_{p_0q_0} + B^{s_1}_{p_1q_1})$-valued holomorphic function on S_0 with

$$\sup_{n\ge 0} \sup_{z\in S} \|d_n H(z)\|_{L_\infty} \le 2 \sup_{n\ge 0} \sup_{z\in S} \|g_n(z)\|_{L_\infty} < \infty. \tag{5.10}$$

For $\delta > 0$, let $H_\delta(z) = e^{\delta(z-\theta)^2} H(z)$. Then, H_δ also satisfies $H_\delta(\theta) = f$ and (5.5). We now show that H_δ belongs to $\mathcal{F}(B^{s_0}_{p_0 q_0}, B^{s_1}_{p_1 q_1})$. For $j \in \{0,1\}$, noting that

$$\mathrm{Re}\rho_1(j+it) = s\frac{p}{p_j} - s_j, \quad \mathrm{Re}\rho_2(j+it) = \frac{p}{p_j}, \quad \mathrm{Re}\rho_3(j+it) = \frac{q}{q_j} - \frac{p}{p_j},$$

we have

$$|v^{s_j}_{n-1} g_n(j+it)|^{p_j} = |v^s_{n-1} d_n f|^p (\|v^s_{n-1} d_n f\|^p_{L_p})^{(qp_j/pq_j)-1}.$$

Hence, we have

$$\|v^{s_j}_{n-1} g_n(j+it)\|_{L_{p_j}} = (\|v^s_{n-1} d_n f\|_{L_p})^{q/q_j}.$$

Therefore,

$$\begin{aligned}
(5.11) \qquad & \|H(j+it)\|_{B^{s_j}_{p_j q_j}} \\
& \le \left\|(\|v^{s_j}_{n-1} g_n(j+it)\|_{L_{p_j}})_{n\ge 0}\right\|_{\ell_{q_j}} + \left\|(\|v^{s_j}_{n-1} E_{n-1}[g_n(j+it)]\|_{L_{p_j}})_{n\ge 0}\right\|_{\ell_{q_j}} \\
& \le 2\left\|(\|v^s_{n-1} d_n f\|_{L_p})_{n\ge 0}\right\|^{1/q_j}_{\ell_q} = 2.
\end{aligned}$$

By (5.8) and (5.11), we obtain $H_\delta \in \mathcal{F}(B^{s_0}_{p_0 q_0}, B^{s_1}_{p_1 q_1})$ with

$$(5.12) \qquad \|H_\delta\|_{\mathcal{F}(B^{s_0}_{p_0 q_0}, B^{s_1}_{p_1 q_1})} \le 2\max(e^{\delta\theta^2}, e^{\delta(1-\theta)^2}), \quad H_\delta(\theta) = f.$$

Thus, by (5.12), (5.9) and (5.10), we obtain (i).

We now show (ii). In this case, we define ρ_1, ρ_2 and ρ_3 by

$$\begin{aligned}
\rho_1(z) &= \left(s\frac{q}{q_0} - s_0\right)(1-z) + \left(s\frac{q}{q_1} - s_1\right) z, \\
\rho_2(z) &= \frac{q}{q_0}(1-z) + \frac{q}{q_1} z, \\
\rho_3(z) &= \left(\frac{p}{p_0} - \frac{q}{q_0}\right)(1-z) + \left(\frac{p}{p_1} - \frac{q}{q_1}\right) z.
\end{aligned}$$

Let $f \in \mathcal{T}$ such that $\|f\|_{F^s_{pq}} = 1$. For $n \ge 0$, $z \in S$ and $\omega \in \Omega$, define $g_n(z,\omega)$ by

$$g_n(z,\omega) = v_{n-1}(\omega)^{\rho_1(z)} |d_n f(\omega)|^{\rho_2(z)} \mathrm{sgn}(d_n f(\omega)) \, (1 + G_n(\omega))^{\rho_3(z)}$$

where $G_n(\omega)$ denotes

$$G_n(\omega) = \|(v_{k-1}(\omega)^s d_k f(\omega))_{0\le k\le n}\|_{\ell_q}.$$

Then, by the same way as in (5.7), we obtain martingales $H(z) = (H_n(z))_{n\ge 0}$ such that $H(\theta) = f$, $H(z) \in \mathcal{T}$ for all $z \in S$ and that $z \mapsto H(z)$ is holomorphic from S_0 into $F^{s_0}_{p_0 q_0} + F^{s_1}_{p_1 q_1}$. Furthermore, $H_\delta(z) = e^{\delta(z-\theta)^2} H(z)$ satisfies $H_\delta(\theta) = f$ and (5.5) for every $\delta > 0$.

We now show that H_δ belongs to $\mathcal{F}(F^{s_0}_{p_0 q_0}, F^{s_1}_{p_1 q_1})$ for every $\delta > 0$. We first show it in case where $q_0, q_1 < \infty$. Since $\rho_3(\theta) = 0$, we have $\rho_3(0)\rho_3(1) < 0$. We may assume that $\rho_3(0) < 0$ because the other case is proved by the same way. Note that

$$\mathrm{Re}\rho_1(j+it) = s\frac{q}{q_j} - s_j, \quad \mathrm{Re}\rho_2(j+it) = \frac{q}{q_j}, \quad \mathrm{Re}\rho_3(j+it) = \frac{p}{p_j} - \frac{q}{q_j}$$

for $j = 0, 1$. Then, by the assumption $\rho_3(0) < 0$, we have

$$\begin{aligned}(v_{n-1}^{s_0}|g_n(it)|)^{q_0} &= (v_{n-1}^{s}|d_nf|)^q(1+G_n)^{\rho_3(0)q_0} \\ &\le (v_{n-1}^{s}|d_nf|)^q G_n^{\rho_3(0)q_0} \\ &= (v_{n-1}^{s}|d_nf|)^q \left(\sum_{k=0}^{n}(v_{k-1}^{s}|d_kf|)^q\right)^{(pq_0/qp_0)-1}\end{aligned}$$

and

$$\begin{aligned}(v_{n-1}^{s_1}|g_n(1+it)|)^{q_1} &= (v_{n-1}^{s}|d_nf|)^q(1+G_n)^{\rho_3(1)q_1} \\ &\le C(v_{n-1}^{s}|d_nf|)^q\{1+G_n^{\rho_3(1)q_1}\} \\ &= C(v_{n-1}^{s}|d_nf|)^q \left\{1+\left(\sum_{k=0}^{n}(v_{k-1}^{s}|d_kf|)^q\right)^{(pq_1/qp_1)-1}\right\},\end{aligned}$$

where C is a positive constant depending only on $\rho_3(1)q_1$. Using Lemma 5.3 and the assumption $\rho_3(0) < 0$, which is equivalent to $pq_0 < qp_0$, we have

$$\begin{aligned}\left(\sum_{n=0}^{\infty}(v_{n-1}^{s_0}|g_n(it)|)^{q_0}\right)^{1/q_0} &\le \left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^q\left(\sum_{k=0}^{n}(v_{k-1}^{s}|d_kf|)^q\right)^{(pq_0/qp_0)-1}\right)^{1/q_0} \\ &\le \frac{1}{\min((pq_0/qp_0),1)}\left\{\left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^q\right)^{1/q}\right\}^{p/p_0} \\ &= \frac{qp_0}{pq_0}\left\{\left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^q\right)^{1/q}\right\}^{p/p_0}.\end{aligned}$$

Similarly, we have

$$\tag{5.13}\begin{aligned}&\left(\sum_{n=0}^{\infty}(v_{n-1}^{s_1}|g_n(1+it)|)^{q_1}\right)^{1/q_1} \\ &\le C^{1/q_1}\left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^q\right)^{1/q_1} + C^{1/q_1}\left\{\left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^q\right)^{1/q}\right\}^{p/p_1}.\end{aligned}$$

Since $\|f\|_{F_{pq}^s} = 1$, we have

$$\left\|\left\{\left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^q\right)^{1/q}\right\}^{p/p_j}\right\|_{L_{p_j}} = \left\|\left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^q\right)^{1/q}\right\|_{L_p}^{p/p_j} = 1$$

for $j = 0, 1$. Furthermore, since the assumption $\rho_3(0) < 0$ is equivalent to $p/p_1 > q/q_1$, we have

$$\begin{aligned}\left\|\left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^{q}\right)^{1/q_1}\right\|_{L_{p_1}} &= \left\|\left\{\left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^{q}\right)^{1/q}\right\}^{q/q_1}\right\|_{L_{p_1}} \\ &\le \left\|\left\{\left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^{q}\right)^{1/q}\right\}^{p/p_1}\right\|_{L_{p_1}}^{qp_1/pq_1} \\ &= \left\|\left(\sum_{n=0}^{\infty}(v_{n-1}^{s}|d_nf|)^{q}\right)^{1/q}\right\|_{L_p}^{q/q_1} = 1.\end{aligned}$$

Hence, by Lemma 4.1, we have

(5.14)

$$\begin{aligned}&\|H(j+it)\|_{F^{s_j}_{p_jq_j}} \\ &= \left\|\left(\sum_{n=0}^{\infty}(v_{n-1}^{s_j}|h_n(j+it)|)^{q_j}\right)^{1/q_j}\right\|_{L_{p_j}} \\ &\le \left\|\left(\sum_{n=0}^{\infty}(v_{n-1}^{s_j}|g_n(j+it)|)^{q_j}\right)^{1/q_j}\right\|_{L_{p_j}} + \left\|\left(\sum_{n=0}^{\infty}(v_{n-1}^{s_j}|E_{n-1}[g_n(j+it)]|)^{q_j}\right)^{1/q_j}\right\|_{L_{p_j}} \\ &\le (1+C_{p_j,q_j})\left\|\left(\sum_{n=0}^{\infty}(v_{n-1}^{s_j}|g_n(j+it)|)^{q_j}\right)^{1/q_j}\right\|_{L_{p_j}} \le C',\end{aligned}$$

where C' is a positive constant depending only on p_0, p_1, q_0, q_1 and θ. Therefore, we obtain $H_\delta \in \mathcal{F}(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})$ with

$$\|H_\delta\|_{\mathcal{F}(F^{s_0}_{p_0q_0},F^{s_1}_{p_1q_1})} \le C' \max(e^{\delta\theta^2}, e^{\delta(1-\theta)^2}).$$

Hence, we have the desired conclusion for the case where $q_0, q_1 < \infty$.

For the case where $q_0 < \infty$ and $q_1 = \infty$, we replace (5.13) by

$$\sup_{n\ge0}|v_{n-1}^{s_1}g_n(1+it)| = \sup_{n\ge0}(1+G_n)^{p/p_1} \le C(1+\|(v_{n-1}^{s}d_nf)_{n\ge0}\|_{\ell_q}^{p/p_1})$$

where C is a positive constant depending only on p/p_1. Furthermore, we replace (5.14) for $j=1$ by

$$\begin{aligned}\|H(1+it)\|_{F^{s_1}_{p_1\infty}} &= \left\|\sup_{n\ge0} v_{n-1}^{s_1}|h_n(1+it)|\right\|_{L_{p_1}} \\ &\le (1+C_{p_1,\infty})\left\|\sup_{n\ge0} v_{n-1}^{s_1}|g_n(1+it)|\right\|_{L_{p_1}} \\ &\le C(1+C_{p_1,\infty})\left(1+\left\|\|(v_{n-1}^{s}d_nf)_{n\ge0}\|_{\ell_q}^{p/p_1}\right\|_{L_{p_1}}\right) \le C'.\end{aligned}$$

The rest of the proof is the same as in the case of $q_0, q_1 < \infty$. We have the desired conclusion for the case where $q_0 < \infty$ and $q_1 = \infty$. Similarly, we can prove the case where $q_0 = \infty$ and $q_1 < \infty$.

We now prove the case where $q_0 = q_1 = \infty$. We replace (5.13) by

$$\sup_{n\geq 0} |v_{n-1}^{s_j} g_n(j+it)| = \sup_{n\geq 0} (1+G_n)^{p/p_j} \leq C(1 + \|(v_{n-1}^s d_n f)_{n\geq 0}\|_{\ell_q}^{p/p_j})$$

where C is a positive constant depending only on p/p_0 and p/p_1.

Then, we replace (5.14) by

$$\begin{aligned}\|H(j+it)\|_{F_{p_j\infty}^{s_j}} &= \left\| \sup_{n\geq 0} v_{n-1}^{s_j} |h_n(j+it)| \right\|_{L_{p_j}} \\ &\leq (1 + C_{p_j,\infty}) \left\| \sup_{n\geq 0} v_{n-1}^{s_j} |g_n(j+it)| \right\|_{L_{p_j}} \\ &\leq C(1 + C_{p_j,\infty}) \left(1 + \left\| \|(v_{n-1}^s d_n f)_{n\geq 0}\|_{\ell_q}^{p/p_j} \right\|_{L_{p_j}} \right) \leq C'.\end{aligned}$$

The rest of the proof is the same as in the case of $q_0, q_1 < \infty$. The proof is completed. □

We now give the proof of Theorem 2.3.

Proof of Theorem 2.3. Combining $\theta \in (0,1)$ and $\min(q_0, q_1) < \infty$, we have $q < \infty$. Hence, combining Lemma 5.4 and Lemma 5.5, we obtain that $\|f\|_{[B_{p_0q_0}^{s_0}, B_{p_1q_1}^{s_1}]_\theta} \leq C_1 \|f\|_{B_{pq}^s}$ for all $f \in B_{pq}^s$, where C_1 is the constant in Lemma 5.5. Similarly, if $1 < p_0, p_1 < \infty$, then we have $\|f\|_{[F_{p_0q_0}^{s_0}, F_{p_1q_1}^{s_1}]_\theta} \leq C_1 \|f\|_{F_{pq}^s}$ for all $f \in F_{pq}^s$. Therefore, we only have to show the converse inequality.

We first give a proof for the case of martingale Besov spaces. Let $f \in [B_{p_0q_0}^{s_0}, B_{p_1q_1}^{s_1}]_\theta$. Let $F \in \mathcal{F}(B_{p_0q_0}^{s_0}, B_{p_1q_1}^{s_1})$ such that $F(\theta) = f$. From the fact $\|d_n g\|_{L_{p_j}} \leq C \|g\|_{B_{p_jq_j}^{s_j}}$, where C is a positive constant independent of $g = (g_n)_{n\geq 0} \in \mathcal{M}$, we have that $v_{n-1}^{s_0(1-z)+s_1 z} d_n F(z)$ belongs to $\mathcal{F}(L_{p_0}, L_{p_1})$ by a standard argument. Hence, by Lemma 5.1 with the fact $[L_{p_0}, L_{p_1}]_\theta = L_p$ ([2, Theorem 5.1.1]), we obtain that

$$\|v_{n-1}^s d_n F(\theta)\|_{L_p} \leq a_n^{1-\theta} b_n^\theta \tag{5.15}$$

where

$$\begin{aligned} a_n &= \frac{1}{1-\theta} \int_{-\infty}^{\infty} \|v_{n-1}^{s_0} d_n F(it)\|_{L_{p_0}} \mu_0(\theta, t)\, dt, \\ b_n &= \frac{1}{\theta} \int_{-\infty}^{\infty} \|v_{n-1}^{s_1} d_n F(1+it)\|_{L_{p_1}} \mu_1(\theta, t)\, dt. \end{aligned}$$

Using Minkowski's inequality and the fact that

$$\frac{1}{1-\theta} \int_{-\infty}^{\infty} \mu_0(\theta, t)\, dt = \frac{1}{\theta} \int_{-\infty}^{\infty} \mu_1(\theta, t)\, dt = 1, \tag{5.16}$$

we have

$$\begin{aligned} \|(a_n)_{n\geq 0}\|_{\ell_{q_0}} &\leq \frac{1}{1-\theta} \int_{-\infty}^{\infty} \|F(it)\|_{B_{p_0q_0}^{s_0}} \mu_0(\theta, t)\, dt \leq \|F\|_{\mathcal{F}(B_{p_0q_0}^{s_0}, B_{p_1q_1}^{s_1})}, \\ \|(b_n)_{n\geq 0}\|_{\ell_{q_1}} &\leq \frac{1}{\theta} \int_{-\infty}^{\infty} \|F(1+it)\|_{B_{p_1q_1}^{s_1}} \mu_1(\theta, t)\, dt \leq \|F\|_{\mathcal{F}(B_{p_0q_0}^{s_0}, B_{p_1q_1}^{s_1})}. \end{aligned} \tag{5.17}$$

Therefore, using (5.15), Hölder's inequality and (5.17), we obtain

$$\begin{aligned}
\|f\|_{B^s_{pq}} &= \|F(\theta)\|_{B^s_{pq}} \\
&= \left\|(\|v^s_{n-1} d_n F(\theta)\|_{L_p})_{n\ge 0}\right\|_{\ell_q} \\
&\le \left\|(a_n^{1-\theta} b_n^{\theta})_{n\ge 0}\right\|_{\ell_q} \\
&\le \|(a_n)_{n\ge 0}\|_{\ell_{q_0}}^{1-\theta} \|(b_n)_{n\ge 0}\|_{\ell_{q_1}}^{\theta} \\
&\le (\|F\|_{\mathcal{F}(B^{s_0}_{p_0q_0}, B^{s_1}_{p_1q_1})})^{1-\theta} (\|F\|_{\mathcal{F}(B^{s_0}_{p_0q_0}, B^{s_1}_{p_1q_1})})^{\theta} = \|F\|_{\mathcal{F}(B^{s_0}_{p_0q_0}, B^{s_1}_{p_1q_1})}.
\end{aligned}$$

Thus, we obtain the desired conclusion for the case of martingale Besov spaces.

We next give the proof for the case of martingale Triebel-Lizorkin spaces.

Let $f \in [F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1}]_\theta$. Let $G \in \mathcal{F}(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})$ such that $G(\theta) = f$. Let $h \in \mathcal{T}$ such that $\|h\|_{F^{-s}_{p'q'}} = 1$. Noting that $1 < p_0, p_1 < \infty$, we use Lemma 5.5 to take $H \in \mathcal{F}(F^{-s_0}_{p_0'q_0'}, F^{-s_1}_{p_1'q_1'})$ such that $H(\theta) = \overline{h}$, $\|H\|_{\mathcal{F}(F^{-s_0}_{p_0'q_0'}, F^{-s_1}_{p_1'q_1'})} \le C_1$ and that H satisfies (5.5). Then define $D_h(z) = \sum_{n=0}^{\infty} E\,[d_n G(z) d_n H(z)]$. Since H satisfies these conditions mentioned above, we have that $D_h \in \mathcal{F}(\mathbb{C}, \mathbb{C})$. Moreover, as in (4.1), we have

$$\begin{aligned}
|D_h(j+it)| &\le \|G(j+it)\|_{F^{s_j}_{p_jq_j}} \, \|H(j+it)\|_{F^{-s_j}_{p_j'q_j'}} \\
&\le \|G\|_{\mathcal{F}(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})} \, \|H\|_{\mathcal{F}(F^{-s_0}_{p_0'q_0'}, F^{-s_1}_{p_1'q_1'})} \\
&\le C_1 \|G\|_{\mathcal{F}(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})}
\end{aligned} \tag{5.18}$$

where $j = 0, 1$. Using Lemma 5.1, (5.18) and (5.16), we obtain that

$$\begin{aligned}
&|D_h(\theta)| \\
&\le \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} |D_h(it)|\mu_0(\theta,t)\,dt\right)^{1-\theta} \left(\frac{1}{\theta}\int_{-\infty}^{\infty} |D_h(1+it)|\mu_0(\theta,t)\,dt\right)^{\theta} \\
&\le (C_1\|G\|_{\mathcal{F}(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})})^{1-\theta} (C_1\|G\|_{\mathcal{F}(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})})^{\theta} = C_1\|G\|_{\mathcal{F}(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})}
\end{aligned}$$

for all $h \in \mathcal{T}$ such that $\|h\|_{F^{-s}_{p'q'}} = 1$ and for all $G \in \mathcal{F}(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})$ such that $G(\theta) = f$. Therefore, we have

$$\sup_{h\in\mathcal{T}:\|h\|_{F^{-s}_{p'q'}}=1} |D_h(\theta)| \le C_1 \|f\|_{[F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1}]_\theta}. \tag{5.19}$$

For $g = (g_n)_{n\ge 0} \in \mathcal{M}$ and $N \ge 0$, let $g^N = (g_{n\wedge N})_{n\ge 0}$, where $n \wedge N = \min(n, N)$. Define

$$(F^s_{pq})^N = \{g^N : g \in F^s_{pq}\}, \quad \mathcal{T}^N = \{g^N : g \in \mathcal{T}\}.$$

Then, it is clear that $(F^s_{pq})^N$ is a closed subspace of F^s_{pq}. Moreover, by the same way as in Lemma 5.4, we have that $\mathcal{T}^N$ is dense in $(F^{-s}_{p'q'})^N$, even if $q' = \infty$. Hence, by Theorem 2.2 and (5.19), we have

$$\|f^N\|_{F^s_{pq}} \le C \sup_{h\in\mathcal{T}^N:\|h\|_{F^{-s}_{p'q'}}=1} |D_h(\theta)| \le CC_1 \|f\|_{[F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1}]_\theta}$$

where C and C_1 are positive constants in Theorem 2.2 and in Lemma 5.5 respectively. Using monotone convergence theorem, we obtain

$$\|f\|_{F^s_{pq}} = \sup_{N\ge 0} \|f^N\|_{F^s_{pq}} \le CC_1 \|f\|_{[F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1}]_\theta}$$

for all $f \in [F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1}]_\theta$. Therefore, we obtain the desired conclusion for the case of martingale Triebel-Lizorkin spaces. The proof is completed. □

6 Proof of Theorem 2.4.

In this section, we give a proof of Theorem 2.4.

Proof of Theorem 2.4. Since $|d_nf| = |E_n(f - f_{n-1})| \le E_n|f - f_{n-1}|$, we have

$$\|f\|_{B^s_{pq}} \le \left\|(\|v^s_{n-1}E_n|f - f_{n-1}|\|_{L_p})_{n\ge0}\right\|_{\ell_q},$$
$$\|f\|_{F^s_{pq}} \le \left\|\|(v^s_{n-1}E_n|f - f_{n-1}|)_{n\ge0}\|_{\ell_q}\right\|_{L_p}.$$

We now show the converse inequalities. Let δ_1 be the constant in (2.1). We first show (2.6) for $q = \infty$. By (2.1) and the assumption $s > 0$, we have

$$v^s_{n-1}|f - f_{n-1}| \le v^s_{n-1}\sum_{k=n}^{\infty}|d_kf| \le \sum_{k=n}^{\infty}\delta_1^{s(n-k)}|v^s_{k-1}d_kf|. \tag{6.1}$$

From Jensen's inequality for E_n and (6.1), we have

$$\left\|v^s_{n-1}E_n|f - f_{n-1}|\right\|_{L_p} \le \left\|v^s_{n-1}|f - f_{n-1}|\right\|_{L_p} \le \sum_{k=n}^{\infty}\delta_1^{s(n-k)}\|v^s_{k-1}d_kf\|_{L_p} \le \frac{\|f\|_{B^s_{p\infty}}}{1 - \delta_1^{-s}}.$$

Therefore, we have (2.6) for $q = \infty$.

To show (2.7) for $q = \infty$, let $G = \sup_{n\ge0} v^s_{n-1}|d_nf|$. Then, by (6.1), we have

$$v^s_{n-1}E_n|f - f_{n-1}| \le \sum_{k=n}^{\infty}\delta_1^{s(n-k)}E_n|v^s_{k-1}d_kf| \le \frac{E_nG}{1 - \delta_1^{-s}}.$$

Therefore, using Doob's inequality, we have

$$\begin{aligned}\left\|\sup_{n\ge0} v^s_{n-1}E_n|f - f_{n-1}|\right\|_{L_p} &\le (1 - \delta_1^{-s})^{-1}\left\|\sup_{n\ge0} E_nG\right\|_{L_p}\\ &\le \frac{p}{(p-1)(1 - \delta_1^{-s})}\|G\|_{L_p}\\ &= \frac{p}{(p-1)(1 - \delta_1^{-s})}\|f\|_{F^s_{p\infty}}.\end{aligned}$$

Thus, we have (2.7) for $q = \infty$.

We next show (2.6) for $0 < q < \infty$. If $q \le 1$, then we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\delta_1^{s(n-k)}\|v^s_{k-1}d_kf\|_{L_p}\right)^q &\le \sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\delta_1^{sq(n-k)}\|v^s_{k-1}d_kf\|^q_{L_p}\\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\delta_1^{sq(n-k)}\right)\|v^s_{k-1}d_kf\|^q_{L_p}\\ &\le (1 - \delta_1^{-sq})^{-1}\sum_{k=0}^{\infty}\|v^s_{k-1}d_kf\|^q_{L_p}.\end{aligned}$$

If $1 < q < \infty$, then, denoting by q' the conjugate exponent of q, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\delta_1^{s(n-k)}\|v_{k-1}^s d_k f\|_{L_p}\right)^q &= \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\delta_1^{s(n-k)/q'}\delta_1^{s(n-k)/q}\|v_{k-1}^s d_k f\|_{L_p}\right)^q \\ &\le \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\delta_1^{s(n-k)}\right)^{q/q'}\sum_{k=n}^{\infty}\delta_1^{s(n-k)}\|v_{k-1}^s d_k f\|_{L_p}^q \\ &= (1-\delta_1^{-s})^{-q/q'}\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\delta_1^{s(n-k)}\right)\|v_{k-1}^s d_k f\|_{L_p}^q \\ &\le (1-\delta_1^{-s})^{-q}\sum_{k=0}^{\infty}\|v_{k-1}^s d_k f\|_{L_p}^q.\end{aligned} \tag{6.2}$$

Therefore, we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\delta_1^{s(n-k)}\|v_{k-1}^s d_k f\|_{L_p}\right)^q \le C^q\sum_{k=0}^{\infty}\|v_{k-1}^s d_k f\|_{L_p}^q \tag{6.3}$$

where C is a positive constant depending only on s, q and δ_1. Combining Remark 4.1, (6.3) and (6.1), we have

$$\left(\sum_{n=0}^{\infty}\left\|v_{n-1}^s E_n|f-f_{n-1}|\right\|_{L_p}^q\right)^{1/q} \le \left(\sum_{n=0}^{\infty}\left\|v_{n-1}^s|f-f_{n-1}|\right\|_{L_p}^q\right)^{1/q} \le C\|f\|_{B_{pq}^s}.$$

Thus, we obtain (2.6).

We now show (2.7) for $1 \le q < \infty$. Similarly as in (6.2), we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\delta_1^{s(n-k)}v_{k-1}^s|d_k f|\right)^q \le (1-\delta_1^{-s})^{-q}\sum_{k=0}^{\infty}(v_{k-1}^s|d_k f|)^q. \tag{6.4}$$

Combining (6.1) and (6.4), we have

$$\left(\sum_{n=0}^{\infty}(v_{n-1}^s|f-f_{n-1}|)^q\right)^{1/q} \le (1-\delta_1^{-s})^{-1}\left(\sum_{k=0}^{\infty}(v_{k-1}^s|d_k f|)^q\right)^{1/q}.$$

Therefore, using Lemma 4.1, we have

$$\begin{aligned}\left\|\left(\sum_{n=0}^{\infty}(v_{n-1}^s E_n|f-f_{n-1}|)^q\right)^{1/q}\right\|_{L_p} &\le C_{p,q}\left\|\left(\sum_{n=0}^{\infty}(v_{n-1}^s|f-f_{n-1}|)^q\right)^{1/q}\right\|_{L_p} \\ &\le \frac{C_{p,q}}{1-\delta_1^{-s}}\left\|\left(\sum_{k=0}^{\infty}(v_{k-1}^s|d_k f|)^q\right)^{1/q}\right\|_{L_p}.\end{aligned}$$

We have the desired conclusion.

□

7 Proofs of Theorems 2.5 and 2.6. In this section, we give proofs of Theorems 2.5 and 2.6. As is described in Section 2, we postulate following conditions:

(7.1) Every σ-algebra $\mathcal{F}_n$ is generated by countable atoms.

(7.2) $\{\mathcal{F}_n\}_{n\geq 0}$ is regular.

(7.3) If $B \in A(\mathcal{F}_{n-1})$, $B' \in A(\mathcal{F}_n)$ and $B' \subset B$, then $P(B') < P(B)$ for every $n \geq 1$.

(7.4) $\mathcal{F}_0 = \{\emptyset, \Omega\}$,

where $A(\mathcal{F}_n)$ stands for the set of all atoms in $\mathcal{F}_n$. Define $\mathcal{F}_n$-measurable functions b_n and v_n by

$$b_n = \sum_{B\in A(\mathcal{F}_n)} P(B)\chi_B, \quad v_n = b_n^{-1}. \tag{7.5}$$

By [11, Lemma 3.3], b_n satisfy

$$\left(1+\frac{1}{R}\right) b_n \leq b_{n-1} \leq Rb_n \tag{7.6}$$

where R is the constant in (2.10). Hence, $v = (v_n)_{n\geq 0}$ in (7.5) belongs to $\mathcal{V}$.

We start with the following lemma.

Lemma 7.1. *Let $p_0, p_1 \in (0,\infty)$ with $p_0 < p_1$. Let n be a non-negative integer. Let $\alpha = 1/p_0 - 1/p_1$. Let $f = (f_n)_{n\geq 0} \in \mathcal{M}$. If $d_n f \in L_{p_1}$, then*

$$\|d_n f\|_{L_{p_1}} \leq R^\alpha \|v_{n-1}^\alpha d_n f\|_{L_{p_0}} \tag{7.7}$$

with convention $v_{-1} = v_0$ and $f_{-1} = 0$, where R is the constant in (2.10).

Proof. If $n = 0$, then $\|d_n f\|_{L_{p_1}} = \|v_{n-1}^\alpha d_n f\|_{L_{p_0}}$ because $d_0 f$ is constant and $v_{-1} = v_0 = 1$ by $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Since $R \geq 2$, we have (7.7) for $n = 0$.

For $n \geq 1$, let $B \in A(\mathcal{F}_n)$. Then, since $d_n f$ is constant on B, we have

$$\|\chi_B d_n f\|_{L_{p_1}} = P(B)^{1/p_1}\|\chi_B d_n f\|_{L_\infty} = P(B)^{1/p_1-1/p_0}\|\chi_B d_n f\|_{L_{p_0}}. \tag{7.8}$$

Using (7.8), $\alpha = 1/p_0 - 1/p_1$ and $v_n \leq Rv_{n-1}$, we have

$$\|\chi_B d_n f\|_{L_{p_1}} = \|\chi_B v_n^\alpha d_n f\|_{L_{p_0}} \leq R^\alpha \|\chi_B v_{n-1}^\alpha d_n f\|_{L_{p_0}}. \tag{7.9}$$

Using (7.9), we have

$$\begin{aligned} \|d_n f\|_{L_{p_1}}^{p_1} &= \sum_{B\in A(\mathcal{F}_n)} \|\chi_B d_n f\|_{L_{p_1}}^{p_1} \\ &\leq R^{\alpha p_1} \sum_{B\in A(\mathcal{F}_n)} \|\chi_B v_{n-1}^\alpha d_n f\|_{L_{p_0}}^{p_1} \\ &\leq R^{\alpha p_1} \left(\sum_{B\in A(\mathcal{F}_n)} \|\chi_B v_{n-1}^\alpha d_n f\|_{L_{p_0}}^{p_0} \right)^{p_1/p_0} = R^{\alpha p_1}\|v_{n-1}^\alpha d_n f\|_{L_{p_0}}^{p_1}. \end{aligned}$$

We have the desired conclusion. □

We next show the following lemma.

Lemma 7.2. *Let $p \in (0,\infty)$ and $\alpha \in (0,\infty)$. Let n be a non-negative integer. Let $f \in F^{\alpha}_{p\infty}$ with $\|f\|_{F^{\alpha}_{p\infty}} = 1$. Then,*

$$|v^{\alpha}_{n-1} d_n f| \le R^{1/p} v^{1/p}_{n-1}$$

with convention $v_{-1} = v_0$ and $f_{-1} = 0$, where R is the constant in (2.10)*.*

Proof. Let $B \in A(\mathcal{F}_n)$. Since $v^{\alpha}_{n-1} d_n f$ is constant on B, we have

$$\begin{aligned}
\chi_B |v^{\alpha}_{n-1} d_n f| &= \chi_B \left(\frac{1}{P(B)} \int_B |v^{\alpha}_{n-1}(\omega) d_n f(\omega)|^p \, dP(\omega) \right)^{1/p} \\
&\le \frac{\chi_B}{P(B)^{1/p}} \left(\int_\Omega |v^{\alpha}_{n-1}(\omega) d_n f(\omega)|^p \, dP(\omega) \right)^{1/p} \\
&\le \chi_B v_n^{1/p} \left(\int_\Omega \sup_{n\ge 0} |v^{\alpha}_{n-1}(\omega) d_n f(\omega)|^p \, dP(\omega) \right)^{1/p} \\
&\le \chi_B R^{1/p} v^{1/p}_{n-1} \|f\|_{F^{\alpha}_{p\infty}} = \chi_B R^{1/p} v^{1/p}_{n-1}.
\end{aligned}$$

The proof is completed. □

We now show Theorems 2.5 and 2.6.

Proof of Theorem 2.5. By Theorem 2.1, we only have to give a proof for the case where $s = 0$. We first show that

$$\|f\|_{B^{0}_{p_1 q}} \le R^{\alpha} \|f\|_{B^{\alpha}_{p_0 q}}. \tag{7.10}$$

Indeed, using Lemma 7.1, we have (7.10) as follows:

$$\|f\|_{B^{0}_{p_1 q}} = \left(\sum_{n=0}^{\infty} \|d_n f\|^q_{L_{p_1}} \right)^{1/q} \le R^{\alpha} \left(\sum_{n=0}^{\infty} \|v^{\alpha}_{n-1} d_n f\|^q_{L_{p_0}} \right)^{1/q} = R^{\alpha} \|f\|_{B^{\alpha}_{p_0 q}}.$$

We next show

$$\|f\|_{F^{0}_{p_1 q}} \le R^{\alpha} \|f\|_{F^{\alpha}_{p_0 \infty}}. \tag{7.11}$$

Let $f \in F^{\alpha}_{p_0\infty}$ with $\|f\|_{F^{\alpha}_{p_0\infty}} = 1$. Let

$$F(\omega) = \sup_{n\ge 0} |v^{\alpha}_{n-1}(\omega) d_n f(\omega)|, \quad G(\omega) = \left(\sum_{n=0}^{\infty} |d_n f(\omega)|^q \right)^{1/q}.$$

We show

$$G \le C F^{p_0/p_1} \tag{7.12}$$

where C is a positive constant depending only on p_0, p_1, q and R. Indeed, using Lemma 7.2 with $p = p_0$ and $\alpha = 1/p_0 - 1/p_1$, we have

$$\begin{aligned}
|d_n f| &= v^{-\alpha}_{n-1} |v^{\alpha}_{n-1} d_n f| \\
&\le \min(v^{-\alpha}_{n-1} F, v^{-\alpha}_{n-1} R^{1/p_0} v^{1/p_0}_{n-1}) \\
&= \min(v^{-\alpha}_{n-1} F, R^{1/p_0} v^{1/p_1}_{n-1}).
\end{aligned}$$

Therefore, noting the convention $v_{-1} = v_0$ and the fact $(1+1/R)v_{n-1} \le v_n \le Rv_{n-1}$, we have (7.12) as follows:

$$\begin{aligned}
G^q &\le \sum_{n=0}^{\infty} \min(v_{n-1}^{-\alpha}F, R^{1/p_0}v_{n-1}^{1/p_1})^q \\
&\le 2\sum_{n=1}^{\infty} \min(v_{n-1}^{-\alpha}F, R^{1/p_0}v_{n-1}^{1/p_1})^q \\
&\le 2\sum_{n=1}^{\infty} \frac{1}{\log(1+1/R)} \int_{v_{n-1}}^{v_n} \min(v_{n-1}^{-\alpha}F, R^{1/p_0}v_{n-1}^{1/p_1})^q \,\frac{dt}{t} \\
&\le \frac{2}{\log(1+1/R)} \sum_{n=1}^{\infty} \int_{v_{n-1}}^{v_n} \min(R^{\alpha}t^{-\alpha}F, R^{1/p_0}t^{1/p_1})^q \,\frac{dt}{t} \\
&\le \frac{2R^{q/p_0}}{\log(1+1/R)} \int_1^{\infty} \min(t^{-\alpha}F, t^{1/p_1})^q \,\frac{dt}{t} \\
&\le \frac{2R^{q/p_0}}{q\log(1+1/R)} \left(\frac{1}{\alpha} + p_1\right) F^{qp_0/p_1}.
\end{aligned}$$

By (7.12), we have

$$\|f\|_{F^0_{p_1q}} = \|G\|_{L_{p_1}} \le C\|F\|_{L_{p_0}}^{p_0/p_1} = C\|f\|_{F^{\alpha}_{p_0\infty}}^{p_0/p_1} = C.$$

We have the desired conclusion. □

Proof of Theorem 2.6. By Theorem 2.1 and Proposition 3.2, we only have to show $B^0_{p\infty} \hookrightarrow B^{-1/p}_{\infty\infty}$. As in (7.8), we have

$$\|v_{n-1}^{-1/p}d_nf\|_{L_\infty} \le R^{1/p}\|v_n^{-1/p}d_nf\|_{L_\infty} \le R^{1/p} \sup_{B\in A(\mathcal{F}_n)} \|\chi_B d_nf\|_{L_p} \le R^{1/p}\|f\|_{B^0_{p\infty}}.$$

We obtain the desired conclusion. □

8 Proof of Theorem 2.7 and Corollary 2.8.

In this section, we prove Theorem 2.7 and Corollary 2.8. To do this, we need the following John-Nirenberg type lemma.

Lemma 8.1. *Let $p \in (0,\infty)$ and $q \in [1,\infty)$. Then, the following equivalence holds:*

$$\|f\|_{F^0_{\infty q}} \sim \sup_{n\ge 0} \left\| E_n \left[\left(\sum_{k=n}^{\infty} |d_kf|^q \right)^{p/q} \right]^{1/p} \right\|_{L_\infty} \qquad (f \in F^0_{\infty q}).$$

For the proof of Lemma 8.1, we refer to [17, Theorem 2].

Lemma 8.2. *Suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. Furthermore, assume that $\{\mathcal{F}_n\}_{n\ge 0}$ is regular with* (2.11) *and* (2.12). *Let $v = (v_n)_{n\ge 0}$ be the sequence of functions defined in* (2.9). *Let $s > 0$. Then, $B^s_{\infty\infty} = \Lambda_1^-(s)$ with equivalent norms.*

Proof. By Theorem 2.4 and the regularity of $\{\mathcal{F}_n\}_{n\geq 0}$, we have

$$\begin{aligned}
\|f\|_{B^s_{\infty\infty}} &\sim \sup_{n\geq 0} \left\|v^s_{n-1} E_n|f - f_{n-1}|\right\|_{L_\infty} \\
&\sim \sup_{n\geq 0} \|v^s_n E_n|f - f_{n-1}|\|_{L_\infty} \\
&= \sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{P(B)^{1+s}} \int_B |f(\omega) - (E_{n-1}f)(\omega)|\, dP(\omega) \\
&= \|f\|_{\Lambda_1^-(s)}.
\end{aligned}$$

We have the desired conclusion. □

Proof of Theorem 2.7. We obtain (2.16) and (2.18) from (2.13) and (2.14) respectively. To show (2.17), let $f \in F^0_{pq}$ and $\alpha = 1/p$. Noting that $|d_k I_\alpha f| = |b^\alpha_{k-1} d_k f| \leq b^\alpha_{n-1}|d_k f|$ for $k \geq n$ and using the regularity of $\{\mathcal{F}_n\}_{n\geq 0}$, we have

$$\left(\sum_{k=n}^\infty |d_k I_\alpha f|^q\right)^{1/q} \leq b^\alpha_{n-1} \left(\sum_{k=n}^\infty |d_k f|^q\right)^{1/q} \leq R^\alpha b^\alpha_n \left(\sum_{k=0}^\infty |d_k f|^q\right)^{1/q}.$$

Then, for $B \in A(\mathcal{F}_n)$, we have

$$\begin{aligned}
E\left[\chi_B \left(\sum_{k=n}^\infty |d_k I_\alpha f|^q\right)^{p/q}\right] &\leq R^{\alpha p} P(B)^{\alpha p} E\left[\chi_B \left(\sum_{k=0}^\infty |d_k f|^q\right)^{p/q}\right] \\
&\leq R P(B) \|f\|^p_{F^0_{pq}}
\end{aligned}$$

by $\alpha = 1/p$. Since $\mathcal{F}_n$ is generated by $A(\mathcal{F}_n)$, this means

$$E_n\left[\left(\sum_{k=n}^\infty |d_k I_\alpha f|^q\right)^{p/q}\right]^{1/p} \leq R^\alpha \|f\|_{F^0_{pq}}. \tag{8.1}$$

Combining (8.1), Lemma 8.1 and Theorem 2.1, we obtain the desired conclusion. □

Proof of Corollary 2.8. Taking $s = 0$, $q = 2$ in (2.16) and combining with the fact $L_p(\ell_2) \hookrightarrow L_p(\ell_\infty)$, we obtain (2.19). Similarly, taking $s = 0$, $q = 2$ in (2.17), we obtain (2.20). Taking $s = 0$ in (2.18), we obtain (2.21) by Lemma 8.2 and by the fact $L_p(\ell_2) \hookrightarrow L_p(\ell_\infty)$. The proof is completed. □

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