A MODIFIED ENDPOINT ESTIMATE OF THE KUNZE-STEIN PHENOMENON ASSOCIATED WITH COMPLEX SEMISIMPLE LIE GROUPS.

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Abstract

The endpoint estimates for the Kunze-Stein phenomenon associated with real rank one semisimple Lie groups and the Jacobi hypergroup were respectively obtained by A. Ionescu and J. Liu. Recently, in the case of the Jacobi hypergroup, an alternative proof using the Abel transform was obtained by the first author. In this paper we apply the same argument to the complex semisimple Lie groups and prove a modified endpoint estimate for the Kunze-stein phenomenon associated with the complex semisimple Lie groups.

1 Introduction

Let G be a noncompact connected semisimple Lie group of real rank one and $G = KAK$ the Cartan decomposition of G. The endpoint estimate of the Kunze-Stein phenomenon is given as follows:

$$\|f * g\|_{L^{2,\infty}(G)} \leq c\|f\|_{L^{2,1}(G)}\|g\|_{L^{2,1}(G)} \tag{1}$$

for all functions f, g on G, where $\|\cdot\|_{L^{p,q}(G)}$ is the norm of the Lorentz space $L^{p,q}(G)$ (see [4]). This estimate is also true for the Jacobi hypergroup (see [6] and cf. [5]). However, when the real rank of G is greater than one, we don't know whether the above estimate is true or not. In this paper under the assumption that G is complex, we obtain a modified estimate:

$$\|f * M_m g\|_{L^{2,\infty}(\Delta)} \leq c\|f\|_{L^{2,1}(\Delta)}\|g\|_{L^{2,1}(\Delta)} \tag{2}$$

for all K-bi-invariant functions f, g on G, where $L^{p,q}(\Delta)$ is the subspace of K-bi-invariant functions in $L^{p,q}(G)$ and M_m is a Fourier multiplier on G corresponding to m on $\mathfrak{a}^*$ and satisfying a good property. We call such Fourier multipliers as of type $(\sqrt{\Delta}, \infty)$ (see Definition 3.1) and we give a criterion on m by which M_m is of type $(\sqrt{\Delta}, \infty)$. In §4 we give some examples of m satisfying the criterion for $G = Sp(4, \mathbf{C})$ and $SL(3, \mathbf{C})$.

2 Preliminaries

Let G be a connected complex semisimple Lie group. Let $G = KAN$ and $G = KAK$ be the Iwasawa and Cartan decompositions of G, where K, A and N are maximal compact, vector and nilpotent subgroups of G respectively. Let $\mathfrak{g}$ and $\mathfrak{a}$ be the Lie algebras of G and A respectively. Then $A = \exp \mathfrak{a}$. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$ and Σ_+ the set of all positive roots of $(\mathfrak{g}, \mathfrak{a})$. We put $\rho(H) = \sum_{\alpha \in \Sigma_+} \alpha(H)$. Let W be the Weyl group of (G, A) and $\mathfrak{a}_+$ the positive Weyl chamber of $\mathfrak{a}$. We put $A_+ = \exp \mathfrak{a}_+$. Let dg, da, dk, dn, dH be the invariant measures on $G, A, K, N, \mathfrak{a}$ respectively, normalized as in [8], §1. Especially, $d(\exp H) = dH$ and $dg = e^{2\rho(\log a)} dk da dn$. We define for $H \in \mathfrak{a}$,

$$D(\exp H) = \sum_{w \in W} \det w \cdot e^{w\rho(H)} = \prod_{\alpha \in \Sigma_+} (e^{\alpha(H)} - e^{-\alpha(H)}),$$

and put $\Delta = D^2$. Then $\Delta(\exp H) = O(e^{2\rho(H)})$. We define for $\lambda \in \mathfrak{a}^*$,

$$\pi(\lambda) = \prod_{\alpha \in \Sigma_+} \lambda(H_\alpha),$$

where H_α is determined as an element in the root space of α by setting $\alpha(H) = \langle H, H_\alpha \rangle$ for all $H \in \mathfrak{a}$. For K-bi-invariant functions f on G, it follows that

$$\int_G f(g) dg = C_0 \int_{\mathfrak{a}} f(\exp H) \Delta(\exp H) dH.$$

For a positive function w on A, we denote by $L^p(w)$, $1 \le p \le \infty$, the space of K-bi-invariant functions of G satisfying

$$\int_{\mathfrak{a}} |f(\exp H)|^p w(\exp H) dH < \infty.$$

Similarly, the Lorentz space $L^{p,q}(w)$ is defined by the space consisting of all K-bi-invariant functions f on G such that $f \circ \exp$ belongs to the $w \circ \exp$ -weighted $L^{p,q}$ Lorentz space on $\mathfrak{a}$ (cf. [3]).

We overview some basic facts on the spherical and Abel transforms on G. We refer to [2], §14, [7], §4 and [8], §7 for the details. For $\lambda \in \mathfrak{a}^*$ the spherical function ϕ_λ on G is given by

$$D(\exp H) \phi_\lambda(\exp H) = \frac{\pi(\rho)}{\pi(i\lambda)} \sum_{w \in W} \det w \cdot e^{iw\lambda(H)}. \tag{3}$$

For K-bi-invariant functions f on G, the spherical transform $\widehat{f}$ on $\mathfrak{a}_+$ and the Abel transform $\mathcal{A}f$ on A_+ are respectively defined by

$$\widehat{f}(\lambda) = \int_G f(x)\phi_\lambda(g^{-1})dg,$$
$$\mathcal{A}f(\exp H) = e^{\rho(H)} \int_N f(\exp H \cdot n)dn.$$

We often identify a K-bi-invariant function f on G and a function F on A_+ with W-invariant functions on A which are denoted by the same symbols. Then it is known that for $\lambda \in \mathfrak{a}^*$,

$$\widehat{f}(\lambda) = (\mathcal{A}f \circ \exp)\tilde{}(\lambda) = \int_{\mathfrak{a}} \mathcal{A}f(\exp H)e^{-i\lambda(H)}dH, \tag{4}$$

where $\tilde{}$ denotes the classical Fourier transform on $\mathfrak{a}$. Especially, it follows that

$$\mathcal{A}(f * g) = \mathcal{A}f \otimes \mathcal{A}g, \tag{5}$$

where $*$ and $\otimes$ are the convolutions on G and A respectively.

Let $\pi(\partial_H)$ be the differential operator on $\mathfrak{a}$ with constant coefficients defined as follows: For a smooth function ϕ on $\mathfrak{a}$,

$$\pi(\partial_H)\phi(H) = \prod_{\alpha \in \sum_+} \langle \alpha, \phi'(H) \rangle,$$

where $\phi'(H) \in \mathfrak{a}^*$ is the differential at H of ϕ. Then

$$\pi(-i\lambda)\widehat{f}(\lambda) = C_0\pi(\rho)|W|(D \cdot f \circ \exp)\tilde{}(\lambda)$$

(see [8], (25)) and, since $\pi(-i\lambda)\widehat{f}(\lambda) = (\pi(-\partial_H)\mathcal{A}f \circ \exp)\tilde{}$, it follows that

$$f(\exp H) = \frac{C}{D(\exp H)}\pi(-\partial_H)\mathcal{A}f(\exp H), \tag{6}$$

where $C^{-1} = C_0\pi(\rho)|W|$ (cf. [8], Theorem 6). Especially, noting (5), we see that

$$f * g(\exp H) = \frac{C}{D(\exp H)}\pi(-\partial_H)\mathcal{A}f \otimes \mathcal{A}g(\exp H). \tag{7}$$

In the following we use the letter c to denote different constants.

3 A version of the endpoint estimate

We introduce a Fourier multiplier which will be used to modify the endpoint estimate (1). Let m be a bounded W-invariant function on $\mathfrak{a}$. For $f \in L^2(\Delta)$ the corresponding Fourier multiplier M_m is defined as $\widehat{M_m f}(\lambda) = m(\lambda)\widehat{f}(\lambda)$. Clearly, M_m is also a Fourier multiplier on $L^2(\mathfrak{a})$.

Definition 3.1. *M_m is of type $(\sqrt{\Delta}, \infty)$ if M_m satisfies*

$$\|M_m(\mathcal{A}f \circ \exp)\|_{L^\infty(\mathfrak{a})} \leq c\|f\|_{L^1(\sqrt{\Delta})} \tag{8}$$

for all $f \in L^1(\sqrt{\Delta})$.

The following two lemmas will be used in the proof of the modified endpoint estimate (2).

Lemma 3.2. *Let f be a smooth K-bi-invariant function on G and $f \in L^1(\sqrt{\Delta})$. Then there exists a positive constant c such that*

$$\|\pi(-\partial_H)\mathcal{A}f \circ \exp\|_{L^1(\mathfrak{a})} \leq c\|f\|_{L^1(\sqrt{\Delta})}. \tag{9}$$

Proof. Since $\sqrt{\Delta} = D$, this lemma is obvious from (6). □

Lemma 3.3. *Let $E(\exp H) = \prod_{\alpha\in\Sigma_+} e^{\alpha(H)} = e^{\rho(H)}$. For all functions $f \in L^1(E)$,*

$$\|f\|_{L^1(E)} \leq c\|f\|_{L^{2,1}(E^2)}.$$

Proof. Let $\Sigma_+^0 = \{\alpha_i \mid 1 \leq i \leq n = \dim \mathfrak{a}\}$ denote the set of positive simple roots. Mapping $H \in \mathfrak{a}$ to $(\alpha_i(H))_{1\leq i\leq n} \in \mathbf{R}^n$, we identify $\mathfrak{a}$ with $\mathbf{R}^n$ and $\mathfrak{a}_+$ with a domain $\mathbf{R}^n_{++}$ in $\mathbf{R}^n_+$. Let $\delta = E \circ \exp$. Then it is enough to prove that for functions F on $\mathbf{R}^n_{++}$, $\|F\|_{L^1(\delta)} \leq c\|F\|_{L^{2,1}(\delta^2)}$. There exist $d_i > 0$, $1 \leq i \leq n$, for which $\delta(H) = \prod_{\alpha\in\Sigma_+} e^{\alpha(H)} = \prod_{i=1}^n e^{d_i\alpha_i(H)}$. We may suppose that F is the characteristic function χ_S of a set S in $\mathbf{R}^n_{++}$ and S is a rectangle $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$. Since the volume of $\{x \mid |\chi_S(x)| > \lambda\}$ with respect to δ^2 is given by $V(S) = \int_S \delta^2(x)dx$ if $0 \leq \lambda \leq 1$ and 0 otherwise, the rearrangement function $\chi_S^*(t)$ is 1 if $0 \leq t \leq V(S)$ and 0 otherwise. Then

$$\|\chi_S\|_{L^{2,1}(\delta^2)} = c\int_0^\infty |\chi_S^*(t)|t^{\frac{1}{2}}\frac{dt}{t} = cV(S)^{\frac{1}{2}}.$$

On the other hand, it follows that

$$\begin{aligned}\|\chi_S\|_{L^1(\delta)} =& \prod_{i=1}^n \int_{a_i}^{b_i} e^{d_i\alpha_i(H)}dH \leq c\prod_{i=1}^n (e^{2d_ib_i} - e^{2d_ia_i})^{\frac{1}{2}} \\ =& \Big(\prod_{i=1}^n \int_{a_i}^{b_i} e^{2d_i\alpha_i(H)}dH\Big)^{\frac{1}{2}} = cV(S)^{\frac{1}{2}}.\end{aligned}$$

Therefore, the desired result follows. □

Our main theorem can be stated as follows:

Theorem 3.4. *Let G be a connected complex semisimple Lie groups. Let M_m be a Fourier multiplier of type $(\sqrt{\Delta}, \infty)$. Then (2) holds for all $f, g \in L^{2,1}(\Delta)$.*

Proof. Similarly as in the proof of [6], in order to show (2), it suffices from the duality of Lorentz spaces to prove that

$$\Big|\int f * M_m g(\exp H) h(\exp H)\Delta(\exp H) dH\Big| \\ \leq c\|f\|_{L^{2,1}(\Delta)}\|g\|_{L^{2,1}(\Delta)}\|h\|_{L^{2,1}(\Delta)} \qquad (10)$$

for all $h \in L^{2,1}(\Delta)$. Let R be a compact set of $\mathfrak{a}$ containing the origin inside. We note that the integral of the left hand is written as $M_m(f * g * h)(e)$. If one of f, g, h were supported on $\exp R$, say f, it follows that

$$\begin{aligned}|M_m(f * g * h)(e)| &\leq \|f * g\|_{L^2(\Delta)}\|M_m h\|_{L^2(\Delta)} \leq \|f\|_{L^1(\Delta)}\|g\|_{L^2(\Delta)}\|h\|_{L^2(\Delta)} \\ &\leq \Big(\int_R \Delta(\exp H) dH\Big)^{\frac{1}{2}} \|f\|_{L^2(\Delta)}\|g\|_{L^2(\Delta)}\|h\|_{L^2(\Delta)}.\end{aligned}$$

Since $L^{2,1}(\Delta) \subset L^2(\Delta)$, the desired estimate follows. Therefore, we may suppose that f, g, h are all supported on the outside $(\exp R)^c$ of $\exp R$. It follows from (7), (8) and Lmma 3.2 that

$$\begin{aligned}&\Big|\int f * M_m g(\exp H) h(\exp H)\Delta(\exp H) dH\Big| \\ \leq& c\|\pi(-\partial_H)\mathcal{A}f \otimes M_m\mathcal{A}g(\exp H)\|_{L^\infty(\mathfrak{a})}\|h\|_{L^1(\sqrt{\Delta})} \\ \leq& c\|\pi(-\partial_H)\mathcal{A}f \circ \exp\|_{L^1(\mathfrak{a})}\|M_m\mathcal{A}g \circ \exp\|_{L^\infty(\mathfrak{a})}\|h\|_{L^1(\sqrt{\Delta})} \\ \leq& c\|f\|_{L^1(\sqrt{\Delta})}\|g\|_{L^1(\sqrt{\Delta})}\|h\|_{L^1(\sqrt{\Delta})}.\end{aligned}$$

If a function a is supported on $(\exp R)^c$, then it follows Lemma 3.3 that

$$\|a\|_{L^1(\sqrt{\Delta})} \leq c\|a\|_{L^1(E)} \leq c\|a\|_{L^{2,1}(E^2)} \leq c\|a\|_{L^{2,1}(\Delta)}$$

Therefore, the desired (10) follows. □

Now we shall give a criterion by which M_m is of type $(\sqrt{\Delta}, \infty)$. We recall that the spherical function $\phi_\lambda(\exp H)$ is, as a function of λ, the Fourier transform of a compactly supported L^1 function $A_H(S) = A(S, H)$ on $\mathfrak{a}$:

$$D(\exp H)\phi_\lambda(\exp H) = \int_{\mathfrak{a}} A(S, H) e^{i\lambda(S)} dS$$

(see [1]). Hence it follows from (4) that for all K-bi-invariant $f \in L^1(G)$, $\mathcal{A}f(\exp S)$ is given by

$$\mathcal{A}f(\exp S) = \int_{\mathfrak{a}} f(\exp H)D(\exp H)A(S,H)dH.$$

We apply M_m to the first variable of $A(S,H)$ and denote it by $M_{m,1}A(S,H)$. If $M_{m,1}A(S,H)$ belongs to $L^1(\mathfrak{a})$ as a function of S, then

$$m(\lambda)D(\exp H)\phi_\lambda(\exp H) = \int_{\mathfrak{a}} M_{m,1}A(S,H)e^{i\lambda(S)}dS$$

and thus

$$M_m(\mathcal{A}f \circ \exp)(S) = \int_{\mathfrak{a}} f(\exp H)D(\exp H)M_{m,1}A(S,H)dH.$$

Therefore, (8) follows if there exists $c > 0$ such that $\|M_{m,1}A(\cdot,H)\|_{L^\infty(\mathfrak{a})} < c$ for all $H \in \mathfrak{a}$. We see the following.

Corollary 3.5. *Let us suppose that*

$$m(\lambda)D(\exp H)\phi_\lambda(\exp H) = \int_{\mathfrak{a}} B(S,H)e^{i\lambda(S)}dS, \tag{11}$$

where $B(S,H) \in L^1(\mathfrak{a})$ as a function of S and $B \in L^\infty(\mathfrak{a}^2)$. Then M_m is of type $(\sqrt{\Delta},\infty)$ and thus, (2) holds for $f,g \in L^{2,1}(\Delta)$.

4 Examples

(I) The rank one cases: When $\dim A = 1$, it is easy to see that for $H \in \mathfrak{a}$,

$$D(\exp H)\phi_\lambda(\exp H) = c\frac{\sin\lambda(H)}{\lambda} = c\int_{-H}^{H} e^{i\lambda(S)}dS.$$

Hence, for $m(\lambda) = 1$, $B(S,H)$ is the characteristic function of $[-H,H]$. Therefore the identity operator is of type $(\sqrt{\Delta},\infty)$. The endpoint estimate of the Kunze-Stein phenomenon (1) holds without modification.

(II) The case of $Sp(4,\mathbf{C})$: We shall obtain a multiplier M_m of type $(\sqrt{\Delta},\infty)$. $\mathfrak{a}$ is given by $\mathfrak{a} = \{H(a_1,a_2) = \text{diag}\ (a_1,a_2,-a_1,-a_2) \mid a_1,a_2 \in \mathbf{C}\}$, where diag is a diagonal matrix. We define $e_i \in \mathfrak{a}^*$, $i = 1,2$, by $e_i(H(a_1,a_2)) = a_i$. Then $\Sigma_+ = \{2e_1, 2e_2, e_1+e_2, e_1-e_2\}$. Let $\alpha = e_1 - e_2$ and $\beta = 2e_2$. We denote by s_γ the reflection on $\mathfrak{a}^*$ with respect to $\gamma \in \Sigma_+$. Then the Weyl group W is given by $W = \{I, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta, (s_\alpha s_\beta)^2\}$. We parametrize $\lambda \in \mathfrak{a}^*$ as $\lambda = \lambda_1(2\alpha+\beta) + \lambda_2\beta = 2\lambda_1 e_1 + 2\lambda_2 e_2$. Then the action of $w \in W$ on λ is given as follows.

$w \in W$	we_1	we_2	$\det w$	$\frac{1}{2}w\lambda(H(a_1,a_2))$
I	e_1	e_2	1	$\lambda_1 a_1 + \lambda_2 a_2$
s_α	e_2	e_1	-1	$\lambda_1 a_2 + \lambda_2 a_1$
s_β	e_1	$-e_2$	-1	$\lambda_1 a_1 - \lambda_2 a_2$
$s_\alpha s_\beta$	e_2	$-e_1$	1	$\lambda_1 a_2 - \lambda_2 a_1$
$s_\beta s_\alpha$	$-e_2$	e_1	1	$-\lambda_1 a_2 + \lambda_2 a_1$
$s_\alpha s_\beta s_\alpha$	$-e_1$	e_2	-1	$-\lambda_1 a_1 + \lambda_2 a_2$
$s_\beta s_\alpha s_\beta$	$-e_2$	$-e_1$	-1	$-\lambda_1 a_2 - \lambda_2 a_1$
$(s_\alpha s_\beta)^2$	$-e_1$	$-e_2$	1	$-\lambda_1 a_1 - \lambda_2 a_2$

We denote the partial sum of $\sum_{w\in W} \det w \cdot e^{iw\lambda(H)}$ by

$$I(w_1, w_2, \ldots, w_l) = \sum_{w=w_1,w_2,\ldots,w_l} \det w \cdot e^{iw\lambda(H)}.$$

Then it follows that

$$\begin{aligned} I(I, s_\alpha) &= 2ie^{i(\lambda_1+\lambda_2)(a_1+a_2)} \sin(\lambda_1-\lambda_2)(a_1-a_2) \\ I(s_\beta, s_\alpha s_\beta) &= -2ie^{i(\lambda_1-\lambda_2)(a_1+a_2)} \sin(\lambda_1+\lambda_2)(a_1-a_2) \\ I(s_\beta s_\alpha, s_\alpha s_\beta s_\alpha) &= 2ie^{-i(\lambda_1-\lambda_2)(a_1+a_2)} \sin(\lambda_1+\lambda_2)(a_1-a_2) \\ I(s_\beta s_\alpha s_\beta, (s_\alpha s_\beta)^2) &= -2ie^{-i(\lambda_1+\lambda_2)(a_1+a_2)} \sin(\lambda_1-\lambda_2)(a_1-a_2). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{w\in W} \det w \cdot e^{iw\lambda(H)} = 4(&-\sin(\lambda_1+\lambda_2)(a_1+a_2)\sin(\lambda_1-\lambda_2)(a_1-a_2) \\ &+\sin(\lambda_1-\lambda_2)(a_1+a_2)\sin(\lambda_1+\lambda_2)(a_1-a_2)). \end{aligned}$$

Since $\lambda = 2\lambda_1 e_1 + 2\lambda_2 e_2$, $\pi(i\lambda) = 64\lambda_1\lambda_2(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)$ and thus, it follows from (3) that

$$\begin{aligned} D(\exp H)\phi_\lambda(\exp H) = &\frac{1}{16}\frac{1}{\lambda_1\lambda_2} \\ &\times\Big(-\frac{\sin(\lambda_1+\lambda_2)(a_1+a_2)}{\lambda_1+\lambda_2}\frac{\sin(\lambda_1-\lambda_2)(a_1-a_2)}{\lambda_1-\lambda_2} \\ &\quad+\frac{\sin(\lambda_1-\lambda_2)(a_1+a_2)}{\lambda_1-\lambda_2}\frac{\sin(\lambda_1+\lambda_2)(a_1-a_2)}{\lambda_1+\lambda_2}\Big). \end{aligned}$$

Now let

$$m(\lambda) = \sin^2\lambda_1 \sin^2\lambda_2.$$

Clearly, m is W-invariant and

$$m(\lambda)D(\exp H)\phi_\lambda(\exp H) = c\frac{\sin^2\lambda_1}{\lambda_1}\frac{\sin^2\lambda_2}{\lambda_2} \times\Big(-\frac{\sin(\lambda_1+\lambda_2)(a_1+a_2)}{\lambda_1+\lambda_2}\frac{\sin(\lambda_1-\lambda_2)(a_1-a_2)}{\lambda_1-\lambda_2} +\frac{\sin(\lambda_1-\lambda_2)(a_1+a_2)}{\lambda_1-\lambda_2}\frac{\sin(\lambda_1+\lambda_2)(a_1-a_2)}{\lambda_1+\lambda_2}\Big).$$

We see that

$$\frac{\sin^2\lambda_1}{\lambda_1}\frac{\sin^2\lambda_2}{\lambda_2}$$

is the Fourier transform of $u(x,y) = -\frac{1}{4}\mathrm{sgn}\, x\cdot \mathrm{sgn}\, y$ times the characteristic function of $\{(x,y) \mid |x|\le 1, |y|\le 1\}$ and

$$\frac{\sin(\lambda_1+\lambda_2)(a_1+a_2)}{\lambda_1+\lambda_2}\frac{\sin(\lambda_1-\lambda_2)(a_1-a_2)}{\lambda_1-\lambda_2}$$

is the Fourier transform of $v(x,y) = \frac{1}{2}$ times the characteristic function of a compact set $\{(x,y) \mid |x+y|\le |a_1+a_2|, |x-y|\le |a_1-a_2|\}$. Hence $B(S,H)$ in (11) is a constant multiple of

$$u\otimes v(S) + u\otimes s_\beta v(S).$$

We note that $\|u\|_{L^1(\mathfrak{a}^2)} = 1$ and $\|v\|_{L^\infty(\mathfrak{a}^2)} = \frac{1}{2}$ and thus, $\|u\otimes v\|_{L^\infty(\mathfrak{a}^2)} \le \|u\|_{L^1(\mathfrak{a}^2)}\|v\|_{L^\infty(\mathfrak{a}^2)} \le \frac{1}{2}$. Similarly, $\|u\otimes s_\beta v\|_{L^\infty(\mathfrak{a}^2)} \le \frac{1}{2}$. Therefore, we see that $B(S,H)$ satisfies the condition of Corollary 3.5 and thus, M_m is a Fourier multiplier of type $(\sqrt{\Delta},\infty)$.

(III) The case of $SL(3,\mathbf{C})$: We shall obtain a multiplier M_m of type $(\sqrt{\Delta},\infty)$. $\mathfrak{a}$ is given by $\mathfrak{a} = \{H(a_1,a_2) = \mathrm{diag}\ (a_1,a_2,-(a_1+a_2)) \mid a_1,a_2\in\mathbf{C}\}$. We define $e_i\in\mathfrak{a}^*$, $i=1,2$, by $e_i(H(a_1,a_2)) = a_i$ and $e_3(H(a_1,a_2)) = -(a_1+a_2)$. Then $\Sigma_+ = \{e_1-e_2, e_2-e_3, e_1-e_3\}$. Let $\alpha = e_1-e_2$ and $\beta = e_2-e_3$. We denote by s_γ the reflection on $\mathfrak{a}^*$ with respect to $\gamma\in\Sigma_+$. Then the Weyl group W is given by $W = \{I, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha\}$. We parametrize $\lambda\in\mathfrak{a}^*$ as $\lambda = \lambda_1\frac{4}{3}(2\alpha+\beta)+\lambda_2\frac{4}{3}(-\alpha+\beta) = 4\lambda_1 e_1 + 4\lambda_2 e_2$. Then the action of $w\in W$ on λ is given as follows.

$w\in W$	we_1	we_2	$\det w$	$\frac{1}{4}w\lambda(H(a_1,a_2))$
I	e_1	e_2	1	$\lambda_1a_1+\lambda_2a_2$
s_α	e_2	e_1	-1	$\lambda_1a_2+\lambda_2a_1$
s_β	e_1	e_3	-1	$\lambda_1a_1-\lambda_2(a_1+a_2)$
$s_\alpha s_\beta$	e_2	e_3	1	$\lambda_1a_2-\lambda_2(a_1+a_2)$
$s_\beta s_\alpha$	e_3	e_1	1	$-\lambda_1(a_1+a_2)+\lambda_2a_1$
$s_\alpha s_\beta s_\alpha$	e_3	e_2	-1	$-\lambda_1(a_1+a_2)+\lambda_2a_2$

We see that

$$\begin{aligned} I(I, s_\alpha) &= 2ie^{2i(\lambda_1+\lambda_2)(a_1+a_2)} \sin 2(\lambda_1 - \lambda_2)(a_1 - a_2) \\ I(s_\beta, s_\alpha s_\beta) &= -2ie^{2i(\lambda_1-2\lambda_2)(a_1+a_2)} \sin 2\lambda_1(a_1 - a_2) \\ I(s_\beta s_\alpha, s_\alpha s_\beta s_\alpha) &= 2ie^{2i(-2\lambda_1+\lambda_2)(a_1+a_2)} \sin 2\lambda_2(a_1 - a_2). \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{w\in W} \det w \cdot e^{iw\lambda(H)} \\ &= 4(ie^{2i(\lambda_1+\lambda_2)(a_1+a_2)} \cos(\lambda_1 - \lambda_2)(a_1 - a_2) \sin(\lambda_1 - \lambda_2)(a_1 - a_2) \\ &\quad -e^{-i(\lambda_1+\lambda_2)(a_1+a_2)} \sin 2\lambda_1(a_1 - a_2) \sin 3(\lambda_1 - \lambda_2)(a_1 + a_2) \qquad (12) \\ &\quad -ie^{2i(-2\lambda_1+\lambda_2)(a_1+a_2)} \cos(\lambda_1 + \lambda_2)(a_1 - a_2) \sin(\lambda_1 - \lambda_2)(a_1 - a_2)) \\ &= 4(2i \sin(\lambda_1 - \lambda_2)(a_1 - a_2)e^{2i(\lambda_1+\lambda_2)(a_1+a_2)} \sin \lambda_1(a_1 - a_2) \sin \lambda_2(a_1 - a_2) \\ &\quad -e^{-i(\lambda_1+\lambda_2)(a_1+a_2)} \sin 2\lambda_1(a_1 - a_2) \sin 3(\lambda_1 - \lambda_2)(a_1 + a_2) \\ &\quad +2 \sin(\lambda_1 - \lambda_2)(a_1 - a_2)e^{-i(\lambda_1-2\lambda_2)(a_1+a_2)} \sin 3\lambda_1(a_1 + a_2) \\ &\qquad\qquad \times \cos(\lambda_1 + \lambda_2)(a_1 - a_2)). \end{aligned}$$

Here, to derive the first equality we used the double-angle formula to $I(I, s_\alpha)$ and added $\pm 2ie^{2i(-2\lambda_1+\lambda_2)(a_1+a_2)} \sin 2\lambda_1(a_1 - a_2)$ to $I(s_\beta, s_\alpha s_\beta)$ and $I(s_\beta s_\alpha, s_\alpha s_\beta s_\alpha)$ respectively, and for the second equality we added $\mp ie^{2i(\lambda_1+\lambda_2)(a_1+a_2)} \cos(\lambda_1 + \lambda_2)(a_1 - a_2) \sin(\lambda_1 - \lambda_2)(a_1 - a_2)$ to the first and the third terms in (12) respectively.

Now let

$$m(\lambda) = \sin^2 \lambda_1 \sin^2 \lambda_2 \sin^2(\lambda_1 - \lambda_2).$$

Clearly, m is W-invariant. Since $\lambda = 4\lambda_1 e_1 + 4\lambda_2 e_2$, $\pi(i\lambda) = -64i\lambda_1\lambda_2(\lambda_1 - \lambda_2)$ and thus, it follows from (3) that

$$\begin{aligned} &m(\lambda)D(\exp H)\phi_\lambda(\exp H) \\ &= \frac{i}{16} \sin^2 \lambda_1 \frac{\sin^2 \lambda_2}{\lambda_2} \sin^2(\lambda_1 - \lambda_2) \\ &\quad \times \Big(2i\frac{\sin(\lambda_1 - \lambda_2)(a_1 - a_2)}{\lambda_1 - \lambda_2} e^{2i(\lambda_1+\lambda_2)(a_1+a_2)} \frac{\sin \lambda_1(a_1 + a_2)}{\lambda_1} \sin \lambda_2(a_1 - a_2) \\ &\quad -e^{-i(\lambda_1+\lambda_2)(a_1+a_2)} \frac{\sin 2\lambda_1(a_1 - a_2)}{\lambda_1} \frac{\sin 3(\lambda_1 - \lambda_2)(a_1 + a_2)}{\lambda_1 - \lambda_2} \\ &\quad +2\frac{\sin(\lambda_1 - \lambda_2)(a_1 - a_2)}{\lambda_1 - \lambda_2} e^{-i(\lambda_1-2\lambda_2)(a_1+a_2)} \frac{\sin 3\lambda_1(a_1 + a_2)}{\lambda_1} \\ &\qquad\qquad \times \cos(\lambda_1 + \lambda_2)(a_1 - a_2)\Big). \end{aligned}$$

We see that

$$\frac{\sin^2 \lambda_2}{\lambda_2}$$

is the value at $4\lambda_2$ of the one dimensional Fourier transform of $-i\mathrm{sgn}\ y$ times the characteristic function of $\{y \mid |y| \le \frac{1}{2}\}$ and

$$\frac{\sin c(\lambda_1-\lambda_2)(a_1 \mp a_2)}{\lambda_1-\lambda_2}\frac{\sin d\lambda_1(a_1 \pm a_2)}{\lambda_1}$$

is the Fourier transform of 4 times the characteristic function of a compact set $\{(x,y) \mid |y| \le |\frac{c}{4}(a_1 \mp a_2)|, |x+y| \le |\frac{d}{4}(a_1 \pm a_2)|\}$. As Fourier multipliers, other terms correspond to translations of these characteristic functions. Hence we can easily deduce that $B(S,H)$ satisfies the condition of Corollary 3.5. Therefore, M_m is a Fourier multiplier of type $(\sqrt{\Delta}, \infty)$.

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