STRONGLY GRADED RINGS WHICH ARE MAXIMAL ORDERS

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Abstract.

Let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a strongly graded ring of type $\mathbb{Z}$. In [6], it is shown that if R_0 is a maximal order, then so is R. We define a concept of $\mathbb{Z}$-invariant maximal order and show R_0 is a $\mathbb{Z}$-invariant maximal order if and only if R is a maximal order. We provide examples of R_0 which are $\mathbb{Z}$-invariant maximal orders but not maximal orders.

1 Introduction Let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a strongly graded ring of type $\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. We always assume that R_0, the degree zero part, is a prime Goldie ring with its quotient ring Q_0 and $C_0 = \{c \in R_0 \mid c \text{ is regular in } R_0\}$, which is a regular Ore set of R and the ring of fractions Q^g of R at C_0 has the following properties:

(i). $Q^g = \oplus_{n\in\mathbb{Z}} Q_0 R_n (Q_0 R_n = R_n Q_0)$.

(ii). $Q^g = Q_0[X, X^{-1}, \sigma]$ for some automorphism σ of R_0 ([6, 1.3]) and so

(iii). Q^g is a left and right principal ideal ring.

We denote by the quotient ring of R by Q. We define a concept of $\mathbb{Z}$-invariant maximal order in order to get the following three conditions are equivalent: (i) R_0 is a $\mathbb{Z}$-invariant maximal order (ii) R is a maximal order (iii) R is a graded maximal order. We give examples of R_0 which are $\mathbb{Z}$-invariant maximal orders but not maximal orders. We refer the readers to [7] or [8] and [9] for some elementary properties and some definitions of order theory and graded ring theory which are not mentioned in the paper.

2 The proof of Theorem Since Q^g is the quotient ring of R at C_0, the following lemma follows from the proof of [2, Theorem 1.31].

Lemma 1 *Let A be an ideal of R. Then $AQ^g = Q^g A$.*

Lemma 2 *Let A_0 be an ideal of R_0. Then the right ideal $A_0 R$ is an ideal of R if and only if $R_n A_0 = A_0 R_n$ for all $n \in \mathbb{Z}$. In this case, $A_0 R$ is a graded ideal.*

Proof. If $A_0 R$ is an ideal of R, then $R_n A_0 R_{-n} \subseteq A_0$, that is, $R_n A_0 \subseteq A_0 R_n$ for all $n \in \mathbb{Z}$ and so $R_{-n} A_0 \subseteq A_0 R_{-n}$ also follows. Hence $R_n A_0 = A_0 R_n$. Conversely if $R_n A_0 = A_0 R_n$ for all $n \in \mathbb{Z}$, then it is easy to see that $A_0 R$ is an ideal of R. □

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Definition 1

(1). *A left and right R_0-submodule A_0 of Q_0 is called $\mathbb{Z}$-invariant if $R_nA_0 = A_0R_n$ for all $n \in \mathbb{Z}$.*

(2). *R_0 is called a $\mathbb{Z}$-invariant maximal order in Q_0 if $O_l(A_0) = R_0 = O_r(A_0)$ for any nonzero $\mathbb{Z}$-invariant ideal A_0 of R_0.*

(3). (10, p.205) *R is a graded maximal order in Q^g if for each graded over-ring S such that $R \subseteq S \subseteq Q^g$ and $aSb \subseteq R$ for some regular homogeneous elements $a, b \in Q^g$, it follows $R = S$.*

Lemma 3

(1). *Let A_0 and B_0 be $\mathbb{Z}$-invariant left and right R_0-submodules in Q_0. Then A_0B_0 is $\mathbb{Z}$-invariant.*

(2). *Let A_0 be a $\mathbb{Z}$-invariant left R_0-ideal which is a right R_0-submodule in Q_0 and B_0 be a $\mathbb{Z}$-invariant right R_0-ideal which is a left R_0-submodule in Q_0. Then $C_0 = \{r_0 \in R_0 \mid A_0r_0 \subseteq R_0\}$ and $D_0 = \{r_0 \in R_0 \mid r_0B_0 \subseteq R_0\}$) are both $\mathbb{Z}$-invariant.*

Proof.

(1). It is clear.

(2). $R_0 \supseteq A_0C_0$ implies $R_0 \supseteq R_nA_0C_0R_{-n} = A_0R_nC_0R_{-n}$ for all $n \in \mathbb{Z}$ and so $R_nC_0R_{-n} \subseteq C_0$ and also $R_{-n}C_0R_n \subseteq C_0$. Hence $C_0R_n = R_nC_0$ for all $n \in \mathbb{Z}$, that is, C_0 is $\mathbb{Z}$-invariant. Similarly D_0 is $\mathbb{Z}$-invariant.

□

Lemma 4 *The following conditions are equivalent.*

(1). *R_0 is a $\mathbb{Z}$-invariant maximal order.*

(2). *$O_l(A_0) = R_0$ for each $\mathbb{Z}$-invariant left R_0-ideal A_0 which is a right R_0-submodule in Q_0, and $O_r(B_0) = R_0$ for each $\mathbb{Z}$-invariant right R_0-ideal B_0 which is a left R_0-submodule in Q_0.*

Proof.
(2) $\Rightarrow$ (1) This is a special case.
(1) $\Rightarrow$ (2) Let A_0 be a $\mathbb{Z}$-invariant left R_0-ideal which is a right R_0-submodule in Q_0 and let $C_0 = \{ r_0 \in R_0 \mid A_0r_0 \subseteq R_0 \}$. Then A_0C_0 is a $\mathbb{Z}$-invariant ideal of R_0 by Lemma 3. Thus $R_0 = O_l(A_0C_0) \supseteq O_l(A_0) \supseteq R_0$ and so $O_l(A_0) = R_0$ follows. Similarly if B_0 is a $\mathbb{Z}$-invariant right R_0-ideal which is a left R_0-submodule in Q_0, then $O_r(B_0) = R_0$.

□

Theorem 1 *Let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a strongly graded ring of type $\mathbb{Z}$. Then the following conditions are equivalent:*

(1). *R_0 is a $\mathbb{Z}$-invariant maximal order in Q_0.*

(2). *R is a maximal order in Q.*

(3). *R is a graded maximal order in Q^g.*

Proof.
(1) ⇒ (2) Let S be an over-ring of R such that $aSb \subseteq R$ for some regular $a, b \in Q$. We may assume that $a, b \in R$. Put $T = R + RaS$, an over-ring of R with $Tb \subseteq R$. We claim $T = R$. Since TbR is an ideal of R, it follows from Lemma 1 that $TbQ^g = TbRQ^g = uQ^g = Q^g u$ for some regular element $u \in Q^g$ since Q^g is a principal ideal ring. For any $t \in T$, $tu \in TbQ^g = Q^g u$ and so $t \in Q^g$. Thus $T \subseteq Q^g$ follows. For any $n \in \mathbb{Z}$, let $C_n(T) = \{a_n \in Q_0R_n \mid \exists t = a_n + a_{n_1} + \cdots + a_{n_l} \in T \text{ such that } n > n_i (1 \leq i \leq l)\} \cup \{0\}$, which is a left and right R_0-submodule of Q^g. It is easy to see that $C_n(T) = R_nC_0(T) = C_0(T)R_n$. So, in particular, $C_0(T)$ is a $\mathbb{Z}$-invariant over-ring of R_0. To prove that $C_0(T)$ is a left R_0-ideal, write $b = b_k +$ (the lower degree parts). Since $Tb \subseteq R$, it follows that $R_0 \supseteq C_{-k}(T)b_k = C_0(T)R_{-k}b_k$ and so $R_0 \supseteq C_0(T)R_{-k}b_kR_0$. Hence $C_0(T)$ is a left R_0-ideal since $R_{-k}b_kR_0$ is a non-zero ideal of R_0 and is a right R_0-submodule. Thus, by Lemma 4, $O_l(C_0(T)) = R_0$ and $R_0 \subseteq C_0(T) \subseteq O_l(C_0(T)) = R_0$ since $C_0(T)$ is an over-ring of R_0, which implies $R_0 = C_0(T)$ and $R_n = R_nC_0(T) = C_n(T)$ for all $n \in \mathbb{Z}$. Hence $T = R$ follows. Since $aS \subseteq RaS \subseteq T = R$, the left version of the above proof shows that $S = R$. Hence R is a maximal order in Q.
(2) ⇒ (3) This is a special case.
(3) ⇒ (1) Let A_0 be a $\mathbb{Z}$-invariant ideal of R_0. By Lemma 2, A_0R is a graded ideal of R. Thus it follows from [6, Lemma 1.5] that $RO_l(A_0) = O_l(A_0R) = R = O_r(RA_0) = O_r(A_0)R$ and so $O_l(A_0) = R_0 = O_r(A_0)$. Hence R_0 is a $\mathbb{Z}$-invariant maximal order in Q_0. □

Finally, we give some examples of maximal orders R such that R_0 are $\mathbb{Z}$-invariant maximal orders but not maximal orders.

Let R_0 be a hereditary Noetherian prime ring (an HNP ring for short) with its quotient ring Q_0 satisfying the following conditions:

(a). There is a cycle $M_{01}, \ldots, M_{0n}$ $(n \geq 2)$ so that $X = M_{01} \cap \cdots \cap M_{0n}$ is a maximal invertible ideal of R_0.

(b). Any maximal ideal different from $M_{0i} (1 \leq i \leq n)$ is invertible.

See [1] and [5] for examples of HNP rings satisfying (a) and (b) (the simplest example is $\begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, where p is a prime number). Let

$$R = \oplus_{n \in \mathbb{Z}} X^n (R_n = X^n)$$

a strongly graded ring of type $\mathbb{Z}$, and A_0 be an eventually idempotent ideal of R_0. Then there are $M_{0i_1}, \ldots, M_{0i_r}$ $i_j \in \{1, \ldots, n\}$($r < n$) which are the full set of maximal ideals containing A_0. Thus A_0 is not a $\mathbb{Z}$-invariant ideal by [4, Theorem 14].

Hence R_0 is a $\mathbb{Z}$-invariant maximal order since an ideal of R_0 is $\mathbb{Z}$-invariant if and only if it is invertible by [3, Theorem 2.9 and 4.2].

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