ON THE BANG-BANG PRINCIPLE FOR DIFFERENTIAL INCLUSIONS IN A REFLEXIVE SEPARABLE BNANACH SPACE

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Abstract. In this paper, we consider the relation existing between the solutions of the following differential inclusions: (I) $\dot{x} \in \Gamma(t, x), x(0) = 0$ and (II) $\dot{x} \in \operatorname{ext} \Gamma(t, x), x(0) = 0$ defined on a reflexive separable Banach space. In particular, we establish the sufficient conditions which guarantee the set of solutions of (II) is dense in the set of solutions of (I) with respect to the (weak) uniformly continuous topology.

Let $(\mathfrak{X}, \|\cdot\|)$ be a real reflexive separable Banach space and T be a positive real number. Let $\Gamma : [0, T] \times \mathfrak{X} \twoheadrightarrow \mathfrak{X}$ be a correspondence (=multi-valued function). We consider a relation existing between the sets of solutions of the following differential inclusions:

(I) $\dot{x} \in \Gamma(t, x), x(0) = 0$, and
(II) $\dot{x} \in \operatorname{ext} \Gamma(t, x), x(0) = 0$,

where $\operatorname{ext} A$ stands for the weak-closure of the extreme points of A. By a solution of (I) and (II), we mean an absolutely continuous function $x : [0, T] \to \mathfrak{X}$ that satisfies $\dot{x} \in \Gamma(t, x(t))$ a.e. in $t \in [0, T]$ and $x(0) = 0$ in the case of (I) and $\dot{x} \in \operatorname{ext} \Gamma(t, x(t))$ a.e. in $t \in [0, T]$ and $x(0) = 0$ in the case of (II). We denote by $\mathcal{R}$ and $\mathcal{R}_*$ the set solutions of (I) and (II) respectively. Tateishi [5, 6] established the existence of solutions of the differential inclusions (I) under the following assumptions:

(i) Γ is nonempty and weakly compact-valued, i.e., $\Gamma(t, x)$ is nonempty and weakly compact for each $(t, x) \in [0, T] \times \mathfrak{X}$,
(ii) for each fixed $t \in [0, T]$, the correspondence $t \twoheadrightarrow \Gamma(t, x)$ is continuous with respect to the weak topology for $\mathfrak{X}$,
(iii) for each fixed $x \in \mathfrak{X}$, the correspondence $t \twoheadrightarrow \Gamma(t, x)$ is measurable, and
(iv) there exists $M > 0$ such that $\sup\{\|y\| \mid y \in \Gamma(t, x), t \in [0, T], x \in \mathfrak{X}\} \le M$.

Furthermore, Tateishi [6, 7] examined the relations existing between the solutions set of (I) and (III): $\dot{x} \in \overline{\operatorname{co}} \, \Gamma(t, x), x(0) = 0$. The aim of this paper is to establish the relation between the sets of solutions (I) and (II). Bressan [1, 2] established the existence of solutions of both of the problems and obtained the closure result $\mathcal{R} = \overline{\mathcal{R}_*}$ in the case that $\mathfrak{X}$ is a finite dimensional space. In this paper, we generalize his theorem to infinite dimensional spaces.

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1. Preliminaries

In this section, we offer some notations and lemmata used in this paper.

Let $(\mathfrak{X}, \|\cdot\|)$ be a reflexive separable Banach space with $\mathfrak{X}^*$ its dual. We denote by $\mathfrak{X}^w$ the space $\mathfrak{X}$ endowed with the weak topology. Let $\mathfrak{S} = \{x \in \mathfrak{X} \mid \|x\| \leq \max(MT, M)\}$ where M and T are constants which appear in the Introduction section. The set $\mathfrak{S}$ endowed with the relative topology of $\mathfrak{X}^w$ is denoted by $\mathfrak{S}^w$. The following proposition is from Larman and Rogers [4, Theorem 2].

Proposition 1. *Let E be a Hausdorff locally convex topological vector space. Let X be a metrizable compact subset of E. Let V be the linear subspace generated by the set $\overline{\mathrm{co}}\, X$. Then it is possible to introduce a norm on V so that the relative topologies of $\overline{\mathrm{co}}\, X$, as a subset of E, and as a subset of the normed space V, coincide.*

$\mathfrak{X}^w$ is a Hausdorff locally convex topological vector space and $\mathfrak{S}^w$ is a metrizable and compact subset of $\mathfrak{X}^w$. Furthermore, the linear subspace generated by $\mathfrak{S}^w$ is the whole space $\mathfrak{X}$. Hence, we can, by the above proposition, introduce a norm $\|\cdot\|^w$ on $\mathfrak{X}^w$ so that the topology on $\mathfrak{S}^w$ and the relative topology as a subset of the normed vector space $(\mathfrak{X}, \|\cdot\|^w)$ coincide.

We denote by h the Hausdorff distance on $\mathfrak{S}^w$ induced by $\|\cdot\|^w$, that is, $h(A, A') = \max\{\sup_{x\in A'} d(x, A), \sup_{x\in A} d(x, A')\}$ for any closed subsets A, A' of $\mathfrak{S}^w$, where $d(x, A) = \inf\{\|x - y\|^w \mid y \in A\}$. For $A \subset \mathfrak{S}$ and $\alpha > 0$, we set $B[A, \alpha] = \{x \in \mathfrak{S} \mid d(x, A) < \alpha\}$. We denote by μ, the Lebesgue measure defined on the interval $[0, T]$.

Lemma 1. *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space and $\Gamma : \mathfrak{X} \twoheadrightarrow \mathfrak{X}$ be convex, compact-valued and continuous. Then the map $\mathrm{ext}\,\Gamma : \mathfrak{X} \twoheadrightarrow \mathfrak{X}$ is lower hemi-continuous.*

Proof. Let $x_0, y_0 \in \mathfrak{X}$ with $y_0 \in \mathrm{ext}\,\Gamma(x_0)$ and $\{x_n\}$ be a sequence which converges to x_0. We must show that for some subsequence $x_{n'}$ of x_n and some $y_{n'} \in \mathrm{ext}\,\Gamma(x_{n'})$, we have $y_{n'} \to y_0$. Since Γ is continuous, there exists a sequence $y_n \in \Gamma(x_n)$ such that $y_n \to y_0$. Since Γ is compact and convex-valued, the Krein-Milman theorem implies that $\Gamma(x_n) = \overline{\mathrm{co}}\,\mathrm{ext}\,\Gamma(x_n)$. Hence, for each $n \in \mathbb{N}$, there exists $\alpha_n^i \geq 0, \sum_i \alpha_n^i = 1\ (i \in \mathbb{N})$, where only finitely many α_n^i are not equal to zero, and $z_n^i \in \mathrm{ext}\,\Gamma(x_n)$ such that $\|y_n - \sum_i \alpha_n^i z_n^i\| \leq 1/n$. Let $y_n^i \in \Gamma(x_0)$ be such that $\|z_n^i - y_n^i\| \leq h(\Gamma(x_n), \Gamma(x_0))$, where h is the Hausdorff metric defined by $\|\cdot\|$. Since $\Gamma(x_0)$ is compact, there exist, for each fixed i, converging subsequences $y_{n'}^i$ to y_0^i and $\alpha_{n'}^i$ to α_0^i. Then $\sum_i \alpha_0^i y_0^i = y_0$ and since y_0 is an extreme point of $\Gamma(x_0)$, we have each y_0^i is equal to y_0 for all i with $\alpha_0^i > 0$. Let i^* be such that $\alpha_0^{i^*} > 0$. Then $\limsup_{n'} \|z_{n'}^{i^*} - y_0\| \leq \limsup_{n'} \|z_{n'}^{i^*} - y_{n'}^{i^*}\| + \limsup_{n'} \|y_{n'}^{i^*} - y_0\| = 0$. Hence $y_{n'}$ also converges to y_0 and this completes the proof. □

Lemma 2. *Let $F : [0, T] \times \mathfrak{X}^w \twoheadrightarrow \mathfrak{X}^w$ be lower hemi-continuous and $V \subset \mathfrak{X}^w$ be open. Then the correspondence $H : [0, T] \times \mathfrak{X}^w \twoheadrightarrow \mathfrak{X}^w$ defined by $H(t, x) = \overline{F(t, x) \cap V}$ is lower hemi-continuous, where $\overline{A}$ stands for the closure with respect to the weak topology of $\mathfrak{X}$.*

Proof. Let K be a weakly closed subset of $\mathfrak{X}$. Then we have the following implications:

$$H(t, x) \subset K \Leftrightarrow F(t, x) \cap V \subset K \Leftrightarrow F(t, x) \subset K \cup V^c.$$

Since $K \cup V^c$ is weakly closed and F is lower hemi-continuous, the set $\{(t,x) \mid H(t,x) \subset K\} = \{(t,x) \mid F(t,x) \subset K \cup V^c\}$ is closed in $[0,T] \times \mathfrak{X}^w$. It follows that H is lower hemi-continuous. □

Lemma 3. *Let $I_0 \subset [0,T]$ be a measurable set with $\mu(I_0) = \sigma$ and let M and ϵ be given positive real numbers. Then the solution $\psi : [0,T] \to \mathbb{R}$ of the differential equation*

$$\dot{\psi}(t) = \psi(t) + 2M\chi_{I_0}(t) + 4\epsilon, \psi(0) = 0 \tag{1}$$

is positive, monotonically increasing and satisfies the following inequality:

$$\psi(T) \leq 2M\sigma e^T + 4\epsilon(e^T - 1).$$

Proof. It is easy to verify that the solution ψ is positive and monotonically increasing. By calculating the solution ψ of (1) directly, we obtain

$$\psi(T) = 2M \cdot \int_0^T \chi_{I_0}(s) e^{(T-s)} ds + 4\epsilon \int_0^T e^{(T-s)} ds \leq 2M\sigma e^T + 4\epsilon(e^T - 1).$$

□

2. Main Theorem

Theorem 1. *Let $\Gamma : [0,T] \times \mathfrak{X} \twoheadrightarrow \mathfrak{X}$ be a correspondence which satisfies the conditions:*

- (i) *Γ is convex and weakly compact-valued, that is $\Gamma(t,x)$ is convex and weakly compact for each $(t,x) \in [0,T] \times \mathfrak{X}$,*
- (ii) *Γ is continuous, where $\mathfrak{X}$ is endowed with the weak topology.*
- (iii) *$h(\Gamma(t,x), \Gamma(t,y)) \leq \|x - y\|^w$, and*
- (iv) *there exists $M > 0$ such that $\sup\{\|y\| \mid y \in \Gamma(t,x), t \in [0,T], x \in \mathfrak{X}\} \leq M$.*

Then $\mathcal{R} = \overline{\mathcal{R}_*}$, that is the set of solutions of (II) is dense in the set of solutions of (I) with respect to the (weak) uniform convergence topology.

Proof. **Step 1.** Let v be a solution of (I) and let ϵ be a positive real number. Then there exists, an open subset I_0 of $[0,T]$ with $\mu(I_0) < \epsilon$ such that, for all $t \in I_1 = [0,T] \cap I_0^c$, $\dot{v}(t)$ exists and lies in $\Gamma(t, v(t))$, and the restriction $\dot{v}\mid_{I_1}$ of $\dot{v}$ to I_1 is continuous. We may also assume that $[0, \tau_0] \subset I_0$ for some $\tau_0 > 0$.

Step 2. Let $\mathfrak{M}$ be the set of $\{u, \tau\}$ of an absolutely continuous mapping u and a positive constant $0 \leq \tau \leq T$ such that u is defined on the closed interval $[0,\tau]$ and satisfies $\dot{u} \in \operatorname{ext}\Gamma(t, u(t))$ a.e. in $t \in [0,\tau], u(0) = 0$,

$$\|u(\tau) - v(\tau)\|^w \leq \psi(\tau), \text{ and} \tag{2}$$

$$\|u(t) - v(t)\|^w \leq \psi(t) + 2M\epsilon \text{ for all } t \in [0,\tau], \tag{3}$$

where ψ is a solution of (1).

Step 3. Since $[0,\tau_0] \subset I_0$, the pair $\{u, \tau_0\}$ for every solution u of (II) satisfies the above properties, thus the set $\mathfrak{M}$ is nonempty. Let us define a partial ordering $\precsim_{\mathfrak{M}}$ on $\mathfrak{M}$ by $(u_1, \tau_1) \precsim_{\mathfrak{M}} (u_2, \tau_2) \Leftrightarrow \tau_1 \leq \tau_2$ and u_2 is an extension of u_1. Then Zorn's lemma implies that there exists a maximal element (u^*, τ^*) of $\mathfrak{M}$.

Step 4. Since $\epsilon > 0$ is arbitrary, the equations (2) and (3) imply that the solution u^* of (II) can be arbitrarily near to the solution v with respect to the (weak) uniform convergence topology on $[0, \tau^*]$. In the following two steps, we show that τ^* obtained in Step 3 equals T. In this step, we consider the case $\tau^* \in I_0$. Then, since I_0 is open, there exists a positive number δ such that $[\tau^*, \tau^* + \delta] \subset I_0$. Then, we have an absolutely continuous function $w : [\tau^*, \tau^* + \delta] \to X$ satisfying $\dot{w}(t) \in \operatorname{ext}\Gamma(t, w(t))$ for $t \in [\tau^*, \tau^* + \delta]$, $w(\tau^*) = u^*(\tau^*)$. Let us define $w^* : [0, \tau^* + \delta] \to \mathfrak{X}$ by

$$w^*(t) = \begin{cases} u^*(t) & \text{for } t \in [0, \tau^*] \\ w(t) & \text{for } t \in [\tau^*, \tau^* + \delta]. \end{cases}$$

Then, for $t \in [\tau^*, \tau^* + \delta]$, we obtain the estimation:

$$\begin{aligned} &\|w^*(t) - v(t)\|^w \\ &\leq \|w^*(\tau^*) - v(\tau^*)\|^w + \int_{\tau^*}^t \|\dot{w}^*(s) - \dot{v}(s)\|^w ds \\ &\leq \|w^*(\tau^*) - v(\tau^*)\|^w + \int_{\tau^*}^t 2M\chi_{I_0}(s) ds \\ &\leq \psi(\tau^*) + \int_{\tau^*}^t \dot{\psi}(s) ds = \psi(t), \end{aligned}$$

where the first inequality is an immediate consequence of the fundamental theorem of calculus, and the second follows from assumption (iv) and the third from (2) and the definition of ψ. Thus w^* belongs to $\mathfrak{M}$, which contradicts the maximality of u^*.

Step 5. In this step, we consider the case $\tau^* \in I_1$. By assumption (iii) and (2), we have $y^* \in \Gamma(\tau^*, u^*(\tau^*))$ such that $\|y^* - \dot{v}(\tau^*)\|^w \leq \psi(\tau^*)$. Since $\dot{v}^* \mid_{I_1}$ is continuous by assumption and $\operatorname{ext}\Gamma : I_1 \times \mathfrak{X}^w \twoheadrightarrow \mathfrak{X}^w$ is lower hemi-continuous by Lemma 1, we have a positive constant $0 < \delta \leq \epsilon$ such that $\|\dot{v}^*(t) - \dot{v}^*(\tau)\|^w < \epsilon$ for $t \in I_1 \cap [\tau^*, \tau^* + \delta]$, and

(4) $\quad \operatorname{ext}\Gamma(\tau^*, u^*(\tau^*)) \subset B[\operatorname{ext}\Gamma(t, x), \epsilon]$ for $t \in [\tau^*, \tau^* + \delta]$ and $x \in B[u^*(\tau^*), M\delta]$.

By the Krein-Milman theorem, we have: $y^* \in \Gamma(\tau^*, u^*(\tau^*)) = \overline{\operatorname{co}}\operatorname{ext}\Gamma(\tau^*, u^*(\tau^*))$, where $\overline{\operatorname{co}}$ stands for the closed convex hull of A. Thus we obtain, for any $\epsilon > 0$, finite points $y_1, y_2, \ldots, y_m$ in $\operatorname{ext}\Gamma(\tau^*, u^*(\tau^*))$ and nonnegative real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ with $\sum_{i=1}^m \lambda_i = 1$ such that $\|y^* - \sum_i \lambda_i y_i\|^w < \epsilon$. Then we obtain, by Lyapunov's convexity theorem, a set of m measurable partition $J_0, J_1, \ldots, J_m$ such that $t < s$ for $t \in J_i, s \in J_j$ with $i < j, \cup_i J_i = I_1 \cap [\tau^*, \tau^* + \delta]$, and $\mu(J_i) = \lambda_i \mu(I_1 \cap [\tau^*, \tau^* + \delta])$. For $t \in [\tau^*, \tau^* + \delta]$, we set

$$H(t, x) = \begin{cases} \overline{\operatorname{ext}\Gamma(t, x)} & \text{if } t \in I_0 \\ \overline{\operatorname{ext}\Gamma(t, x) \cap B[y_i, \epsilon]} & \text{if } t \in J_i. \end{cases}$$

Then by (4), $y_i \in \operatorname{ext}\Gamma(\tau^*, u^*(\tau^*)) \subset B[\operatorname{ext}\Gamma(t, x), \epsilon]$ for $i = 1, 2, \ldots, m, t \in [\tau^*, \tau^* + \delta]$, and $x \in B[u^*(\tau^*), M\delta]$. It follows that $\operatorname{ext}\Gamma(t, x) \cap B[y_i, \epsilon] \neq \emptyset$ and hence, $H(t, x) \neq \emptyset$ for such pair (t, x). In view of Lemma 2, the restriction of H to each of the product spaces $I_0 \times B[u^*(\tau), M\delta]$ and $J_i \times B[u^*(\tau), M\delta]$ is lower hemi-continuous. Thus H is almost lower hemi-continuous and we have an absolutely continuous function $u_\delta : [\tau^*, \tau^* + \delta] \to$

$\mathfrak{X}$ satisfying the following conditions: $u_\delta(\tau^*) = u^*(\tau^*)$, and $\dot{u}_\delta(t) \in H(t, u_\delta(t))$ a.e. in $[\tau^*, \tau^* + \delta]$ (see, e.g., Deimling [3,Theorem 9.3]). Let us define $w^* : [0, \tau^* + \delta] \to \mathfrak{X}$ by

$$w^*(t) = \begin{cases} u^*(t) & \text{for } t \in [0, \tau^*] \\ u_\delta(t) & \text{for } t \in [\tau^*, \tau^* + \delta]. \end{cases}$$

Then, w^* can be seen to be an element of $\mathfrak{M}$ as follows. First, we verify that w^* satisfies the condition (2) for $\tau = \tau^* + \delta$. Setting $I_0^* = I_0 \cap [\tau^*, \tau^* + \delta], I_1^* = I_1 \cap [\tau^*, \tau^* + \delta]$, we have

$$\begin{aligned}
&\|w^*(\tau^* + \delta) - v(\tau^* + \delta)\|^w \\
&\leq \|w^*(\tau^*) - v(\tau^*)\|^w + \|\int_{\tau^*}^{\tau^*+\delta} [\dot{u}_\delta(t) - \dot{v}(t)]dt\|^w \\
&\leq \psi(\tau^*) + \int_{I_0^*} \|\dot{u}_\delta(t) - \dot{v}(t)\|^w dt + \|\sum_{i=1}^m \int_{J_i} \dot{u}_\delta(t)dt - \int_{I_1^*} \dot{v}(t)dt\|^w \\
&\leq \psi(\tau^*) + 2M\mu(I_0^*) + \sum_{i=1}^m \int_{J_i} \|\dot{u}_\delta(t) - y_i\|^w dt + \mu(I_1^*)\|\sum_{i=1}^m \lambda_i y_i - y^*\|^w \\
&+ \delta\|y^* - \dot{v}(\tau^*)\|^w + \epsilon\mu(I_1^*) \\
&\leq \psi(\tau^*) + 2M\mu(I_0^*) + 4\epsilon\mu(I_1^*) + \delta\|y^* - \dot{v}(\tau^*)\|^w \\
&\leq \psi(\tau^*) + \int_{\tau^*}^{\tau^*+\delta} (2M\chi_{I_0}(t) + 4\epsilon t)dt + \delta\psi(\tau^*) \leq \psi(\tau^* + \delta),
\end{aligned}$$

where the first inequality is a consequence of the fundamental theorem of calculus. In the second inequality, the interval $[\tau^*, \tau^* + \delta]$ splits into I_0^* and I_1^*. The third inequality uses the relation $\mu(J_i) = \lambda_i\mu(I_1^*)$.

Let us now turn to the condition (3): for each t with $\tau^* \leq t \leq \tau^* + \delta$, we have

$$\begin{aligned}
&\|w^*(t) - v(t)\|^w \\
&\leq \|w^*(\tau^*) - v(\tau^*)\|^w + 2M(t - \tau) \\
&\leq \psi(\tau^*) + 2M\epsilon \\
&\leq \psi(t) + 2M\epsilon.
\end{aligned}$$

Step 6. Since w^* belongs to $\mathfrak{M}$, u^* is not a maximal element of $\mathfrak{M}$, which is a contradiction. Thus we conclude that u^* is defined on the whole interval $[0, T]$.

Step 7. We have shown, in the above various steps, that, for each solution v of differential inclusion (II) and each $\epsilon > 0$, there exists a solution u^* of the differential inclusion (I) such that $\|u^*(t) - v(t)\|^w \leq \psi(t) + 2M\epsilon$ for all $t \in [0, T]$ and hence

$$\begin{aligned}
&\sup_{t\in[0,T]} \|u^*(t) - v(t)\|^w \\
&\leq \psi(t) + 2M\epsilon \\
&\leq \epsilon(2Me^T + 4(e^T - 1) + 2M).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this completes the proof of Theorem 1.

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