NORM INEQUALITIES RELATED TO THE MATRIX GEOMETRIC MEAN OF NEGATIVE POWER

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Abstract. In this paper, we show norm inequalities related to the matrix geometric mean of negative power for positive definite matrices: For positive definite matrices A and B,

$$\left\| e^{(1-\beta)\log A+\beta\log B} \right\| \le \|A \natural_\beta B\| \le \left\| A^{1-\beta}B^{\beta} \right\|$$

for every unitarily invariant norm and $-1 \le \beta \le -\frac{1}{2}$, where the β-quasi geometric mean $A \natural_\beta B$ is defined by $A \natural_\beta B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\beta}A^{\frac{1}{2}}$. For our purposes, we show the Ando-Hiai log-majorization of negative power.

1 Introduction. Let $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices and denote the matrix absolute value of any $A \in \mathbb{M}_n$ by $|A| = (A^*A)^{\frac{1}{2}}$. For $A \in \mathbb{M}_n$, we write $A \ge 0$ if A is positive semidefinite and $A > 0$ if A is positive definite, that is, A is positive and invertible. For two Hermitian matrices A and B, we write $A \ge B$ if $A - B \ge 0$, and it is called the Löwner ordering. A norm $\|\cdot\|$ on $\mathbb{M}_n$ is said to be unitarily invariant if $\|UXV\| = \|X\|$ for all $X \in \mathbb{M}_n$ and unitary U, V.

Let A and B be two positive definite matrices. The arithmetic-geometric mean inequality says that

$$A \sharp_\alpha B \le (1-\alpha)A + \alpha B \qquad \text{for all } \alpha \in [0,1], \tag{1.1}$$

where the α-geometric mean $A \sharp_\alpha B$ is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\alpha}A^{\frac{1}{2}} \qquad \text{for all } \alpha \in [0,1],$$

also see [11]. As another matrix geometric mean, we recall that the chaotic geometric mean $A \diamondsuit_\alpha B$ is defined by

$$A \diamondsuit_\alpha B = e^{(1-\alpha)\log A+\alpha\log B} \qquad \text{for all } \alpha \in \mathbb{R},$$

also see [5, Section 3.5]. If A and B commute, then $A \diamondsuit_\alpha B = A \sharp_\alpha B = A^{1-\alpha}B^{\alpha}$ for $\alpha \in [0,1]$. In [4], Bhatia and Grover showed precise norm estimations of the arithmetic-geometric mean inequality (1.1) as follows: For each $\alpha \in [0,1]$ and any unitarily invariant norm $\|\cdot\|$

$$\begin{aligned} \|A \sharp_\alpha B\| &\le \|A \diamondsuit_\alpha B\| \le \left\| B^{\frac{\alpha}{2}}A^{1-\alpha}B^{\frac{\alpha}{2}} \right\| \\ &\le \left\| \frac{1}{2}\left(A^{1-\alpha}B^{\alpha} + B^{\alpha}A^{1-\alpha}\right) \right\| \le \left\| A^{1-\alpha}B^{\alpha} \right\| \le \|(1-\alpha)A + \alpha B\| \end{aligned}$$

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and

$$\begin{aligned}|||A \sharp_\alpha B||| &\le |||A \diamondsuit_\alpha B||| \\ &\le \left|\left|\left|\left(B^{\frac{\alpha p}{2}}A^{(1-\alpha)p}B^{\frac{\alpha p}{2}}\right)^{\frac{1}{p}}\right|\right|\right| \le \left|\left|\left|((1-\alpha)A^p+\alpha B^p)^{\frac{1}{p}}\right|\right|\right| \quad \text{for all } p>0.\end{aligned}$$

For convenience in symbolic expression, we define $A \natural_\beta B$ for $\beta \in [-1,0)$ and positive definite matrices A, B as follows:

$$A \natural_\beta B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\beta A^{\frac{1}{2}} \quad \text{for all } \beta \in [-1,0), \tag{1.2}$$

whose formula is the same as $\sharp_\alpha$. Though $A \natural_\beta B$ for $\beta \in [-1,0)$ are not matrix means in the sense of Kubo-Ando theory [11], it is known in [7] that $A \natural_\beta B$ have matrix mean like properties for any positive definite matrices A and B. Thus we call (1.2) the β-quasi geometric mean for $\beta \in [-1,0)$. For more detail, see [7].

On the other hand, the following reverse arithmetic-geometric mean inequality holds:

$$(1-\beta)A+\beta B \le A \natural_\beta B \qquad \text{for all } \beta \in [-1,0),$$

also see [8]. Though we have no relation among $A \natural_\beta B$, $A \diamondsuit_\beta B$ and $A^{1-\beta}B^\beta$ for $\beta \in [-1,0)$ under the Löwner ordering, it follows from a proof similar to Bhatia-Grover's one in [4] that for each $\beta \in \mathbb{R}$ and any unitarily invariant norm $|||\cdot|||$

$$|||A \diamondsuit_\beta B||| \le \left|\left|\left|B^{\frac{\beta}{2}}A^{1-\beta}B^{\frac{\beta}{2}}\right|\right|\right| \le \left|\left|\left|\frac{1}{2}\left(A^{1-\beta}B^\beta + B^\beta A^{1-\beta}\right)\right|\right|\right| \le \left|\left|\left|A^{1-\beta}B^\beta\right|\right|\right|. \tag{1.3}$$

Also, by the Lie-Trotter formula $\lim_{t\to 0}\left(e^{\frac{t}{2}B}e^{tA}e^{\frac{t}{2}B}\right)^{\frac{1}{t}} = e^{A+B}$ and the Araki-Cordes inequality $|||B^tA^tB^t||| \le |||(BAB)^t|||$ for all $t \in [0,1]$, also see [3, Exercise IX.1.5,Theorem IX.2.10], it follows that for each $\beta \in [-1,0)$

$$|||A \diamondsuit_\beta B||| \le \left|\left|\left|\left(B^{\frac{\beta q}{2}}A^{(1-\beta)q}B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}\right|\right|\right| \tag{1.4}$$

holds for all $q>0$ and $\left|\left|\left|\left(B^{\frac{\beta q}{2}}A^{(1-\beta)q}B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}\right|\right|\right|$ decreases to $|||A \diamondsuit_\beta B|||$ as $q \downarrow 0$. It is natural to ask what is the estimate of the β-quasi geometric mean in the norm inequalities (1.3) and (1.4) for $\beta \in [-1,0)$.

In this paper, we show norm inequalities related to the β-quasi geometric mean of negative power, the chaotic geometric mean $A \diamondsuit_\beta B$ and $A^{1-\beta}B^\beta$ for positive definite matrices A, B. Moreover, we show precise norm estimations of the reverse arithmetic-geometric mean inequality under the assumption $A \ge B$. For our purposes, we need the Ando-Hiai log-majorization of negative power.

2 Preliminaries. In this section, we have some preliminary results on the log majorization of matrices. For Hermitian matrices H, K the weak majorization $H \prec_w K$ means that

$$\sum_{i=1}^k \lambda_i(H) \le \sum_{i=1}^k \lambda_i(K) \qquad \text{for } k=1,2,\ldots,n,$$

where $\lambda_1(H) \ge \cdots \ge \lambda_n(H)$ and $\lambda_1(K) \ge \cdots \ge \lambda_n(K)$ are the eigenvalues of H and K respectively. Further, the majorization $H \prec K$ means that $H \prec_w K$ and the equality holds for $k = n$ in the above, i.e., $\mathrm{Tr}H = \mathrm{Tr}K$. For $A, B \geq 0$ let us write $A \prec_{w(\log)} B$ and refer to the weak log majorization if

$$\prod_{i=1}^{k} \lambda_i(A) \leq \prod_{i=1}^{k} \lambda_i(B) \qquad \text{for } k = 1, 2, \ldots, n.$$

Further the log majorization $A \prec_{(\log)} B$ means that $A \prec_{w(\log)} B$ and the equality holds for $k = n$ in the above, i.e.,

$$\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B) \quad \text{i.e.,} \quad \det A = \det B.$$

Note that when $A, B > 0$ the log majorization $A \prec_{(\log)} B$ is equivalent to $\log A \prec \log B$. It is known that for positive semidefinite $A, B \geq 0$,

$$A \prec_{w(\log)} B \Longrightarrow A \prec_w B \Longrightarrow |||A||| \leq |||B|||$$

for any unitarily invariant norm. See [1, 12] for theory of majorization for matrices.

For each matrix X and $k = 1, 2, \ldots, n$, let $C_k(X)$ denote the k-fold antisymmetric tensor power of X. See [12] for details. Then (1)-(3) below are basic facts, (4) is easily seen from (2) and (3), and (5) follows from the Binet-Cauchy theorem.

Lemma 2.1. (1) $C_k(X^*) = C_k(X)^*$.

(2) $C_k(XY) = C_k(X)C_k(Y)$ *for every pair of matrices* X, Y.

(3) $C_k(X^{-1}) = C_k(X)^{-1}$ *for nonsingular* X.

(4) $C_k(A^p) = C_k(A)^p$ *for every positive definite* $A > 0$ *and all* $p \in \mathbb{R} \setminus \{0\}$.

(5) *For every* $A > 0$, $\prod_{i=1}^{k} \lambda_i(A) = \lambda_1(C_k(A))$ *for* $k = 1, 2, \ldots, n$ *and consequently, for* $A, B > 0$, $\lambda_1(C_k(A)) \leq \lambda_1(C_k(B))$ *for all* $k = 1, \ldots, n$ *if and only if* $A \prec_{w(\log)} B$.

3 Ando-Hiai Log-Majorization of negative power.

For $0 \leq \alpha \leq 1$, the matrix α-geometric mean is the matrix mean corresponding to the matrix monotone function t^α. Note that $A \sharp_\alpha B = B \sharp_{1-\alpha} A$ and if $AB = BA$ then $A \sharp_\alpha B = A^{1-\alpha}B^\alpha$, and $(A, B) \mapsto A \sharp_\alpha B$ is jointly monotone, also see [5, Lemma 3.2].

On the other hand, the β-quasi geometric mean for $\beta \in [-1, 0)$ has the following properties in [7]; for any positive definite matrices A, B and C

(i) **consistency with scalars:** If A and B commute, then $A \natural_\beta B = A^{1-\beta}B^\beta$.

(ii) **homogeneity:** $(\alpha A) \natural_\beta (\alpha B) = \alpha(A \natural_\beta B)$ for all $\alpha > 0$.

(iii) **right reverse monotonicity:** $B \leq C$ implies $A \natural_\beta B \geq A \natural_\beta C$.

We recall the log-majorization theorem due to Ando-Hiai [2]: For each $\alpha \in [0, 1]$

$$A^r \sharp_\alpha B^r \prec_{(\log)} (A \sharp_\alpha B)^r \qquad \text{for } r \geq 1,$$

or equivalently

$$(A^p \sharp_\alpha B^p)^{\frac{1}{p}} \prec_{(\log)} (A^q \sharp_\alpha B^q)^{\frac{1}{q}} \qquad \text{for } 0 < q < p.$$

To show the main theorem related to the β-quasi geometric mean for $\beta \in [-1, 0)$, we need the following Ando-Hiai log-majorization of negative power $\beta \in [-1, 0)$:

Theorem 3.1. *For every positive definite matrices $A, B > 0$ and $\beta \in [-1, 0)$,*

$$A^r \natural_\beta B^r \prec_{(\log)} (A \natural_\beta B)^r \qquad \text{for all } 0 < r \le 1 \tag{3.1}$$

or equivalently

$$(A \natural_\beta B)^r \prec_{(\log)} (A^r \natural_\beta B^r) \qquad \text{for all } r \ge 1, \tag{3.2}$$

$$(A^q \natural_\beta B^q)^{\frac{1}{q}} \prec_{(\log)} (A^p \natural_\beta B^p)^{\frac{1}{p}} \qquad \text{for all } 0 < q \le p. \tag{3.3}$$

Proof. The equivalence of (3.1)-(3.3) is immediate. It is easy to see by Lemma 2.1 that for $k = 1, \dots, n$

$$C_k(A^r \natural_\beta B^r) = C_k(A)^r \natural_\beta C_k(B)^r$$

and

$$C_k((A \natural_\beta B)^r) = (C_k(A) \natural_\beta C_k(B))^r.$$

Also,

$$\det(A^r \natural_\beta B^r) = (\det A)^{r(1-\beta)}(\det B)^{r\beta} = \det(A \natural_\beta B)^r.$$

Hence, in order to prove (3.1), it suffices to show that

$$\lambda_1(A^r \natural_\beta B^r) \le \lambda_1(A \natural_\beta B)^r \qquad \text{for all } 0 < r \le 1. \tag{3.4}$$

For this purpose we may prove that $A \natural_\beta B \le I$ implies $A^r \natural_\beta B^r \le I$, because both sides of (3.4) have the same order of homogeneity for A, B, so that we can multiply A, B by a positive constant.

First let us assume $\frac{1}{2} \le r \le 1$ and write $r = 1 - \varepsilon$ with $0 \le \varepsilon \le \frac{1}{2}$. Let $C = A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}}$. Then $B^{-1} = A^{-\frac{1}{2}} C A^{-\frac{1}{2}}$ and $A \natural_\beta B = A^{\frac{1}{2}} C^{-\beta} A^{\frac{1}{2}}$. If $A \natural_\beta B \le I$, then $C^{-\beta} \le A^{-1}$ so that $A \le C^\beta$ and $A^\varepsilon \le C^{\beta\varepsilon}$ for $0 \le \varepsilon \le \frac{1}{2}$ by Löwner-Heinz inequality. Since $-\beta \in (0, 1]$ and $1 - \varepsilon \in [\frac{1}{2}, 1]$, we now get

$$\begin{aligned}
A^r \natural_\beta B^r &= A^{\frac{1-\varepsilon}{2}} (A^{\frac{\varepsilon-1}{2}} B^{1-\varepsilon} A^{\frac{\varepsilon-1}{2}})^\beta A^{\frac{1-\varepsilon}{2}} \\
&= A^{\frac{1-\varepsilon}{2}} (A^{\frac{1-\varepsilon}{2}} (B^{-1})^{1-\varepsilon} A^{\frac{1-\varepsilon}{2}})^{-\beta} A^{\frac{1-\varepsilon}{2}} \\
&= A^{\frac{1-\varepsilon}{2}} (A^{\frac{1-\varepsilon}{2}} (A^{-\frac{1}{2}} C A^{-\frac{1}{2}})^{1-\varepsilon} A^{\frac{1-\varepsilon}{2}})^{-\beta} A^{\frac{1-\varepsilon}{2}} \\
&= A^{\frac{1-\varepsilon}{2}} (A^{-\frac{\varepsilon}{2}} [A \sharp_{1-\varepsilon} C] A^{-\frac{\varepsilon}{2}})^{-\beta} A^{\frac{1-\varepsilon}{2}} \\
&= A^{\frac{1}{2}-\varepsilon} [A^\varepsilon \sharp_{-\beta} (A \sharp_{1-\varepsilon} C)] A^{\frac{1}{2}-\varepsilon} \\
&\le A^{\frac{1}{2}-\varepsilon} [C^{\beta\varepsilon} \sharp_{-\beta} (C^\beta \sharp_{1-\varepsilon} C)] A^{\frac{1}{2}-\varepsilon},
\end{aligned}$$

using the joint monotonicity of matrix geometric means. Since a direct computation yields

$$C^{\beta\varepsilon} \sharp_{-\beta} (C^\beta \sharp_{1-\varepsilon} C) = C^{\beta(2\varepsilon-1)}$$

and by Löwner-Heinz inequality and $0 \le 1 - 2\varepsilon \le 1$, $C^{-\alpha} \le A^{-1}$ implies $C^{-\beta(1-2\varepsilon)} \le A^{-(1-2\varepsilon)}$ and thus we get

$$A^r \natural_\beta B^r \le A^{\frac{1}{2}-\varepsilon} C^{\beta(2\varepsilon-1)} A^{\frac{1}{2}-\varepsilon} \le A^{\frac{1}{2}-\varepsilon} A^{-1+2\varepsilon} A^{\frac{1}{2}-\varepsilon} = I.$$

Therefore (3.4) is proved in the case of $\frac{1}{2} \le r \le 1$.

When $0 < r < \frac{1}{2}$, writing $r = 2^{-k}(1-\varepsilon)$ with $k \in \mathbb{N}$ and $0 \le \varepsilon \le \frac{1}{2}$, and repeating the argument above we have

$$\begin{aligned}\lambda_1(A^r \natural_\beta B^r) &\le \lambda_1(A^{2^{-(k-1)}(1-\varepsilon)} \natural_\beta B^{2^{-(k-1)}(1-\varepsilon)})^{\frac{1}{2}} \\ &\vdots \\ &\le \lambda_1(A^{1-\varepsilon} \natural_\beta B^{1-\varepsilon})^{2^{-k}} \\ &\le \lambda_1(A \natural_\beta B)^r\end{aligned}$$

and so the proof is complete. □

By Theorem 3.1, we have the following results:

Theorem 3.2. *Let A and B be positive definite matrices and $|||\cdot|||$ any unitarily invariant norm, and $\beta \in [-1,0)$. If f is a continuous non-decreasing function on $[0,\infty)$ such that $f(0) \ge 0$ and $f(e^t)$ is convex, then*

$$|||f(A^r \natural_\beta B^r)||| \le |||f((A \natural_\beta B)^r)||| \qquad \text{for all } 0 < r \le 1.$$

In particular,

$$|||A^r \natural_\beta B^r||| \le |||(A \natural_\beta B)^r||| \qquad \text{for all } 0 < r \le 1$$

or equivalently

$$|||(A \natural_\beta B)^r||| \le |||(A^r \natural_\beta B^r)||| \qquad \text{for all } r \ge 1,$$

$$\left|\left|\left|(A^q \natural_\beta B^q)^{\frac{1}{q}}\right|\right|\right| \le \left|\left|\left|(A^p \natural_\beta B^p)^{\frac{1}{p}}\right|\right|\right| \qquad \text{for all } 0 < q \le p.$$

Proof. By [9, Proposition 4.4.13], if $A \prec_{w(\log)} B$ for positive definite matrices A and B and f is a continuous non-decreasing function on $[0,\infty)$ such that $f(0) \ge 0$ and $f(e^t)$ is convex, then $f(A) \prec_w f(B)$ and so $|||f(A)||| \le |||f(B)|||$. Hence Theorem 3.2 follows from Theorem 3.1. □

Corollary 3.3. *For every positive definite matrices $A, B > 0$ and $\beta \in [-1,0)$,*

$$A \natural_\beta B \le I \quad \text{implies} \quad A^r \natural_\beta B^r \le I \quad \text{for all } 0 < r \le 1.$$

4 Norm inequalities for quasi geometric mean. In this section, we show the main norm inequalities related to the quasi geometric mean for positive definite matrices. By [5, Lemma 5.5], we have the following quasi-geometric mean version of the Lie-Trotter formula: If A and B are positive definite matrices, then for each $\beta \in [-1,0)$

$$A \diamondsuit_\beta B = \lim_{p \to 0} (A^p \natural_\beta B^p)^{\frac{1}{p}} \tag{4.1}$$

and so for each $\beta \in [-1,0)$ $\left|\left|\left|(A^p \natural_\beta B^p)^{\frac{1}{p}}\right|\right|\right|$ decreases to $|||A \diamondsuit_\beta B|||$ as $p \downarrow 0$. Hence we have the following norm inequality for the quasi geometric mean of negative power:

Theorem 4.1. *Let A and B be positive definite matrices. Then for every unitarily invariant norm*

$$|||A \diamondsuit_\beta B||| \le |||A \natural_\beta B||| \qquad \text{for all } \beta \in [-1,0).$$

Proof. By Theorem 3.2, it follows that

$$\left|\left|\left|(A^q \natural_\beta B^q)^{\frac{1}{q}}\right|\right|\right| \leq \left|\left|\left|(A^p \natural_\beta B^p)^{\frac{1}{p}}\right|\right|\right| \qquad \text{for all } 0 < q < p$$

and as $q \to 0$ and $p = 1$ we have the desired inequality by (4.1). □

Theorem 4.2. *Let A and B be positive definite matrices. Then for every unitarily invariant norm*

$$\|A \natural_\beta B\| \leq \|A^{1-\beta}B^\beta\| \qquad \textit{for all } \beta \in [-1, -\tfrac{1}{2}]. \tag{4.2}$$

Proof. For the matrix norm $\|\cdot\|$, by the Araki-Cordes inequality $\|B^tA^tB^t\| \leq \|(BAB)^t\|$ for all $t \in [0,1]$, we have for $-1 \leq \beta \leq -\frac{1}{2}$

$$\begin{aligned}
\|A \natural_\beta B\| &= \left\|A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\beta A^{\frac{1}{2}}\right\| \\
&= \left\|A^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-\beta} A^{\frac{1}{2}}\right\| \\
&\leq \left\|A^{-\frac{1}{2\beta}}A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}A^{-\frac{1}{2\beta}}\right\|^{-\beta} \qquad \text{by } \tfrac{1}{2} \leq -\beta \leq 1 \\
&= \left\|A^{\frac{\beta-1}{2\beta}}B^{-1}A^{\frac{\beta-1}{2\beta}}\right\|^{-\beta} \\
&\leq \left\|A^{1-\beta}B^{2\beta}A^{1-\beta}\right\|^{\frac{1}{2}} \qquad \text{for } \tfrac{1}{2} \leq -\tfrac{1}{2\beta} \leq 1 \\
&= \left\|(A^{1-\beta}B^{2\beta}A^{1-\beta})^{\frac{1}{2}}\right\|
\end{aligned}$$

and this implies

$$\lambda_1(A \natural_\beta B) \leq \lambda_1((A^{1-\beta}B^{2\beta}A^{1-\beta})^{\frac{1}{2}}) = \lambda_1(|B^\beta A^{1-\beta}|).$$

Replacing A and B by (5) of Lemma 2.1, we obtain

$$\prod_{i=1}^k \lambda_i(A \natural_\beta B) \leq \prod_{i=1}^k \lambda_i(|B^\beta A^{1-\beta}|) \qquad \text{for } k = 1, \ldots, n.$$

Hence we have the weak log majorization $A \natural_\beta B \prec_{w(\log)} \left|B^\beta A^{1-\beta}\right|$ and this implies

$$\|A \natural_\beta B\| \leq \left\|\left|B^\beta A^{1-\beta}\right|\right\| = \|B^\beta A^{1-\beta}\| = \|A^{1-\beta}B^\beta\|$$

for every unitarily invariant norm and so we have the desired inequality (4.2). □

Remark 4.3. In Theorem 4.2, the inequality $\|A \natural_\beta B\| \leq \|A^{1-\beta}B^\beta\|$ does not always hold for $-1/2 < \beta < 0$. In fact, if we put $\beta = -\frac{1}{3}$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then we have the matrix norm $\left\|A \natural_{-\frac{1}{3}} B\right\| = 3.385$ and $\left\|A^{\frac{4}{3}}B^{-\frac{1}{3}}\right\| = 3.375$, and so $\|A \natural_\beta B\| > \left\|A^{1-\beta}B^\beta\right\|$.

Theorem 4.4. *Let A and B be positive definite matrices. Then for every unitarily invariant norm*

$$\|A \diamondsuit_\beta B\| \leq \left\|\left(B^{\frac{\beta q}{2}}A^{(1-\beta)q}B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}\right\| \leq \|A \natural_\beta B\| \quad \textit{for } 0 < q \leq \tfrac{1}{2} \textit{ and } \beta \in [-1, 0) \tag{4.3}$$

and

$$\|A \natural_\beta B\| \leq \|A^{1-\beta}B^\beta\| \leq \left\|\left(B^{\frac{\beta p}{2}}A^{(1-\beta)p}B^{\frac{\beta p}{2}}\right)^{\frac{1}{p}}\right\| \quad \textit{for } p \geq 2 \textit{ and } \beta \in [-1, -\tfrac{1}{2}]. \tag{4.4}$$

Proof. Since the first inequality in (4.3) follows from the Lie-Trotter formula (4.1), we show the second inequality in (4.3). By Theorem 3.2, we have $\left|\left|\left|(A^r \natural_\beta B^r)^{\frac{1}{r}}\right|\right|\right| \leq |||A \natural_\beta B|||$ for all $0 < r \leq 1$. For the matrix norm $\|\cdot\|$, by the Araki-Cordes inequality $\|(BAB)^t\| \leq \|B^t A^t B^t\|$ for all $t \geq 1$, we have for $0 < r \leq 1$

$$\begin{aligned}
\|A \natural_\beta B\| &\geq \|A^r \natural_\beta B^r\|^{\frac{1}{r}} \\
&= \|B^r \natural_{1-\beta} A^r\|^{\frac{1}{r}} \\
&= \left\|B^{\frac{r}{2}}(B^{-\frac{r}{2}}A^r B^{-\frac{r}{2}})^{1-\beta}B^{\frac{r}{2}}\right\|^{\frac{1}{r}} \\
&\geq \left\|B^{\frac{\beta r}{2(1-\beta)}} A^r B^{\frac{\beta r}{2(1-\beta)}}\right\|^{\frac{1-\beta}{r}} \qquad \text{by } 0 < \tfrac{1}{1-\beta} < 1 \\
&\geq \left\|B^{\frac{\beta r}{4}} A^{\frac{(1-\beta)r}{2}} B^{\frac{\beta r}{4}}\right\|^{\frac{2}{r}} \qquad \text{by } \tfrac{1}{2} < \tfrac{1-\beta}{2} \leq 1.
\end{aligned}$$

If we put $q = \frac{r}{2}$, then we have $\left\|\left(B^{\frac{\beta q}{2}} A^{(1-\beta)q} B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}\right\| \leq \|A \natural_\beta B\|$ for $0 < q \leq \frac{1}{2}$ and this implies

$$\lambda_1(\left(B^{\frac{\beta q}{2}} A^{(1-\beta)q} B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}) \leq \lambda_1(A \natural_\beta B).$$

Replacing A and B by (5) of Lemma 2.1, we obtain

$$\prod_{i=1}^{k} \lambda_1(\left(B^{\frac{\beta q}{2}} A^{(1-\beta)q} B^{\frac{\beta q}{2}}\right)^{\frac{1}{q}}) \leq \prod_{i=1}^{k} \lambda_1(A \natural_\beta B) \qquad \text{for } k = 1, \ldots, n,$$

which gives the second inequality in (4.3).

Next, for $s \geq 1$, it follows from Theorem 3.2 that $|||A \natural_\beta B||| \leq \left|\left|\left|(A^s \natural_\beta B^s)^{\frac{1}{s}}\right|\right|\right|$. For the matrix norm $\|\cdot\|$, we have

$$\begin{aligned}
\|A \natural_\beta B\| &\leq \|A^s \natural_\beta B^s\|^{\frac{1}{s}} \\
&= \left\|A^{\frac{s}{2}}(A^{\frac{s}{2}}B^{-s}A^{\frac{s}{2}})^{-\beta}A^{\frac{s}{2}}\right\|^{\frac{1}{s}} \\
&\leq \left\|A^{\frac{-(1-\beta)s}{2\beta}} B^{-s} A^{\frac{-(1-\beta)s}{2\beta}}\right\|^{-\frac{\beta}{s}} \qquad \text{by } \tfrac{1}{2} \leq -\beta \leq 1 \\
&\leq \left\|A^{(1-\beta)s} B^{2\beta s} A^{(1-\beta)s}\right\|^{\frac{1}{2s}} \qquad \text{by } \tfrac{1}{2} \leq -\tfrac{1}{2\beta} \leq 1.
\end{aligned}$$

If we put $p = 2s$, then we have

$$\begin{aligned}
\|A \natural_\beta B\| &\leq \left\|A^{\frac{(1-\beta)p}{2}} B^{\beta p} A^{\frac{(1-\beta)p}{2}}\right\|^{\frac{1}{p}} \\
&= \mathrm{spr}(A^{\frac{(1-\beta)p}{2}} B^{\beta p} A^{\frac{(1-\beta)p}{2}})^{\frac{1}{p}} \\
&= \mathrm{spr}(B^{\frac{\beta p}{2}} A^{(1-\beta)p} B^{\frac{\beta p}{2}})^{\frac{1}{p}} \\
&= \left\|B^{\frac{\beta p}{2}} A^{(1-\beta)p} B^{\frac{\beta p}{2}}\right\|^{\frac{1}{p}} \\
&= \left\|\left(B^{\frac{\beta p}{2}} A^{(1-\beta)p} B^{\frac{\beta p}{2}}\right)^{\frac{1}{p}}\right\|
\end{aligned}$$

for $p \geq 2$, where $\mathrm{spr}(X)$ is the spectral radius of X. By the argument similar to above, we have the inequality (4.4). □

Let A and B be positive definite matrices in $\mathbb{M}_n$ and $\beta \in [-1, 0)$. Since there is the case that $(1-\beta)A+\beta B$ is not positive semidefinite, we have no relation between $\|(1-\beta)A + \beta B\|$ and $\|A \natural_\beta B\|$ though $(1-\beta)A + \beta B \leq A \natural_\beta B$. Suppose that $A \geq B$. Then $(1-\beta)A^p + \beta B^p$ is positive definite for all $p \in (0, 1]$. In particular $0 < (1-\beta)A + \beta B \leq A \natural_\beta B$ and so $\|(1-\beta)A + \beta B\| \leq \|A \natural_\beta B\|$ for every unitarily invariant norm. Thus under the assumption $A \geq B$, we consider the refinement of this norm inequality. For this, we need the following result due to J. I. Fujii [6]: A real valued continuous function f on an interval J is matrix concave if and only if

$$f((1-\beta)H + \beta K) \leq (1-\beta)f(H) + \beta f(K) \tag{4.5}$$

for all Hermitian matrices H and K with $\sigma(H), \sigma(K)$ and $\sigma((1-\beta)H + \beta K) \subset J$ and $\beta \in [-1, 0)$.

Let $0 < q < p \leq 1$. Then the function $f(t) = t^{\frac{q}{p}}$ on $[0, \infty)$ is matrix concave and by (4.5)

$$((1-\beta)A^p + \beta B^p)^{\frac{q}{p}} \leq (1-\beta)A^q + \beta B^q. \tag{4.6}$$

Note that $(1-\beta)A^p + \beta B^p > 0$ for all $p \in (0, 1]$ since $A \geq B$. This implies that

$$\lambda_i\left(((1-\beta)A^p + \beta B^p)^{\frac{q}{p}}\right) \leq \lambda_i\left((1-\beta)A^q + \beta B^q\right) \quad \text{for all } i = 1, \ldots, n.$$

Taking q-th roots of both sides, we obtain

$$\lambda_i\left(((1-\beta)A^p + \beta B^p)^{\frac{1}{p}}\right) \leq \lambda_i\left(((1-\beta)A^q + \beta B^q)^{\frac{1}{q}}\right) \quad \text{for all } i = 1, \ldots, n$$

and so $\left\|((1-\beta)A^p + \beta B^p)^{\frac{1}{p}}\right\|$ is a decreasing function of p.

On the other hand, taking the logarithm of both sides in (4.6) and by (4.5), we obtain

$$\begin{aligned} \log\left((1-\beta)A^p + \beta B^p\right)^{\frac{1}{p}} &\leq \frac{1}{q}\log\left((1-\beta)A^q + \beta B^q\right) \\ &\leq (1-\beta)\log A + \beta \log B \end{aligned}$$

and this implies

$$\lambda_i\left(\log((1-\beta)A^p + \beta B^p)^{\frac{1}{p}}\right) \leq \lambda_i\left((1-\beta)\log A + \beta \log B\right) \quad \text{for all } i = 1, \ldots, n.$$

Taking the exponent of both sides, we obtain

$$\lambda_i\left(((1-\beta)A^p + \beta B^p)^{\frac{1}{p}}\right) \leq \lambda_i\left(e^{(1-\beta)\log A + \beta \log B}\right) \quad \text{for all } i = 1, \ldots, n$$

and so

$$\left\|((1-\beta)A^p + \beta B^p)^{\frac{1}{p}}\right\| \leq \left\|e^{(1-\beta)\log A + \beta \log B}\right\|$$

for all $p \in (0, 1]$. Summing up, we obtain the following result:

Theorem 4.5. *Let A and B be positive definite matrices in $\mathbb{M}_n$ such that $A \geq B$ and $\beta \in [-1, 0)$. Then for every unitarily invariant norm*

$$\|(1-\beta)A + \beta B\| \leq \left\|((1-\beta)A^p + \beta B^p)^{\frac{1}{p}}\right\| \leq \|A \diamondsuit_\beta B\| \leq \|A \natural_\beta B\|$$

for all $p \in (0, 1]$.

Finally, as an application, we show a refinement of the generalized Golden-Thompson inequality in terms of the quasi geometric means. Let H and K be Hermitian matrices. The Golden-Thompson trace inequality is

$$\mathrm{Tr}[e^{H+K}] \leq \mathrm{Tr}[e^H e^K].$$

Hiai-Petz [10] proved the complemented Golden-Thompson inequality:

$$\left|\!\left|\!\left| e^H \sharp_\alpha e^K \right|\!\right|\!\right| \leq \left|\!\left|\!\left| e^{(1-\alpha)H+\alpha K} \right|\!\right|\!\right| \quad \text{for all } \alpha \in [0,1]$$

for every unitarily invariant norm. By Theorem 4.1 and Theorem 4.2, we have a refinement of the Golden-Thompson inequality in terms of the quasi geometric means:

$$\left|\!\left|\!\left| e^{(1-\beta)H+\beta K} \right|\!\right|\!\right| \leq \left|\!\left|\!\left| e^H \natural_\beta e^K \right|\!\right|\!\right| \leq \left|\!\left|\!\left| e^{(1-\beta)H} e^{\beta K} \right|\!\right|\!\right|$$

for all $\beta \in [-1, -\frac{1}{2}]$ and so

$$\mathrm{Tr}[e^{H+K}] \leq \mathrm{Tr}[e^{\frac{1}{1-\beta}H} \natural_\beta e^{\frac{1}{\beta}K}] \leq \mathrm{Tr}[e^H e^K].$$

In particular, if we put $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$, then we have

$$\mathrm{Tr}[e^{2H} \sharp_{\frac{1}{2}} e^{2K}] \leq \mathrm{Tr}[e^{H+K}] \leq \mathrm{Tr}[e^{\frac{2}{3}H} \natural_{-\frac{1}{2}} e^{-2K}] \leq \mathrm{Tr}[e^H e^K].$$

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