O-UNION AND O-DECOMPOSITION ON HYPER K-ALGEBRAS

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Abstract. In this paper, we define a O-union of two hyper K-algebras and O-decomposition of a hyper K-algebra. In general, the O-union of two hyper K-algebra is not a hyper K-algebra. But, if a hyper K-algebra $(H, \circ, 0)$, be the O-union of two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, we investigate which properties of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ is transferred to $(H, \circ, 0)$ and conversely. Also we show that a hyper K-algebra $(H, \circ, 0)$ where $x \in x \circ y$ can be decomposed into a positive implicative hyper BCK-algebra and a hyper K-algebra.

1. Introduction

The concept of BCK-algebra that is a generalization of set difference and propositional calculi was established by Imai and Iséki [3] in 1966. In Ref. [4], Jun et al. applied the hyper structures BCK-algebra. In 1934, Marty [5] introduced for the first time the hyper structure theory in the 8th congress of Scandinavian Mathematicians. In Ref. [2], Borzooei et al. introduced the generalization of BCK-algebra and hyper BCK-algebra, called hyper K-algebra. They studied properties of hyper K-algebra. In this article, the aim is to define the O-union and O-decomposition on hyper K-algebras. Section 2, concerns definitions and theorems that are needed in the sequel. Section 3, we give O-union's definition of two hyper K-algebras and O-decomposition of a hyper K-algebra into two hyper K-algebras and finally in Section 4, we study transferable properties on O-Union (decomposition) hyper K-algebras.

2. Preliminaries

In this section we give some definitions and theorems that are needed in the sequel.

Definition 2.1. [2] Let H be a set containing 0 and the function $\circ : H \times H \to P^*(H)(:= P(H) \setminus \emptyset)$ is called a hyper operation on H. Then $(H, \circ, 0)$ is called a hyper K-algebra (hyper BCK-algebra) if it satisfies HK1-HK5 (BHK1-BHK4).

$$\begin{array}{ll} \mathrm{H}K1 : (x \circ z) \circ (y \circ z) < x \circ y, & \mathrm{B}HK1 : (x \circ z) \circ (y \circ z) \ll x \circ y, \\ \mathrm{H}K2 : (x \circ y) \circ z = (x \circ z) \circ y, & \mathrm{B}HK2 : (x \circ y) \circ z = (x \circ z) \circ y, \\ \mathrm{H}K3 : x < x, & \mathrm{B}HK3 : x \circ H \ll x, \\ \mathrm{H}K4 : x < y, y < x \Rightarrow x = y, & \mathrm{B}HK4 : x \ll y, y \ll x \Rightarrow x = y. \\ \mathrm{H}K5 : 0 < x. & \end{array}$$

for all $x, y, z \in H$, where $x < y (x \ll y) \Leftrightarrow 0 \in x \circ y$. For any $A, B \subseteq H$, $A < B$ if there exist $a \in A$ and $b \in B$ such that $a < b$. Moreover, $A \ll B$ if for all $a \in A$ there exist $b \in B$ such that $a \ll b$. A hyper K-algebra $(H, \circ, 0)$ is bounded if there exist an element $e \in H$ such that $x < e$ for all $x \in H$.

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Definition 2.2. [2] Let S be a nonempty set of a hyper K-algebra $(H, \circ, 0)$ containing 0. If S is a hyper K-algebra with respect to the hyper operation $\circ$ on H, we say that S is a hyper K-subalgebra of H.

Theorem 2.3. [2] *Let S be a nonempty set of a hyper K-algebra $(H, \circ, 0)$. Then S is a hyper K-subalgebra of H iff $x \circ y \subseteq S$ for all $x, y \in S$.*

Theorem 2.4. [7] *Let H be a set containing 0, $P_0(H) := \{A \subseteq H : 0 \in A\}$ and $S = \{f | f : H \to P_0(H)$ is a function$\}$. Then $\circ_f : H \times H \to P^*(H)$ where*

$$x \circ_f y := \begin{cases} f(x), & if x = y, \\ \{x\}, & if x \neq y. \end{cases}$$

is a hyperoperation. Moreover, the following statements are equivalent:

(1) *$(H, o_f, 0)$ is a hyper K-algebra,*
(2) *$f(x) \, o_f \, y = f(x)$ for all $y \neq x, y \in H$,*
(3) *$x \neq y$ and $y \in f(x)$ imply $y \in f(y)$ and $f(y) \subseteq f(x)$.*

This hyper K-algebra is called a quasi union hyper K-algebra.

Theorem 2.5. [7] *Let $(H, \circ, 0)$ be a quasi union hyper K-algebra. Then the following statements are equivalent:*

(1) *H is a positive implicative hyper K-algebra,*
(2) *$f(x) = \{0\}$ or $f(x) = \{0, x\}$ for all $x \in H$,*
(3) *H is a hyper BCK-algebra.*

Definition 2.6. [2, 9] Let I be a subset of a hyper K-algebra containing 0. Then I is said to be a hyper K-ideal (weak hyper K-ideal) of H if $x \circ y < I$ ($x \circ y \subseteq I$) and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Notation: Let A and I be nonempty subsets of a hyper K-algebra H. We set $AR_1I := A \subseteq I$, $AR_2I := A \cap I \neq \emptyset$, and $AR_3I := A < I$.

Definition 2.7. [1] A nonempty subset of a hyper K-algebra H such that $0 \in I$, for all $x, y, z \in H$, and $i, j, k \in \{1, 2, 3\}$ is said to be

(1) implicative hyper K-ideal of H if $((x \circ z) \circ (y \circ x)) < I$, $z \in I \Rightarrow x \in I$,
(2) positive implicative hyper K-ideal of type (i, j, k) if $(x \circ y) \circ zR_iI$ and $y \circ zR_jI$ imply that $x \circ zR_kI$,
(3) commutative hyper K-ideal of type (i, j) if $(x \circ y) \circ zR_iI$, $z \in I$ imply that $x \circ (y \circ (y \circ x))R_jI$.

Theorem 2.8. [1] *Let I be a hyper K-ideal of hyper K-algebra H. Then I is an implicative hyper K-ideal iff $x \circ (y \circ x) < I$ implies that $x \in I$, for any $x, y \in H$.*

3. O-union and O-decomposition on the hyper K-algebras

In this section, at first we define O-union of two hyper K-algebras and O-decomposition of a hyper K-algebra into two hyper K-algebras, and then we study transferable properties on O-Union (decomposition) hyper K-algebras.

Definition 3.1. Let $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are two hyper K-algebras and $\circ := \circ_1 \cup \circ_2$ i.e. $x \circ y = (x \circ_1 y) \cup (x \circ_2 y)$. If $(H, \circ, 0)$ be a hyper K-algebra then we say $(H, \circ, 0)$ is O-union of two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Moreover, a hyper K-algebra

$(H, \circ, 0)$ is called O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ if $\circ = \circ_1 \cup \circ_2$, for all $x, y \in H$. If $\circ$ be different from $\circ_1$ and $\circ_2$, we say that $(H, \circ, 0)$ is a proper O-decomposition.

Example 3.2. The hyper K-algebra $(H, \circ, 0)$

$\circ$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,1,2}	{0,1,2}
2	{2}	{1,2}	{0,1,2}

can be O-decomposed into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ as follows:

$\circ_1$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,2}	{0,1,2}
2	{2}	{2}	{0,1,2}

$\circ_2$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,1,2}	{0,2}
2	{2}	{1,2}	{0,1,2}

The O-decomposition of a hyper K-algebra $(H, \circ, 0)$ is not unique, since the hyper K-algebra $(H, \circ, 0)$ in example 3.2 is O-decomposed as follows:

$\circ_3$	0	1	2
0	{0}	{0,1}	{ 0,1}
1	{1}	{0,1}	{0}
2	{2}	{2}	{0,1}

$\circ_4$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,1,2}	{1,2}
2	{2}	{1,2}	{0,2}

The following example shows that a hyper K-algebra $(H, \circ, 0)$ can not be O-decomposed into two proper hyper K-algebras.

Example 3.3.

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{2}	{0}

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{2}	{0}

$\circ_2$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{2}	{0}

The following example shows that O-union of two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ is not a hyper K-algebra.

Example 3.4. Let $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are hyper K-algebras as follows. Then $(H, \circ, 0)$, the O-union of them is not hyper K-algebra, because $1 < 2, 2 < 1$ but $1 \neq 2$.

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,1}	{0,1,2}

$\circ_2$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0}
2	{2}	{2}	{0,2}

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0,1}
2	{2}	{0,1,2}	{0,1,2}

Theorem 3.5. *Any O-union of two quasi union hyper K-algebras is a quasi union hyper K-algebra.*

Proof. Suppose $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are two quasi union hyper K-algebras, therefore there are two functions $f, g : H \to P_0(H)$ and $\circ : H \times H \to P^*(H)$ such that

$$x \circ_f y := \begin{cases} (f \cup g)(x) & , if x = y; \\ \{x\} & , if x \neq y. \end{cases}$$

It is clear that $\circ$ is a hyperopration, we show that H is a quasi union hyper K-algebra. Let $y \in (f \cup g)(x) = (x \circ_1 x) \cup (x \circ_2 x)$ for any $x, y \in H$. So $y \in x \circ_1 x$ or $y \in x \circ_2 x$. Since

$(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are two quasi union hyper K-algebras, we get $y \in y \circ_1 y \subseteq x \circ_1 x$ or $y \in y \circ_2 y \subseteq x \circ_2 x$. Therefore $y \in (f \cup g)(y) \subseteq (f \cup g)(x)$, and the proof is completed. □

Theorem 3.6. *Let $(H, \circ, 0)$ be a hyper K-algebra such that $x \in x \circ y$ for all $x, y \in H$. Then H is O-decomposition into a positive implicative hyper BCK-algebra $(H, \circ_1, 0)$ and a hyper K-algebra $(H, \circ_2, 0)$.*

Proof. Let $(H, \circ, 0)$ be a hyper K-algebra, since $x \in x \circ y$ we can define $\circ_1 : H \times H \to H$ as follows:

$$x \circ_1 y := \begin{cases} \{x\} & , if x \neq y; \\ \{0\} & , if x = y. \end{cases}$$

It is clear that $(H, \circ_1, 0)$ is a quasi union hyper K-algebra. By Theorem 2.5(1) and (2), $(H, \circ_1, 0)$ is a positive implicative hyper BCK-algebra. So $(H, \circ, 0)$ is written as O-decomposition into a hyper BCK-algebra $(H, \circ_1, 0)$ and at least a hyper K-algebra $(H, \circ_2, 0)$ where $\circ_2 = \circ$. □

Example 3.7. The hyper K-algebra $(H, \circ, 0)$ with following cayley table is O-decomposition into a hyper BCK-algebra $(H, \circ_1, 0)$ and a proper hyper K-algebra $(H, \circ_2, 0)$.

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0,1}
2	{2}	{2}	{0,2}

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{2}	{0}

$\circ_2$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0}
2	{2}	{2}	{0,2}

The following example shows that the condition $x \in x \circ y$ in the theorem 3.6 is necessary.

Example 3.8. By the following cayley table, $(H, \circ, 0)$ is a hyper K-algebra,

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1,3}	{2}
2	{2}	{0,2}	{0,2}	{2}
3	{3}	{3}	{3}	{0}

If $\circ = \circ_1 \cup \circ_2$ then there are 36 hyper oprations on H for $\circ_1$ as follows:

$\circ_1$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0} or {0,1}	{1} or {3}or {1,3}	{2}
2	{2}	{0} or {2} or {0,2}	{0} or {0,2}	{2}
3	{3}	{3}	{3}	{0}

by checking all these cases, we see that $(H, \circ_1, 0)$ is not a hyper BCK-algebra. So $(H, \circ, 0)$ is not written as O-decomposition into a BCK-algebra and a hyper K-algebra.

4. Transferable properties

In this section we study transferable properties on O-Union (decomposition) hyper K-algebras.

Theorem 4.1. *Let $(H, \circ, 0)$ be O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Then S is subalgebra of $(H, \circ, 0)$ if and only if S is subalgebra of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$.*

Proof. It is clear. □

Theorem 4.2. *Let $(H, \circ, 0)$ be O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Then I is a weak hyper K-ideal of $(H, \circ, 0)$ if and only if I is a weak hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$.*

Proof. Suppose I be a weak hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$, $x \circ y \subseteq I$ and $y \in I$. Then $x \circ_1 y \subseteq I$ and $x \circ_2 y \subseteq I$ for all $x, y \in H$. Since I is a weak hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$ then $x \in I$.
Conversely, suppose I be a weak hyper K-ideal of $(H, \circ, 0)$ and $x \circ_1 y \subseteq I$ or $x \circ_2 y \subseteq I$ and $y \in I$. If $x \circ_i y \nsubseteq I$ for some $i \in \{1, 2\}$, then I is a weak hyper K-ideal of $(H, \circ_1, 0)$ or $(H, \circ_2, 0)$, otherwise $x \circ_i y \subseteq I$ for any $i \in \{1, 2\}$ and we have $x \circ y = x \circ_1 y \cup x \circ_2 y \subseteq I$, therefore $x \in I$. □

Theorem 4.3. *Let $(H, \circ, 0)$ is O-decomposition into two hyper K-algebras $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Then*

(1) *If e be a upper bound of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, then e is a upper bound of H.*
(2) *If I be a hyper K-ideal of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, then I is a hyper K-ideal of H.*
(3) *If I be an implicative hyper K-ideal of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, then I is an implicative hyper K-ideal of H.*

Proof. (1): By hypothesis we have $0 \in x \circ_1 e$ and $0 \in x \circ_2 e$ for all $x \in H$. So $0 \in x \circ e$ and e is a upper bound of H.
(2): Let $x \circ y < I$ and $y \in I$, so $x \circ_1 y < I$ or $x \circ_2 y < I$, since I is hyper K-ideal of $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ we get $x \in I$.
(3): Let $x \circ (y \circ x) < I$, so $x \circ_1 (y \circ_1 x) < I$ or $x \circ_2 (y \circ_2 x) < I$, by assumption we have $x \in I$. □

The following example shows that the converse of theorem 4.3 (1) is not true in general.

Example 4.4. Let $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ are hyper K-algebras as follows and $(H, \circ, 0)$ be O-union of them. Then the two hyper K-algebras are not bounded but 1 is a upper bound of $(H, \circ, 0)$.

$\circ_1$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{0}	{0,2}	{2}
3	{3}	{3}	{0,1,3}	{0,1,3}

$\circ_2$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{2}
3	{3}	{0,1,2}	{0,1,3}	{0,1,2,3}

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{0,2}	{0,2}	{2}
3	{3}	{0,1,2,3}	{0,1,3}	{0,1,2,3}

The following example shows that, in the theorem 4.3 (2) and (3) we can not use "or" instead of "and".

Example 4.5. Let $(H, \circ, 0)$, $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ be as follows. Then $I = \{0, 1\}$ is a hyper K-ideal of $(H, \circ_1, 0)$, but I is not a hyper K-ideal of $(H, \circ_2, 0)$ and $(H, \circ, 0)$. Also I is an implicative hyper K-ideal of $(H, \circ_1, 0)$ and it is not implicative hyper K-ideal of $(H, \circ_2, 0)$ and $(H, \circ, 0)$. Because $2 \circ_2 (2 \circ_2 2) < I$ but $2 \notin I$.

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{2}	{0}

$\circ_2$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{0}	{0,2}

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{1}
2	{2}	{0,2}	{0,2}

Theorem 4.6. *Let $(H,\circ,0)$ be O-decomposition into two hyper K-algebras $(H,\circ_1,0)$ and $(H,\circ_2,0)$. Then*

(1) *If I be a hyper K-ideal of $(H,\circ,0)$, then I is a hyper K-ideal of $(H,\circ_1,0)$ or $(H,\circ_2,0)$.*
(2) *If I be an implicative hyper K-ideal of $(H,\circ,0)$, then I is an implicative hyper K-ideal of $(H,\circ_1,0)$ or $(H,\circ_2,0)$.*
(3) *If I be a positive implicative hyper K-ideal of type (i,j,k) in $(H,\circ,0)$, where $i,j,k\in\{1,2,3\}$. Then I is a positive implicative hyper K-ideal of the same type in $(H,\circ_1,0)$ or $(H,\circ_2,0)$.*

Proof. (1): Suppose $x\circ_1 y<I$ or $x\circ_2 y<I$ and $y\in I$ for all $x,y\in H$, then $x\circ y<I$. Since I is hyper K-ideal of H, we have $x\in I$, i.e. I is a hyper K-ideal of $(H,\circ_1,0)$ or $(H,\circ_2,0)$.
(2): Suppose $x\circ_1(y\circ_1 x)<I$ for all $x,y\in H$, then $x\circ(y\circ x)<I$. Since I is an implicative hyper K-ideal of H, by Theorem 2.8, $x\in I$ and I is an implicative hyper K-ideal of $(H,\circ_1,0)$ or $(H,\circ_2,0)$.
(3): It is sufficient to prove for type $(1,1,1)$, the proof for other types is similar. If $(x\circ_i y)\circ_i z\nsubseteq I$ or $y\circ_i z\nsubseteq I$ for some $i\in\{1,2\}$, then I is positive implicative hyper K-ideal of type $(1,1,1)$ in $(H,\circ_1,0)$ or $(H,\circ_2,0)$. Otherwise if $(x\circ_1 y)\circ_1 z\subseteq I$, $(x\circ_2 y)\circ_2 z\subseteq I$, $y\circ_1 z\subseteq I$ and $y\circ_2 z\subseteq I$, then $(x\circ y)\circ z\subseteq I$ and $y\circ z\subseteq I$. Since I is a positive implicative hyper K-ideal of type $(1,1,1)$ in $(H,\circ,o)$, then $x\circ z\subseteq I$ and we get $x\circ_1 z\subseteq I$ and $x\circ_2 z\subseteq I$. Therefore in general I is positive implicative hyper K-ideal of type $(1,1,1)$ in $(H,\circ_1,o)$ or $(H,\circ_2,o)$ and the proof is completed. □

The following example shows that the converse of theorem 4.6 (3) is not true in general.

Example 4.7. Consider the following hyper K-algebras $(H,\circ,0)$, $(H,\circ_1,0)$ and $(H,\circ_2,0)$. Then $(H,\circ,0)$ is O-decomposition into $(H,\circ_1,0)$ and $(H,\circ_2,0)$, and $I=\{0,1\}$ is positive implicative hyper K-ideal of type $(2,1,2)$ in $(H,\circ_1,0)$, but I is not positive implicative hyper K-ideal of type $(2,1,2)$ in $(H,\circ_2,0)$, since $(2\circ_2 1)\circ_2 0\cap I\neq\emptyset$ and $1\circ_2 0\subseteq I$ but $2\circ_2 0\cap I=\emptyset$. Hence I is not positive implicative hyper K-ideal of type $(2,1,2)$ in H.

$\circ$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1,2}	{0,1,2}

$\circ_1$	0	1	2
0	{0}	{0}	{ 0}
1	{1}	{0}	{1}
2	{2}	{2}	{0,2}

$\circ_2$	0	1	2
0	{0}	{0,1,2}	{ 0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

Theorem 4.8. *Let $(H,\circ,0)$ be O-decomposition into two hyper K-algebras $(H,\circ_1,0)$ and $(H,\circ_2,0)$. If I be a positive implicative hyper K-ideal of types $(1,1,2)$ and $(1,1,3)$ in $(H,\circ_1,0)$ or $(H,\circ_2,0)$. Then I is a positive implicative hyper K-ideal of the same type in H.*

Proof. Let I is a positive implicative hyper K-ideal of type $(1,1,2)$ in $(H,\circ_1,0)$ or $(H,\circ_2,0)$, $(x\circ y)\circ z\subseteq I$ and $y\circ z\subseteq I$, then $(x\circ_1 y)\circ_1 z\subseteq I$, $(x\circ_2 y)\circ_2 z\subseteq I$, $y\circ_1 z\subseteq I$ and $y\circ_2 z\subseteq I$. By hypothesis we get that $x\circ_1 z\cap I\neq\emptyset$ or $x\circ_2 z\cap I\neq\emptyset$, hence $x\circ z\cap I\neq\emptyset$ and I is a positive implicative hyper K-ideal of type $(1,1,2)$ in $(H,\circ,0)$. The proof for type $(1,1,3)$ is similar. □

Theorem 4.9. *Let $(H,\circ,0)$ be O-decomposition into two hyper K-algebras $(H,\circ_1,0)$ and $(H,\circ_2,0)$. If I be a positive implicative hyper K-ideal of type $(1,1,1)$, $(1,j,k)$ or $(i,1,k)$*

where $i, j \in \{1, 2, 3\}$ *and* $k \neq 1$ *in* $(H, \circ_1, 0)$ *and* $(H, \circ_2, 0)$*. Then* I *is a positive implicative hyper K-ideal of the same type in* H*.*

Proof. Let I is a positive implicative hyper K-ideal of type $(1, 1, 1)$ in $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$, so $(x \circ_1 y) \circ_1 z \subseteq I, (x \circ_2 y) \circ_2 z \subseteq I$, $y \circ_1 z \subseteq I$ and $y \circ_2 z \subseteq I$. Since I is a positive implicative hyper K-ideal of type $(1, 1, 1)$ in $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, we have $x \circ_1 z \subseteq I$ and $x \circ_2 z \subseteq I$, so $x \circ z \subseteq I$. The proof for the others is similar. □

Theorem 4.9 is not true for other cases, the following example shows this for type $(2, 2, 3)$.

Example 4.10. The hyper K-algebra $(H, \circ, 0)$ is O-decomposition into $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$ as follows and $I = \{0, 1\}$ is a positive implicative of type $(2, 2, 3)$ in $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, but I is not a positive implicative of type $(2, 2, 3)$ in H. Since $(2 \circ 3) \circ 1 \cap I \neq \emptyset$, $3 \circ 1 \cap I \neq \emptyset$ and $2 \circ 1 \not\subseteq I$.

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{0,1,2}
3	{3}	{0,3}	{3}	{0,3}

$\circ_1$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{2}
3	{3}	{0}	{3}	{0,3}

$\circ_2$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{0,1}
3	{3}	{3}	{3}	{0,3}

Theorem 4.11. *Let* $(H, \circ, 0)$ *be O-decomposition into two hyper K-algebras* $(H, \circ_1, 0)$ *and* $(H, \circ_2, 0)$ *and the nonempty subset* I *of* H *be a commutative hyper K-ideal of type* $(i, j); i, j \in \{1, 2, 3\}$ *in* H*. Then* I *is a commutative hyper K-ideal of type* (i, j) *in* $(H, \circ_1, 0)$ *or* $(H, \circ_2, 0)$*.*

Proof. We prove theorem for type $(2, 2)$ and the proof for the other types is similar. Let $(x \circ_1 y) \circ_1 z \cap I \neq \emptyset$ or $(x \circ_1 y) \circ_1 z \cap I \neq \emptyset$ and $z \in I$, so $(x \circ y) \circ z \cap I \neq \emptyset$. Since I is a commutative hyper K-ideal of type $(2, 2)$ in H, we have $x \circ (y \circ (y \circ x)) \cap I \neq \emptyset$. Thus $x \circ_1 (y \circ_1 (y \circ_1 x)) \cap I \neq \emptyset$ or $x \circ_2 (y \circ_2 (y \circ_2 x)) \cap I \neq \emptyset$. □

Theorem 4.12. *Let* $(H, \circ, 0)$ *be O-decomposition into two hyper K-algebras* $(H, \circ_1, 0)$ *and* $(H, \circ_2, 0)$ *and the nonempty subset* I *of* H *be a commutative hyper K-ideal of type* $(1, 1)$ *or* $(i, j); i \in \{1, 2, 3\}, j \in \{2, 3\}$ *in* $(H, \circ_1, 0)$ *and* $(H, \circ_2, 0)$*, then* I *is a commutative hyper K-ideal of the same type in* H*.*

Proof. We prove theorem for type $(1, 1)$ and the proof for the other types is similar. Let $(x \circ y) \circ z \subseteq I$ and $z \in I$. So $(x \circ_1 y) \circ_1 z \subseteq I$ and $(x \circ_2 y) \circ_2 z \subseteq I$. Since I is a commutative hyper K-ideal of type $(1, 1)$ in $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, we have $x \circ_1 (y \circ_1 (y \circ_1 x)) \subseteq I$ and $x \circ_2 (y \circ_2 (y \circ_2 x)) \subseteq I$. Finally $x \circ (y \circ (y \circ x)) \subseteq I$ and I is a commutative hyper K-ideal of type $(1, 1)$ in H. □

The following example shows that, in the theorem 4.12 we can not use "or" instead of "and".

Example 4.13. Consider the following hyper K-algebras $(H, \circ, 0)$, $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$. Then $(H, \circ, 0)$ is O-decomposition into $(H, \circ_1, 0)$ and $(H, \circ_2, 0)$, and $I = \{0, 1\}$ is a commutative hyper K-ideal of type $(1, 1)$ in $(H, \circ_1, 0)$, but I is not commutative hyper K-ideal of type $(1,1)$ in $(H, \circ_2, 0)$. Since $(1 \circ_2 0) \circ_2 0 \subseteq I$ and $1 \circ_2 (0 \circ_2 (0 \circ_2 1)) = \{0,1,2\} \nsubseteq I$. Finally $(1 \circ 0) \circ 0 \subseteq I$ but $1 \circ (0 \circ (0 \circ 1)) = \{0,1,2\} \nsubseteq I$, so I is not a commutative hyper K-ideal of type $(1,1)$ in H.

$\circ$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,1,2}	{0,1,2}
2	{2}	{1,2}	{0,1,2}

$\circ_1$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{2}	{0,2}

$\circ_2$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,1,2}	{0,2}
2	{2}	{1,2}	{0,1,2}

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