CONVERGENCE OF NETS IN POSETS VIA AN IDEAL

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Abstract. It is well known that the meaning of the convergence in posets stings the interest of many investigators such as R. F. Anderson, J. C. Mathews and V. Olejček (see, for example [13,14]). Among others, the notions of the order-convergence and of the o_2-convergence in posets were studied in details, presenting necessary and sufficient conditions under of which these convergences are topological. Many researchers give a special attention to the study of these convergences in different posets, inserting new knowledge in the classical theory of posets's convergence. In this paper, we introduce the ideal-order-convergence in posets, proving results which are based on this notion. We insert topologies in posets and we study their properties. We also give a sufficient and necessary condition for the ideal-order-convergence in a poset to be topological. The introduction of a weaker form of the ideal-order-convergence in posets, called ideal-o_2-convergence, completes our study.

Introduction

The order-convergence in posets was introduced by G. Birkhoff [1]. In general, the order-convergence is not topological, that is a poset X may not have a topology τ so that nets order-converge if and only if they converge with respect to the topology τ on X [14,22]. Then, much attention was paid to those posets in which the order-convergence is topological [15–17,23]. Also, modifications of the order-convergence was studied in [13,18,20,22,23].

Meanwhile with the study of the order-convergence in posets, the notion of the o_2-convergence was communicated by the authors in [13,18]. In fact, the o_2-convergence is a generalization of the order-convergence and, as the order-convergence, the o_2-convergence is also, not topological in general. Also in [20], many sufficient and necessary conditions were given so that this kind of convergence be topological.

On the other hand, in recent years, a lot of papers have been written on statistical convergence and ideal convergence in metric and topological spaces (see, for instance, [2,3, 7–9,12]).

In the present paper we introduce and study the notion of convergence of nets in posets via an ideal. We proceed with the following enumeration: In Section 1, we recall some definitions which will be used in the rest of the paper. In Section 2, we define the notion of the ideal-order-convergence in posets proving classical results for the notion of convergence. In Section 3, we introduce topologies in posets and we give a sufficient and necessary condition for the ideal-order-convergence in a poset to be topological. In Section 4, we study the ideal-order-convergence in Cartesian products of posets. Finally, in Section 5, the concepts of the ideal-o_2-convergence and the topological ideal-o_2-convergence in posets are developed.

1 Preliminaries

In this section we recall some definitions that are needed in the sequel and we refer to [1] for more details. We shall frequently denote posets by their underlying sets, and we

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write X for $(X, \leqslant)$. We will also use the following symbols $(a,b) = \{x \in X : a < x < b\}$, $[a,b] = \{x \in X : a \leqslant x \leqslant b\}$, $(a,b] = \{x \in X : a < x \leqslant b\}$, and $[a,b) = \{x \in X : a \leqslant x < b\}$. In addition, by writing $A \subseteq_{fin} B$ we mean that the set A is a finite subset of the set B.

(1) A subset A of a poset X is said to be directed if $A \neq \emptyset$, and for any $a_1, a_2 \in A$ there exists $a \in A$ such that $a_1 \leqslant a$ and $a_2 \leqslant a$.

(2) A subset A of a poset X is said to be filtered if $A \neq \emptyset$, and for any $a_1, a_2 \in A$ there exists $a \in A$ such that $a \leqslant a_1$ and $a \leqslant a_2$.

If $(D_1, \leqslant_1)$ and $(D_2, \leqslant_2)$ are directed sets, then the Cartesian product $D_1 \times D_2$ is directed by $\leqslant$, where $(d_1, d_2) \leqslant (d_1', d_2')$ if and only if $d_1 \leqslant_1 d_1'$ and $d_2 \leqslant_2 d_2'$.

A net in a set X is an arbitrary function x from a non-empty directed preordered set D to X. If $x(d) = x_d$, for all $d \in D$, then the net x will be denoted by the symbol $(x_d)_{d \in D}$.

Let X be a topological space. A net $(x_d)_{d \in D}$ in X is said to *topology-converge* to a point $x \in X$, if for every open neighborhood U of x, $x_d \in U$ eventually. In this case we write $(x_d)_{d \in D} \xrightarrow{t} x$.

A net $(y_\lambda)_{\lambda \in \Lambda}$ in X is said to be a *semi-subnet* of the net $(x_d)_{d \in D}$ in X if there exists a function $\varphi : \Lambda \to D$ such that $y = x \circ \varphi$, or equivalently, $y_\lambda = x_{\varphi(\lambda)}$ for every $\lambda \in \Lambda$. We write $(y_\lambda)^{\varphi}_{\lambda \in \Lambda}$ to indicate the fact that φ is the function mentioned above.

A family $\mathcal{I}$ of subsets of a non-empty set D is called an *ideal* if $\mathcal{I}$ has the following properties:

(1) $\emptyset \in \mathcal{I}$.

(2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

(3) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

The ideal $\mathcal{I}$ is called *non-trivial* if $D \notin \mathcal{I}$.

Suppose that $(y_\lambda)^{\varphi}_{\lambda \in \Lambda}$ is a semi-subnet of the net $(x_d)_{d \in D}$ in X. For every ideal $\mathcal{I}$ of the directed set D, we consider the family $\{A \subseteq \Lambda : \varphi(A) \in \mathcal{I}\}$. This family is an ideal on Λ which will be denoted by $\mathcal{I}_\Lambda(\varphi)$.

A *filter* $\mathcal{F}$ in a non-empty set X is a family of subsets of X that has the following properties:

(1) $X \in \mathcal{F}$.

(2) If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.

(3) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

If $\emptyset \notin \mathcal{F}$, we say that $\mathcal{F}$ is a *proper* filter.

Given a filter $\mathcal{F}$ on a set X, let $M = \{(x, F) \in X \times \mathcal{F} : x \in F\}$ and for $(x, F), (y, G) \in M$ define $(x, F) \geqslant (y, G)$ if and only if $F \subseteq G$. It is easily seen that $\geqslant$ directs M. The map $s_{\mathcal{F}} : M \to X$ with $s_{\mathcal{F}}(x, F) = x$, is a net in X, which is called the *net associated with* $\mathcal{F}$. If (X, τ) is a topological space, then $\mathcal{F} \xrightarrow{t} x \in X$ with respect to τ if and only if $s_{\mathcal{F}} \xrightarrow{t} x$ with respect to τ.

Dually, given a net $s : M \to X$ on a set X, define

$$\mathcal{F}_s = \{F \subseteq X : \{s(m) : m \geqslant m_0\} \subseteq F \text{ for some } m_0 \in M\}.$$

Then $\mathcal{F}_s$ is a filter on X, which is called the *filter associated with* s. If (X, τ) is a topological space, then $s \xrightarrow{t} x$ with respect to τ if and only if $\mathcal{F}_s \xrightarrow{t} x$ with respect to τ.

Definition 1.1 [9] Let X be a topological space. A net $(x_d)_{d \in D}$ in X is said to $\mathcal{I}$-*topology-converge* to a point $x \in X$, where $\mathcal{I}$ is an ideal on D, if for every open neighborhood U of x, $\{d \in D : x_d \notin U\} \in \mathcal{I}$. In this case we write $(x_d)_{d \in D} \xrightarrow{\mathcal{I}-t} x$.

Definition 1.2 [1] Let X be a poset. A net $(x_d)_{d\in D}$ in X is said to *order-converge* to a point $x \in X$ if there exist subsets A and B of X such that:

(1) A is directed and B is filtered.

(2) $x = \bigvee A = \bigwedge B$.

(3) For every $a \in A$ and $b \in B$, there exists $d_0 \in D$ such that $a \leqslant x_d \leqslant b$ hold for all $d \geqslant d_0$.

In this case we write $(x_d)_{d\in D} \xrightarrow{o} x$.

Given a poset X, by $\mathcal{T}_X^o$ we denote the set consisting of all subsets U of X satisfying the following property: If $(x_d)_{d\in D} \xrightarrow{o} x \in U$, then there exists $d_0 \in D$ such that $x_d \in U$ for every $d \geqslant d_0$. The set $\mathcal{T}_X^o$ forms a topology on X, which is called *the order topology on* X (see [21,23]).

Definition 1.3 [19] Let X be a poset and $x, y, z \in X$. We define:

(1) $x \ll y$, if for any directed subset $A \subseteq X$, for which $\bigvee A$ exists and $y \leqslant \bigvee A$, there is $a \in A$ such that $x \leqslant a$.

(2) $z \rhd y$, if for any filtered subset $B \subseteq X$, for which $\bigwedge B$ exists and $\bigwedge B \leqslant y$, there is $b \in B$ such that $b \leqslant z$.

Clearly, if $x, y, z \in X$, then the following implications hold: $x \ll y \Rightarrow x \leqslant y$, and $z \rhd x \Rightarrow z \geqslant x$.

Definition 1.4 [19] A poset X is called *doubly continuous* if for each element $x \in X$, the set $\{a \in X : a \ll x\}$ is directed, the set $\{b \in X : b \rhd x\}$ is filtered and

$$x = \bigvee\{a \in X : a \ll x\} = \bigwedge\{b \in X : b \rhd x\}.$$

Definition 1.5 [23] The order-convergence in a poset X is called *topological*, if there exists a topology τ on X such that for every net $(x_d)_{d\in D}$ in X and $x \in X$ we have $(x_d)_{d\in D} \xrightarrow{o} x$ if and only if $(x_d)_{d\in D} \xrightarrow{t} x$ with respect to τ.

Proposition 1.6 [23] Let X be a complete lattice. If X satisfies the two infinite distributivity (the meet-infinite distributivity and the join-infinite distributivity) laws, then the following are equivalent:

(1) The order-convergence on X is topological.

(2) X is doubly continuous.

(3) X is a completely distributive lattice.

In the next we recall some definitions and results from [16].

Definition 1.7 Let X be a poset and $x, y, z \in X$. We define:

(1) $x \ll_S y$, if for every directed subset D of X with $\bigvee D = y$, there exists $d \in D$ such that $x \leqslant d$.

(2) $z \rhd_S y$, if for every filtered subset G of X with $\bigwedge G = y$, there exists $g \in G$ such that $z \geqslant g$.

Clearly, if $x, y, z \in X$, then the following implications hold: $x \ll y \Rightarrow x \ll_S y \Rightarrow x \leqslant y$, and $z \rhd x \Rightarrow z \rhd_S x \Rightarrow z \geqslant x$. Also for a poset X and $x \in X$ we use the following symbols: $\Downarrow_S x = \{a \in X : a \ll_S x\}$, $\Uparrow_S x = \{b \in X : x \ll_S b\}$, $\downarrow\downarrow_S x = \{c \in X : x \rhd_S c\}$ and $\uparrow\uparrow_S x = \{d \in X : d \rhd_S x\}$.

Definition 1.8 A poset X is called

(1) *S-doubly continuous* if for each element $x \in X$, the sets $\Downarrow_S x$ and $\Uparrow\!\Uparrow_S x$ are directed and filtered, respectively and $\bigvee \Downarrow_S x = \bigwedge \Uparrow\!\Uparrow_S x = x$, and

(2) *S^*-doubly continuous* if it is S-doubly continuous, and for every $x \in X$, $y \in \Downarrow_S x$ and $z \in \Uparrow\!\Uparrow_S x$, there exist $y_0 \in \Downarrow_S x$ and $z_0 \in \Uparrow\!\Uparrow_S x$ such that $[y_0, z_0] \subseteq \Uparrow_S y \cap \Downarrow\!\Downarrow_S z$.

Proposition 1.9 If X is a doubly continuous poset, then X is S^*-doubly continuous.

Definition 1.10 Let X be a poset.

(1) A filter $\mathcal{F}$ in X *order-converges to x* in the sense of Birkhoff if there exist a directed set D and a filtered set G such that $\bigvee D = x = \bigwedge G$ and $[a, b] \in \mathcal{F}$ for all $a \in D$ and $b \in G$. In this case, we write $\mathcal{F} \xrightarrow{O} x$.

(2) A subset U of X is called a *B-open* set if for any filter $\mathcal{F}$ that order converges to $x \in U$, there exists $F \in \mathcal{F}$ such that $F \subseteq U$. The set $\mathcal{T}_X$ of all B-open subsets of X forms a topology on X, which is called the *B-topology* on X.

Proposition 1.11 Let X be a poset and $U \subseteq X$. Then, $U \in \mathcal{T}_X$ if and only if for any directed subset D of X and any filtered subset G of X with $\bigvee D = \bigwedge G = x \in U$, there exist $d_0 \in D$ and $g_0 \in G$ such that $[d_0, g_0] \subseteq U$.

Theorem 1.12 For a poset X, the order-convergence in X is topological if and only if X is an S^*-doubly continuous poset.

2 Ideal-oder convergence

In this section we introduce the ideal-order-convergence in posets and prove some of its properties.

Definition 2.1 Let X be a poset. A net $(x_d)_{d \in D}$ in X is said to *$\mathcal{I}$-order-converge* to a point $x \in X$, where $\mathcal{I}$ is an ideal on D, if there exist subsets A and B of X such that:

(1) A is directed and B is filtered.

(2) $x = \bigvee A = \bigwedge B$.

(3) For every $a \in A$ and $b \in B$, $\{d \in D : x_d \notin [a, b]\} \in \mathcal{I}$.

Notation 2.2 Let $(x_d)_{d \in D}$ be a net in a poset X and let $\mathcal{I}$ be a non-trivial ideal on D. If $(x_d)_{d \in D}$ $\mathcal{I}$-order-converges to $x \in X$, then the point x is called the *$\mathcal{I}$-o-limit* of the net $(x_d)_{d \in D}$. In this case we write $(x_d)_{d \in D} \xrightarrow{\mathcal{I}-o} x$.

The ideal-convergences with respect to non-trivial ideals can be reduced to convergences of semi-subnets. More precisely, the following fact holds:

Proposition 2.3 *Let $(x_d)_{d \in D}$ be a net in a poset X and $\mathcal{I}$ a non-trivial ideal on D. Then there exists a semi-subnet $(y_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda \in \Lambda_{\mathcal{I}}}$ of $(x_d)_{d \in D}$ such that for every $A \subseteq X$,*

$$\{d \in D : x_d \notin A\} \in \mathcal{I} \text{ if and only if there exists } \lambda_0 \in \Lambda_{\mathcal{I}} \text{ such that } y_\lambda \in A \text{ for all } \lambda \geqslant \lambda_0.$$

In particular, for $x \in X$ and a topology τ on X,

(1) $(x_d)_{d \in D} \xrightarrow{\mathcal{I}-t} x$ *with respect to τ if and only if* $(y_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda \in \Lambda_{\mathcal{I}}} \xrightarrow{t} x$ *with respect to τ.*

(2) $(x_d)_{d \in D} \xrightarrow{\mathcal{I}-o} x$ *if and only if* $(y_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda \in \Lambda_{\mathcal{I}}} \xrightarrow{o} x$.

Proof. Set $\Lambda_{\mathcal{I}} = \{(d, I) \in D \times \mathcal{I} : d \notin I\}$ and define a preorder $\leqslant$ on $\Lambda_{\mathcal{I}}$ by letting $(d, I) \leqslant (d', I')$ if and only if $I \subseteq I'$ for $(d, I), (d', I') \in \Lambda_{\mathcal{I}}$. Since $\mathcal{I}$ is non-trivial, $(\Lambda_{\mathcal{I}}, \leqslant)$ is directed. Let $\varphi_{\mathcal{I}} : \Lambda_{\mathcal{I}} \to D$ such that $(d, I) \mapsto d$ be the projection. Then the semi-subnet $(y_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda \in \Lambda_{\mathcal{I}}}$ of $(x_d)_{d \in D}$ is as required. Indeed, let $\{d \in D : x_d \notin A\} \in \mathcal{I}$ for some $A \subseteq X$. If we set $I_0 = \{d \in D : x_d \notin A\}$ and $\lambda_0 = (d_0, I_0)$, then for each $\lambda = (d, I) \geqslant \lambda_0$ (i.e. $I \supseteq I_0$) we have $y_\lambda = x_d \in A$.

Conversely, let that for some $A \subseteq X$ there exists $\lambda_0 = (d_0, I_0) \in \Lambda_{\mathcal{I}}$ such that $y_\lambda = x_d \in A$ for all $\lambda = (d, I) \geqslant \lambda_0$. Then $\{d \in D : x_d \notin A\} \subseteq I_0 \in \mathcal{I}$.
(1) Take $A = U$ an arbitrary τ-open neighborhood of x.
(2) Take $A = [a, b]$ an arbitrary interval. □

Proposition 2.4 *Suppose that the net $(x_d)_{d \in D}$ in X $\mathcal{I}$-order-converges to $x, y \in X$, where $\mathcal{I}$ is a non-trivial ideal on D. Then, $x = y$.*

Proof. It follows directly from Proposition 2.3 and the fact that a limit of order-convergence is uniquely determined (see Remark 1 in p.15 of [11]). □

Example 2.5 Let $(x_d)_{d \in D}$ be a net in a poset X and $x \in X$. We consider the family

$$\{A \subseteq D : A \subseteq \{d \in D : d \ngeqslant d_0\} \text{ for some } d_0 \in D\}.$$

This family is a non-trivial ideal on D which will be denoted by $\mathcal{I}_D$. The net $(x_d)_{d \in D}$ order-converges to x if and only if $(x_d)_{d \in D} \xrightarrow{\mathcal{I}_D - o} x$.

Example 2.6 Let $X = \{x\} \cup \{a_i : i \in \mathbb{N}\}$, where $\mathbb{N}$ denotes the set of all natural numbers. The order $\leqslant$ on X is defined as follows:

(O1) $a_i < x$, for every $i \in \mathbb{N}$.

(O2) For all $i, j \in \mathbb{N}$, if $i < j$, then $a_i < a_j$.

Then, $(a_i)_{i \in \mathbb{N}} \xrightarrow{o} x$. Indeed, for the subsets $A = \{a_i : i \in \mathbb{N}\}$ and $B = \{x\}$ of X we have:

(1) A is directed and B is filtered.

(2) $x = \bigvee A = \bigwedge B$.

(3) For every $i \in \mathbb{N}$, there exists $j_0 \in \mathbb{N}$ ($j_0 = i$) such that $a_i \leqslant a_j \leqslant x$ hold for all $j \geqslant j_0$.

Generally, for every admissible ideal $\mathcal{I}$ on $\mathbb{N}$, namely, $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$, we have $(a_i)_{i \in \mathbb{N}} \xrightarrow{\mathcal{I} - o} x$. Let $\mathcal{I}_e$ be the ideal of even numbers on $\mathbb{N}$. Then, the net $(a_i)_{i \in \mathbb{N}}$ does not $\mathcal{I}_e$-order-converge to x.

Proposition 2.7 *If $(x_d)_{d \in D}$ is a net with $x_d = x$ for every $d \in D$, then $(x_d)_{d \in D} \xrightarrow{\mathcal{I} - o} x$ holds for every ideal $\mathcal{I}$ of D.*

Proof. The sets $A = B = \{x\}$ satisfy the conditions of Definition 2.1. Particularly, for the condition (3) we have $\{d \in D : x_d = x \notin \{x\}\} = \emptyset \in \mathcal{I}$. □

Proposition 2.8 *If $(x_d)_{d \in D} \xrightarrow{\mathcal{I} - o} x$, then for every semi-subnet $(y_\lambda)^{\varphi}_{\lambda \in \Lambda}$ of the net $(x_d)_{d \in D}$ we have $(y_\lambda)^{\varphi}_{\lambda \in \Lambda} \xrightarrow{\mathcal{I}_\Lambda(\varphi) - o} x$.*

Proof. Let $(y_\lambda)^{\varphi}_{\lambda \in \Lambda}$ be a semi-subnet of the net $(x_d)_{d \in D}$. Suppose that A and B are subsets of X such that:

(1) A is directed and B is filtered.

(2) $x = \bigvee A = \bigwedge B$.

(3) For every $a \in A$ and $b \in B$, $\{d \in D : x_d \notin [a,b]\} \in \mathcal{I}$.

It suffices to prove that for every $a \in A$ and $b \in B$, $\{\lambda \in \Lambda : y_\lambda \notin [a,b]\} \in \mathcal{I}_\Lambda(\varphi)$. Let $C = \{\lambda \in \Lambda : y_\lambda \notin [a,b]\}$. If $C = \emptyset$, then we are done. Suppose that $C \neq \emptyset$. We prove that $\varphi(C) \in \mathcal{I}$. Let $\varphi(\lambda) \in \varphi(C)$, where $\lambda \in C$. Since $y_\lambda = x_{\varphi(\lambda)} \notin [a,b]$, we have $\varphi(\lambda) \in \{d \in D : x_d \notin [a,b]\}$ which means that $\varphi(C) \subseteq \{d \in D : x_d \notin [a,b]\}$. Since $\{d \in D : x_d \notin [a,b]\} \in \mathcal{I}$, $\varphi(C) \in \mathcal{I}$. □

Proposition 2.9 *Let X be a poset and $x, y, z \in X$. If $y \ll_S x$ and $z \rhd_S x$, then for every net $(x_d)_{d\in D}$ in X, which $\mathcal{I}$-order-converges to x, where $\mathcal{I}$ is a non-trivial ideal on D, $\{d \in D : x_d \notin [y,z]\} \in \mathcal{I}$.*

Proof. Let $y \ll_S x$, $z \rhd_S x$ and $(x_d)_{d\in D}$ be a net in X which $\mathcal{I}$-order-converges to x, where $\mathcal{I}$ is a non-trivial ideal on D. Then, there exist subsets A and B of X such that:

(1) A is directed and B is filtered.

(2) $x = \bigvee A = \bigwedge B$.

(3) For each $a \in A$ and $b \in B$, $\{d \in D : x_d \notin [a,b]\} \in \mathcal{I}$.

Since $y \ll_S x$, there exists $a_0 \in A$ such that $y \leqslant a_0$ and since $z \rhd_S x$, there exists $b_0 \in B$ such that $b_0 \leqslant z$. By assumption, for $a_0 \in A$ and $b_0 \in B$ we have that $\{d \in D : x_d \notin [a_0,b_0]\} \in \mathcal{I}$. Since $\{d \in D : x_d \notin [y,z]\} \subseteq \{d \in D : x_d \notin [a_0,b_0]\}$, we have that $\{d \in D : x_d \notin [y,z]\} \in \mathcal{I}$. □

Corollary 2.10 *Let X be a poset and $x, y, z \in X$. If $y \ll x$ and $z \rhd x$, then for every net $(x_d)_{d\in D}$ in X, which $\mathcal{I}$-order-converges to x, where $\mathcal{I}$ is a non-trivial ideal on D, $\{d \in D : x_d \notin [y,z]\} \in \mathcal{I}$.*

Proposition 2.11 *Let X be a S-doubly continuous poset, $(x_d)_{d\in D}$ be a net in X, $x \in X$, and $\mathcal{I}$ be a non-trivial ideal on D. If for every $y, z \in X$ with $y \ll_S x$ and $z \rhd_S x$ we have $\{d \in D : x_d \notin [y,z]\} \in \mathcal{I}$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$.*

Proof. Is a direct consequence of the Definitions 2.1 and 1.8. □

Proposition 2.12 *Let X be a doubly continuous poset, $(x_d)_{d\in D}$ be a net in X, $x \in X$, and $\mathcal{I}$ be a non-trivial ideal on D. If for every $y, z \in X$ with $y \ll x$ and $z \rhd x$ we have $\{d \in D : x_d \notin [y,z]\} \in \mathcal{I}$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$.*

Proof. Is a direct consequence of the Definitions 2.1 and 1.4. □

3 Topologies in posets

In this section we introduce topologies in posets and we give a sufficient and necessary condition for the ideal-order-convergence in a poset to be topological.

Proposition 3.1 *Let X be a set and let $\mathcal{C}_X$ be a class consisting of triads $((x_d)_{d\in D}, x, \mathcal{I})$, where $(x_d)_{d\in D}$ is a net in X, $x \in X$, and $\mathcal{I}$ is a non-trivial ideal on D. The family*

$$\{U \subseteq X : \{d \in D : x_d \notin U\} \in \mathcal{I} \text{ for every } ((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X, x \in U\}$$

is a topology $\tau(\mathcal{C}_X)$ on X.

Proof. Obviously $\emptyset \in \tau(\mathcal{C}_X)$. Moreover, since $\{d \in D : x_d \notin X\} = \emptyset \in \mathcal{I}$, $X \in \tau(\mathcal{C}_X)$. Let $U, V \in \tau(\mathcal{C}_X)$ and $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X$, $x \in U \cap V$. Then, $\{d \in D : x_d \notin U\} \in \mathcal{I}$ and $\{d \in D : x_d \notin V\} \in \mathcal{I}$. Therefore,

$$\{d \in D : x_d \notin U \cap V\} = \{d \in D : x_d \notin U\} \cup \{d \in D : x_d \notin V\} \in \mathcal{I}$$

which means that the intersection $U \cap V \in \tau(\mathcal{C}_X)$. Now, let $U_i \in \tau(\mathcal{C}_X)$, $i \in I$ and $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X$, $x \in \cup_{i\in I} U_i$. Then, $\{d \in D : x_d \notin U_{i_0}\} \in \mathcal{I}$ for some $i_0 \in I$. Since

$$\{d \in D : x_d \notin \cup_{i\in I} U_i\} \subseteq \{d \in D : x_d \notin U_{i_0}\} \in \mathcal{I},$$

we have $\{d \in D : x_d \notin \cup_{i\in I} U_i\} \in \mathcal{I}$. Hence, $\cup_{i\in I} U_i \in \tau(\mathcal{C}_X)$. $\square$

Proposition 3.2 *If $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to $\tau(\mathcal{C}_X)$.*

Proof. Let $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X$ and U be an open neighborhood of x. Since $x \in U \in \tau(\mathcal{C}_X)$, by the definition of the topology $\tau(\mathcal{C}_X)$, we have $\{d \in D : x_d \notin U\} \in \mathcal{I}$. Therefore, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to $\tau(\mathcal{C}_X)$. $\square$

Notation 3.3 For an arbitrary poset X, we denote by $\mathcal{C}_X^o$ the class consisting of triads $((x_d)_{d\in D}, x, \mathcal{I})$, where $(x_d)_{d\in D}$ is a net in X, $x \in X$, and $\mathcal{I}$ is a non-trivial ideal on D such that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$.

Proposition 3.4 *Let X be a poset. Then, $\tau(\mathcal{C}_X^o) = \mathcal{T}_X^o$.*

Proof. Firstly, we prove that $\tau(\mathcal{C}_X^o) \subseteq \mathcal{T}_X^o$. Let $U \in \tau(\mathcal{C}_X^o)$ and a net $(x_d)_{d\in D} \xrightarrow{o} x \in U$. Then by Example 2.5 $(x_d)_{d\in D} \xrightarrow{\mathcal{I}_D-o} x$. By the definition of $\mathcal{I}_D$ it follows that $(x_d)_{d\in D}$ is eventually in U. Thus $U \in \mathcal{T}_X^o$.

We prove the opposite direction $\mathcal{T}_X^o \subseteq \tau(\mathcal{C}_X^o)$. Let $U \in \mathcal{T}_X^o$ and a net $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x \in U$, where $\mathcal{I}$ is a non-trivial ideal on D. Then by Proposition 2.3 the net $(y_\lambda)_{\lambda\in\Lambda_{\mathcal{I}}}^{\varphi_{\mathcal{I}}} \xrightarrow{o} x$. That is, there exists $\lambda_0 \in \Lambda_{\mathcal{I}}$ such that $y_\lambda \in U$ for all $\lambda \geqslant \lambda_0$. Thus $\{d \in D : x_d \notin U\} \in \mathcal{I}$, which means that $U \in \tau(\mathcal{C}_X^o)$. $\square$

The following result is a characterization of open sets in $\mathcal{T}_X^o$.

Lemma 3.5 Let X be a poset and $U \subseteq X$. Then, $U \in \mathcal{T}_X^o$ if and only if for any directed subset D of X and any filtered subset F of X with $\bigvee D = \bigwedge F = x \in U$, there exist $d_0 \in D$ and $f_0 \in F$ such that $[d_0, f_0] \subseteq U$.

Proof. Let $U \in \mathcal{T}_X^o$, D be a directed subset of X, F be a filtered subset of X and $\bigvee D = \bigwedge F = x \in U$. Suppose that for each $d \in D$ and $f \in F$ there exist $g_{(d,f)} \in X$ with $d \leqslant g_{(d,f)} \leqslant f$ and $g_{(d,f)} \notin U$. The Cartesian product $D \times F$ is directed if we define $(d', f') \geqslant (d, f)$ to mean that $d' \geqslant d$ and $f' \leqslant f$. Then, $(g_{(d,f)})_{(d,f)\in D\times F} \xrightarrow{o} x$, and, therefore, the net $(g_{(d,f)})_{(d,f)\in D\times F}$ converges to x, with respect to $\mathcal{T}_X^o$, contradiction. Thus, for some $d_0 \in D$ and $f_0 \in F$ we get $[d_0, f_0] \subseteq U$.

Now, let $U \subseteq X$ and suppose that for any directed subset D of X and any filtered subset F of X with $\bigvee D = \bigwedge F = x \in U$, there exist $d_0 \in D$ and $f_0 \in F$ such that $[d_0, f_0] \subseteq U$. Consider a net $(x_\lambda)_{\lambda\in\Lambda} \xrightarrow{o} x \in U$. Then, by Definition 1.2 there exist a directed subset E of X and a filtered subset G of X with $\bigvee E = \bigwedge G = x$ and for every $e \in E$ and $g \in G$, there exists $\lambda_{e,g} \in \Lambda$ such that $x_\lambda \in [e, g]$ for every $\lambda \geqslant \lambda_{e,g}$. By hypothesis there exist $e_0 \in E$ and $g_0 \in G$ such that $[e_0, g_0] \subseteq U$. Consequently, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in [e_0, g_0] \subseteq U$ for every $\lambda \geqslant \lambda_0$. Hence, by the definition of the topology $\mathcal{T}_X^o$ we have $U \in \mathcal{T}_X^o$. $\square$

Lemma 3.6 *Let X be a poset and $U \subseteq X$. Then, $U \in \tau(\mathcal{C}^o_X)$ if and only if for any directed subset D of X and any filtered subset F of X with $\bigvee D = \bigwedge F = x \in U$, there exist $d_0 \in D$ and $f_0 \in F$ such that $[d_0, f_0] \subseteq U$.*

Proof. The proof is similar to the proof of Lemma 3.5. □

Remark 3.7 *We observe that Proposition 3.4 it follows, alternatively, as a direct consequence of the Lemmas 3.5 and 3.6. Also, given a poset X, in view of Lemma 3.5 and Proposition 1.11 we have that the topology $\mathcal{T}^o_X$ on X is equal to the B-topology on X (see Definition 1.10).*

Corollary 3.8 *Let X be a poset. If $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to $\mathcal{T}^o_X$.*

Proof. Is similar to Proposition 3.2. □

Proposition 3.9 *Let X be a poset. The topology $\mathcal{T}^o_X$ is the finest topology τ on X such that ideal-order-convergence implies ideal-topology-convergence with respect to τ.*

Proof. Let τ be a topology on X such that ideal-order-convergence implies ideal-topology-convergence with respect to τ. We prove that $\tau \subseteq \mathcal{T}^o_X$. Let $U \in \tau$. It suffices to prove that for every $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}^o_X$, $x \in U$ we have that $\{d \in D : x_d \notin U\} \in \mathcal{I}$ (see Proposition 3.4). Let $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}^o_X$. Then, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ and, by assumption, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ. Therefore, $\{d \in D : x_d \notin U\} \in \mathcal{I}$. □

Definition 3.10 The ideal-order-convergence in a poset X is called *topological*, if there exists a topology τ on X such that for every net $(x_d)_{d\in D}$ in X, $x \in X$ and for every non-trivial ideal $\mathcal{I}$ of D, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ if and only if $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ.

Proposition 3.11 *Let X be a poset such that the ideal-order-convergence is topological and let τ be the corresponding topology on X. Then, $\tau \subseteq \mathcal{T}^o_X$.*

Proof. Is a direct consequence of the Proposition 3.9. □

Proposition 3.12 *The ideal-order-convergence in a poset X is topological if and only if the order-convergence in X is topological.*

Proof. Consider a poset X and suppose that the ideal-order-convergence in X is topological. Let $(x_d)_{d\in D}$ be a net in X and $x \in X$. For the non-trivial ideal $\mathcal{I}_D$ of D (see Example 2.5) we have that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}_D-o} x$ if and only if $(x_d)_{d\in D} \xrightarrow{\mathcal{I}_D-t} x$ with respect to some topology τ on X. Therefore, $(x_d)_{d\in D} \xrightarrow{o} x$ if and only if $(x_d)_{d\in D}$ converges to x with respect to τ. Thus, the order-convergence in X is topological.

Conversely, suppose that the order-convergence in X is topological. Let $(x_d)_{d\in D}$ be a net in X, $\mathcal{I}$ a non-trivial ideal on D and $x \in X$. Then by Proposition 2.3 and hypothesis we have the following equivalences: $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ if and only if $(y_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda\in\Lambda_{\mathcal{I}}} \xrightarrow{o} x$ if and only if $(y_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda\in\Lambda_{\mathcal{I}}} \xrightarrow{t} x$ with respect to some topology τ on X if and only if $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ. Thus, the ideal-order-convergence in X is topological. □

As the study of the notion of the ideal-order-convergence is extended, it raises the necessity to clarify, in which posets, is the ideal-order-convergence topological. Following [16] we prove that for a poset X the ideal-order-convergence is topological if and only if X is an S^*-doubly continuous poset.

Proposition 3.13 *Let X be a poset.*

(1) *If $\mathcal{F}$ is a filter on X and $s_{\mathcal{F}}$ is its associated net, then $\mathcal{F} \xrightarrow{O} x \in X$ (in the sense of Definition 1.10) if and only if $s_{\mathcal{F}} \xrightarrow{o} x$ (in the sense of Definition 1.2).*

(2) *If $s : M \to X$ is a net in X and $\mathcal{F}_s$ is its associated filter, then $s \xrightarrow{o} x \in X$ (in the sense of Definition 1.2) if and only if $\mathcal{F}_s \xrightarrow{O} x$ (in the sense of Definition 1.10).*

Proof. (1) Suppose that $\mathcal{F} \xrightarrow{O} x \in X$. Then, there exist a directed set $D \subseteq X$ and a filtered set $G \subseteq X$ such that $\vee D = \wedge G = x$ and $[a, b] = E \in \mathcal{F}$ for all $a \in D$ and $b \in G$. It follows that for every $(f, F) \geqslant (e, E)$ equivalently $F \subseteq E$, we have $s_{\mathcal{F}}(f, F) = f \in F \subseteq E \Rightarrow a \leqslant s_{\mathcal{F}}(f, F) \leqslant b$. Thus $s_{\mathcal{F}} \xrightarrow{o} x$.

Conversely, let $s_{\mathcal{F}} \xrightarrow{o} x \in X$. Then, there exist a directed set $D \subseteq X$ and a filtered set $G \subseteq X$ such that $\vee D = \wedge G = x$ and for every $a \in D$ and $b \in G$ there exists $m_0 = (f_0, F_0) \in M$ such that $a \leqslant s_{\mathcal{F}}(m) \leqslant b$ for all $m \geqslant m_0$. Then, for all $f \in F_0$ we have $a \leqslant s_{\mathcal{F}}(f, F_0) = f \leqslant b$, since $(f, F_0) \geqslant (f_0, F_0)$. Thus, $F_0 \subseteq [a, b]$. So $[a, b] \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{O} x$.

(2) Suppose that $s \xrightarrow{o} x \in X$. Then, there exist a directed set $D \subseteq X$ and a filtered set $G \subseteq X$ such that $\vee D = \wedge G = x$ and for every $a \in D$ and $b \in G$ there exists $m_0 \in M$ such that $a \leqslant s(m) \leqslant b$ for all $m \geqslant m_0$, which means that $[a, b] \supseteq \{s(m) : m \geqslant m_0\} \in \mathcal{F}_s$ and thus $\mathcal{F}_s \xrightarrow{O} x$.

Conversely, let $\mathcal{F}_s \xrightarrow{O} x \in X$. Then, there exist a directed set $D \subseteq X$ and a filtered set $G \subseteq X$ such that $\vee D = \wedge G = x$ and $[a, b] \in \mathcal{F}_s$ for all $a \in D$ and $b \in G$. This means that for some $m_0 \in M$ we have $\{s(m) : m \geqslant m_0\} \subseteq [a, b]$ and thus $s \xrightarrow{o} x$. □

We observe that the coincidence of $\mathcal{T}_X^o$ and B-topology on X is, also, immediate from Proposition 3.13.

Proposition 3.14 *The order-convergence in a poset X (in the sense of Definition 1.2) is topological if and only if the order-convergence in X (in the sense of Definition 1.10) is topological.*

Proof. Is a direct consequence of Proposition 3.13. □

Proposition 3.15 *For a poset X, the ideal-oder convergence is topological for the $\mathcal{T}_X^o$ topology if and only if X is an S^*-doubly continuous poset.*

Proof. Is a direct consequence of Theorem 1.12, Remark 3.7, Proposition 3.12 and Proposition 3.14. □

4 Ideal-order-convergence in Cartesian products of posets

In this section we study ideal-order-convergence in the Cartesian product of two posets X and Y.

For an ideal (resp., filter) $\mathcal{I}$ on a set X, let $\mathcal{I}^*$ denote the dual filter (resp., ideal) on $\mathcal{I}$, that is, $\mathcal{I}^* = \{A \subseteq X : X \setminus A \in \mathcal{I}\}$. For filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on sets D_1 and D_2, respectively, let $\mathcal{F}_1 \times \mathcal{F}_2$ denote the product filter, that is,

$$\mathcal{F}_1 \times \mathcal{F}_2 = \{A \subseteq D_1 \times D_2 : F_1 \times F_2 \subseteq A \text{ for some } F_1 \in \mathcal{F}_1 \text{ and some } F_2 \in \mathcal{F}_2\}.$$

Then the following trivial facts hold:

(1) An ideal (resp., filter) $\mathcal{I}$ on a set X is non-trivial if and only if so is the dual filter (resp., ideal) $\mathcal{I}^*$.

(2) If filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on sets D_1 and D_2, respectively, are non-trivial, so is the product filter $\mathcal{F}_1 \times \mathcal{F}_2$.

Proposition 4.1 *Let D_1, D_2 be two directed sets and let $\mathcal{I}_1$, $\mathcal{I}_2$ be two non-trivial ideals on D_1 and D_2, respectively. The family $(\mathcal{I}_1^* \times \mathcal{I}_2^*)^*$ is a non-trivial ideal on $D_1 \times D_2$, which will denote by $\mathcal{I}_1 \times \mathcal{I}_2$.*

Proof. Is an easy consequence of the above discussion. □

Proposition 4.2 *Let X and Y be two posets. Then, we have $(x_{d_1})_{d_1 \in D_1} \xrightarrow{\mathcal{I}_1-o} x$ and $(y_{d_2})_{d_2 \in D_2} \xrightarrow{\mathcal{I}_2-o} y$, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are two non-trivial ideals of D_1 and D_2, respectively if and only if $((x_{d_1}, y_{d_2}))_{(d_1,d_2) \in D_1 \times D_2} \xrightarrow{\mathcal{I}_1 \times \mathcal{I}_2-o} (x, y)$.*

Proof. Let $(x_{d_1})_{d_1 \in D_1} \xrightarrow{\mathcal{I}_1-o} x$ and $(y_{d_2})_{d_2 \in D_2} \xrightarrow{\mathcal{I}_2-o} y$. We prove that

$$((x_{d_1}, y_{d_2}))_{(d_1,d_2) \in D_1 \times D_2} \xrightarrow{\mathcal{I}_1 \times \mathcal{I}_2-o} (x, y).$$

There exist subsets A_1, B_1 and A_2, B_2 of X and Y, respectively such that:

(1) A_1, A_2 are directed and B_1, B_2 are filtered.

(2) $x = \bigvee A_1 = \bigwedge B_1$ and $y = \bigvee A_2 = \bigwedge B_2$.

(3) For every $a_1 \in A_1$ and $b_1 \in B_1$, $\{d_1 \in D_1 : x_{d_1} \notin [a_1, b_1]\} \in \mathcal{I}_1$.

(4) For every $a_2 \in A_2$ and $b_2 \in B_2$, $\{d_2 \in D_2 : y_{d_2} \notin [a_2, b_2]\} \in \mathcal{I}_2$.

We set $A = A_1 \times A_2$ and $B = B_1 \times B_2$. Then:

(5) A is directed and B is filtered.

(6) $(x, y) = \bigvee A = \bigwedge B$.

Let $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$. We prove that

$$\{(d_1, d_2) \in D_1 \times D_2 : (x_{d_1}, y_{d_2}) \notin [(a_1, a_2), (b_1, b_2)]\} \in \mathcal{I}_1 \times \mathcal{I}_2.$$

It suffices to prove that

$$W = \{(d_1, d_2) \in D_1 \times D_2 : (x_{d_1}, y_{d_2}) \in [(a_1, a_2), (b_1, b_2)]\} \in \mathcal{I}_1^* \times \mathcal{I}_2^*.$$

We set $I_1 = \{d_1 \in D_1 : x_{d_1} \notin [a_1, b_1]\}$ and $I_2 = \{d_2 \in D_2 : y_{d_2} \notin [a_2, b_2]\}$.
Then $D_1 \setminus I_1 = \{d_1 \in D_1 : x_{d_1} \in [a_1, b_1]\} \in \mathcal{I}_1^*$ and $D_2 \setminus I_2 = \{d_2 \in D_2 : y_{d_2} \in [a_2, b_2]\} \in \mathcal{I}_2^*$.
We see that

$$(D_1 \setminus I_1) \times (D_2 \setminus I_2) \subseteq W.$$

Therefore, $W \in \mathcal{I}_1^* \times \mathcal{I}_2^*$.

Conversely, let $((x_{d_1}, y_{d_2}))_{(d_1,d_2) \in D_1 \times D_2} \xrightarrow{\mathcal{I}_1 \times \mathcal{I}_2-o} (x, y)$. We prove that

$$(x_{d_1})_{d_1 \in D_1} \xrightarrow{\mathcal{I}_1-o} x.$$

There exist subsets A and B of $X \times Y$ such that:

(7) A is directed and B is filtered.

(8) $(x, y) = \bigvee A = \bigwedge B$.

(9) For every $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$,
$\{(d_1, d_2) \in D_1 \times D_2 : (x_{d_1}, y_{d_2}) \notin [(a_1, a_2), (b_1, b_2)]\} \in \mathcal{I}_1 \times \mathcal{I}_2$.

We set:
$A_1 = \{x_1 \in X : (x_1, y_1) \in A \text{ for some } y_1 \in Y\}$,
$B_1 = \{x_1 \in X : (x_1, y_1) \in B \text{ for some } y_1 \in Y\}$.
Then A_1 is directed, B_1 is filtered and $x = \bigvee A_1 = \bigwedge B_1$.
We prove that:

(10) For every $a_1 \in A_1$ and $b_1 \in B_1$, $\{d_1 \in D_1 : x_{d_1} \notin [a_1, b_1]\} \in \mathcal{I}_1$.

Let $a_1 \in A_1$ and $b_1 \in B_1$. Then, there exist $a_2, b_2 \in Y$ such that $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$. Hence,

$$\{(d_1, d_2) \in D_1 \times D_2 : (x_{d_1}, y_{d_2}) \notin [(a_1, a_2), (b_1, b_2)]\} \in \mathcal{I}_1 \times \mathcal{I}_2,$$

or equivalently

$$W = \{(d_1, d_2) \in D_1 \times D_2 : (x_{d_1}, y_{d_2}) \in [(a_1, a_2), (b_1, b_2)]\} \in \mathcal{I}_1^* \times \mathcal{I}_2^*.$$

Therefore, there exist $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$ such that $(D_1 \setminus I_1) \times (D_2 \setminus I_2) \subseteq W$. Since

$$\{d_1 \in D_1 : x_{d_1} \in [a_1, b_1]\} \supseteq D_1 \setminus I_1 \in \mathcal{I}_1^*,$$

we have $\{d_1 \in D_1 : x_{d_1} \notin [a_1, b_1]\} \in \mathcal{I}_1$. Similarly, we get $(y_{d_2})_{d_2 \in D_2} \xrightarrow{\mathcal{I}_2 - o} y$. □

Proposition 4.3 *Let X and Y be two posets. Then, $(x_d)_{d \in D} \xrightarrow{\mathcal{I} - o} x$ and $(y_d)_{d \in D} \xrightarrow{\mathcal{I} - o} y$ if and only if $((x_d, y_d))_{d \in D} \xrightarrow{\mathcal{I} - o} (x, y)$.*

Proof. Is similar to the proof of Proposition 4.2. □

Based on the ideas of papers [4–6], we will use Proposition 3.4 to prove the following two propositions.

Proposition 4.4 *Let X and Y be two posets. Then, $\mathcal{T}_X^o \times \mathcal{T}_Y^o \subseteq \mathcal{T}_{X \times Y}^o$.*

Proof. Suppose that $U_X \in \mathcal{T}_X^o$ and $U_Y \in \mathcal{T}_Y^o$. It suffices to prove that $U_X \times U_Y \in \mathcal{T}_{X \times Y}^o$. Let $(((x_d, y_d))_{d \in D}, (x, y), \mathcal{I}) \in \mathcal{C}_{X \times Y}^o$, $(x, y) \in U_X \times U_Y$. From Proposition 4.3, it follows that $((x_d)_{d \in D}, x, \mathcal{I}) \in \mathcal{C}_X^o$ and $((y_d)_{d \in D}, y, \mathcal{I}) \in \mathcal{C}_Y^o$, where $x \in U_X$ and $y \in U_Y$. Therefore,

$$\{d \in D : x_d \notin U_X\} \in \mathcal{I} \ \text{ and } \ \{d \in D : y_d \notin U_Y\} \in \mathcal{I}.$$

Since $\{d \in D : (x_d, y_d) \notin U_X \times U_Y\} = \{d \in D : x_d \notin U_X\} \cup \{d \in D : y_d \notin U_Y\} \in \mathcal{I}$, we conclude that $\{d \in D : (x_d, y_d) \notin U_X \times U_Y\} \in \mathcal{I}$ and, consequently, the product $U_X \times U_Y \in \mathcal{T}_{X \times Y}^o$. □

Proposition 4.5 *Let X and Y be two posets. The Cartesian product topology $\mathcal{T}_X^o \times \mathcal{T}_Y^o$ coincides with the topology $\mathcal{T}_{X \times Y}^o$ if the latter has a base of Cartesian product sets.*

Proof. By Proposition 4.4 it suffices to prove that $\mathcal{T}_{X \times Y}^o \subseteq \mathcal{T}_X^o \times \mathcal{T}_Y^o$. Consider any product $U_X \times U_Y$ which is open in the topology $\mathcal{T}_{X \times Y}^o$. We prove that $U_X \times U_Y \in \mathcal{T}_X^o \times \mathcal{T}_Y^o$. For this purpose we show that $U_X \in \mathcal{T}_X^o$ and $U_Y \in \mathcal{T}_Y^o$. Let $((x_d)_{d \in D}, x, \mathcal{I}) \in \mathcal{C}_X^o$, $x \in U_X$. Let $y \in U_Y$ and consider the net $(y_d)_{d \in D}$, where $y_d = y$ for every $d \in D$. By Propositions 2.7 and 4.3 we have $(((x_d, y_d))_{d \in D}, (x, y), \mathcal{I}) \in \mathcal{C}_{X \times Y}^o$, $(x, y) \in U_X \times U_Y$. Since $U_X \times U_Y \in \mathcal{T}_{X \times Y}^o$, we have $\{d \in D : (x_d, y_d) \notin U_X \times U_Y\} \in \mathcal{I}$. Now, since

$$\{d \in D : x_d \notin U_X\} \subseteq \{d \in D : (x_d, y_d) \notin U_X \times U_Y\},$$

we have $\{d \in D : x_d \notin U_X\} \in \mathcal{I}$. Therefore, $U_X \in \mathcal{T}_X^o$. Similarly, we can see that $U_Y \in \mathcal{T}_Y^o$. □

5 Ideal-o_2-convergence and ideal-o_2-topology

A generalization of the ideal-order-convergence in posets, the so-called ideal-o_2-convergence, is discussed in this section. Moreover, an investigation of the topological ideal-o_2-convergence in posets completes this section.

We will need the following notions.

Definition 5.1 [13,18] Let X be a poset. A net $(x_d)_{d\in D}$ in X is said to *o_2-converge* to a point $x \in X$ if there exist subsets M and N of X such that:

(1) $x = \bigvee M = \bigwedge N$.

(2) For each $m \in M$ and $n \in N$, there exists $d_0 \in D$ such that $m \leqslant x_d \leqslant n$ hold for all $d \geqslant d_0$.

In this case we write $(x_d)_{d\in D} \xrightarrow{o_2} x$.

Definition 5.2 [20] Let X be a poset and $x, y, z \in X$. We define:

(1) $x \ll_\alpha y$, if for every net $(x_d)_{d\in D}$ in X with $(x_d)_{d\in D} \xrightarrow{o_2} y$ there exists $d_0 \in D$ such that $x_d \geqslant x$ for every $d \geqslant d_0$.

(2) $z \rhd_\alpha y$, if for every net $(x_d)_{d\in D}$ in X with $(x_d)_{d\in D} \xrightarrow{o_2} y$ there exists $d_0 \in D$ such that $x_d \leqslant z$ for every $d \geqslant d_0$.

Definition 5.3 [20] A poset X is called *α-doubly continuous* if for each element $x \in X$, $x = \bigvee\{a \in X : a \ll_\alpha x\} = \bigwedge\{b \in X : b \rhd_\alpha x\}$.

Definition 5.4 [10] A poset X is called *O_2-doubly continuous* if it satisfies the following conditions:

(1) X is α-doubly continuous and

(2) if $y \ll_\alpha x$ and $z \rhd_\alpha x$, then there exist $A \subseteq_{fin} \{a \in X : a \ll_\alpha x\}$ and $B \subseteq_{fin} \{b \in X : b \rhd_\alpha x\}$ such that $y \ll_\alpha c$ and $z \rhd_\alpha c$ for each $c \in \bigcap\limits_{m\in A} \bigcap\limits_{n\in B} [m, n]$.

Definition 5.5 Let X be a poset. A net $(x_d)_{d\in D}$ in X is said to *$\mathcal{I}$-o_2-converge* to a point $x \in X$, where $\mathcal{I}$ is a non-trivial ideal on D, if there exist subsets M and N of X such that:

(1) $x = \bigvee M = \bigwedge N$.

(2) For each $m \in M$ and $n \in N$, $\{d \in D : x_d \notin [m, n]\} \in \mathcal{I}$.

Notation 5.6 Let $(x_d)_{d\in D}$ be a net in a poset X and let $\mathcal{I}$ be a non-trivial ideal on D. If $(x_d)_{d\in D}$ $\mathcal{I}$-o_2-converges to $x \in X$, then the point x is called the *$\mathcal{I}$-o_2-limit* of the net $(x_d)_{d\in D}$. In this case we write $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$.

Proposition 5.7 *Let X be a poset, $(x_d)_{d\in D}$ be a net in X and $\mathcal{I}$ a non-trivial ideal on D. Then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$ if and only if $(y_\lambda)^{\varphi_\mathcal{I}}_{\lambda\in\Lambda_\mathcal{I}} \xrightarrow{o_2} x$.*

Proof. Is similar to Proposition 2.3 (2). □

Proposition 5.8 *If a net $(x_d)_{d\in D}$ in X $\mathcal{I}$-o_2-converges to $x, y \in X$, where $\mathcal{I}$ is a non-trivial ideal on D, then $x = y$.*

Proof. It follows directly from Proposition 5.7 and the fact that a limit of o_2-convergence is uniquely determined (see Remark 3 (2) of [20]). □

Proposition 5.9 *Let $(x_d)_{d\in D}$ be a net in a poset X and let $\mathcal{I}$ be a non-trivial ideal on D. If $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$, where $x \in X$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$. Therefore, the $\mathcal{I}$-order-convergence implies the $\mathcal{I}$-o_2-convergence.*

Proof. Is a direct consequence of the Definitions 2.1 and 5.5. □

The converse of Proposition 5.9 is not necessarily true as the following example verifies.

Example 5.10 Let $(\mathbb{Z}, \leqslant)$ be the poset represented by the following diagram:

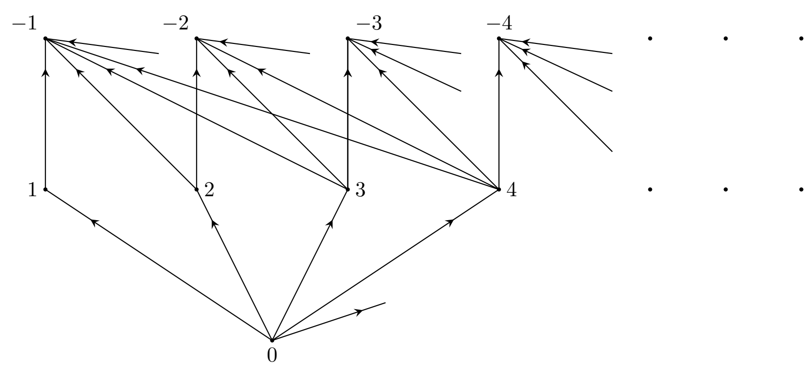

Figure 1: The poset $(\mathbb{Z}, \leqslant)$

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. For the net $(a_n)_{n\in\mathbb{N}}$, where $a_n = n$, $n \in \mathbb{N}$, we have $(a_n)_{n\in\mathbb{N}} \xrightarrow{\mathcal{I}-o_2} 0$. Indeed, for the subsets $M = \{0\}$ and $N = \{-n : n \in \mathbb{N}\}$ of $\mathbb{Z}$ we have:

(1) $0 = \bigvee M = \bigwedge N$.

(2) For every $n \in \mathbb{N}$, $\{m \in \mathbb{N} : a_m \notin [0, -n]\} \in \mathcal{I}$.

But the net $(a_n)_{n\in\mathbb{N}}$ does not $\mathcal{I}$-order-converge to 0, because the subset N of $\mathbb{Z}$ is not filtered.

Remark 5.11 From Proposition 5.9 we can, easily, see that Propositions 2.7, 2.8, Corollary 2.10, and Propositions 4.2, 4.3 are satisfied, also, for the notion of $\mathcal{I}$-o_2-convergence.

Notation 5.12 For an arbitrary poset X, we denote by $\mathcal{C}_X^{o_2}$ the class consisting of triads $((x_d)_{d\in D}, x, \mathcal{I})$, where $(x_d)_{d\in D}$ is a net in X, $x \in X$, and $\mathcal{I}$ is a non-trivial ideal on D such that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$. The corresponding topology $\tau(\mathcal{C}_X^{o_2})$ on X (see Proposition 3.1) is called the *ideal-o_2-topology* on X.

Proposition 5.13 *For any poset X, $\tau(\mathcal{C}_X^{o_2}) = \mathcal{T}_X^{o_2} \subseteq \mathcal{T}_X^{o}$.*

Proof. The equality is similar to the proof of Proposition 3.4 taking into account Proposition 5.7. The inclusion it follows immediately from the definitions. □

Proposition 5.14 *If $((x_d)_{d\in D}, x, \mathcal{I}) \in C_X^{o_2}$, then $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to $\tau(C_X^{o_2})$.*

Proof. It is similar to the proof of Proposition 3.2. □

Remark 5.15 The Corollary 3.8 and the Propositions 3.9, 4.4, 4.5 are satisfied for the ideal-o_2-convergence, replacing the correspondent notions.

Definition 5.16 The ideal-o_2-convergence in a poset X is called *topological*, if there exists a topology τ on X such that for every net $(x_d)_{d\in D}$ in X, $x \in X$ and for every non-trivial ideal $\mathcal{I}$ of D, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$ if and only if $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ.

Proposition 5.17 *The ideal-o_2-convergence in a poset X is topological if and only if the o_2-convergence in X is topological.*

Proof. Is similar to the proof of Proposition 3.12 taking into account Propositions 2.3 and 5.7. □

Proposition 5.18 *Let X be a chain and $x_1, x_2 \in X$. Then, $(x_1, x_2) \in \tau(\mathcal{C}_X^{o_2})$.*

Proof. It suffices to prove that for every $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X^{o_2}$, $x \in (x_1, x_2)$ we have that $\{d \in D : x_d \notin (x_1, x_2)\} \in \mathcal{I}$. Let $((x_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X^{o_2}$. Then, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$. Therefore, there exist subsets M and N of X such that:

(1) $x = \bigvee M = \bigwedge N$.

(2) For each $m \in M$ and $n \in N$, $\{d \in D : x_d \notin [m, n]\} \in \mathcal{I}$.

Let $m_0 \in M$ and $n_0 \in N$ such that $x_1 \leqslant m_0 < x < n_0 \leqslant x_2$. Then,

$$\{d \in D : x_d \notin [m_0, n_0]\} \in \mathcal{I}.$$

Since

$$\{d \in D : x_d \notin (x_1, x_2)\} \subseteq \{d \in D : x_d \notin [m_0, n_0]\},$$

we have $\{d \in D : x_d \notin (x_1, x_2)\} \in \mathcal{I}$. □

Proposition 5.19 Let X be a poset and $x, y, z \in X$. Then, the following statements hold:

(1) $x \ll_\alpha y$ if and only if for every net $(x_d)_{d\in D}$ in X and every non-trivial ideal $\mathcal{I}$ on D such that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} y$ we have $\{d \in D : x_d \ngeqslant x\} \in \mathcal{I}$.

(2) $z \rhd_\alpha y$ if and only if for every net $(x_d)_{d\in D}$ in X and every non-trivial ideal $\mathcal{I}$ on D such that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} y$ we have $\{d \in D : x_d \nleqslant z\} \in \mathcal{I}$.

Proof. (1) ($\Leftarrow$) Let $(x_d)_{d\in D}$ be net in X such that $(x_d)_{d\in D} \xrightarrow{o_2} y$. Consider the ideal $\mathcal{I}_D$. Then, $(x_d)_{d\in D} \xrightarrow{\mathcal{I}_D-o_2} y$ and therefore, $\{d \in D : x_d \ngeqslant x\} \in \mathcal{I}_D$. By the definition of $\mathcal{I}_D$ there exists $d_0 \in D$ such that $\{d \in D : x_d \ngeqslant x\} \subseteq \{d \in D : d \ngeqslant d_0\}$. Therefore, $x_d \geqslant x$ for every $d \geqslant d_0$.

($\Rightarrow$) Let $(x_d)_{d\in D}$ be a net in X and $\mathcal{I}$ a non-trivial ideal on D such that $(x_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} y$. Then, by Proposition 5.7, $(y_\lambda)_{\lambda\in\Lambda_\mathcal{I}}^{\varphi_\mathcal{I}} \xrightarrow{o_2} y$. Thus, there exists $\lambda_0 \in \Lambda_\mathcal{I}$ such that $y_\lambda \geqslant x$ for all $\lambda \geqslant \lambda_0$. By Proposition 2.3 $\{d \in D : x_d \ngeqslant x\} \in \mathcal{I}$.

(2) Is similar to the proof of (1). □

Proposition 5.20 *The ideal-o_2-convergence in a poset X is topological if and only if X is an O_2-doubly continuous poset.*

Proof. According to Theorem 4.11 in [10] and Proposition 5.17 we have the result. □

Corollary 5.21 *The ideal-o_2-convergence in every finite lattice, every chain or antichain is topological.*

Proof. Is a direct consequence of Remark 3.3 in [10] and Proposition 5.20. □

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