HIGHER ORDER APPROXIMATION OF THE DISTRIBUTION OF ANOVA TESTS FOR HIGH-DIMENSIONAL TIME SERIES

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Abstract. Analysis of variance (ANOVA) is tailored for independent observations. Recently, there has been considerable demand for the ANOVA of high-dimensional and dependent observations in many fields. Thus, it is important to analyze the differences among big data's averages of areas from all over the world, such as the financial and manufacturing industries. However, the numerical accuracy of ANOVA for such observations has been inadequately developed. Thus, herein, we study the Edgeworth expansion of distribution of ANOVA tests for high-dimensional and dependent observations. Specifically, we present the second-order approximation of classical test statistics proposed for independent observations. We also provide numerical examples for simulated high-dimensional time-series data.

1 Introduction Analysis of variance (ANOVA) is a type of hypothesis testing method for the null hypothesis of "no treatment effect". It is generally used to test the null hypothesis that the means of three or more populations of within-group means are all equal. Moreover, this method shows whether the within-group means are equal.

ANOVA has a long history in statistics. Gauss founded it in the late 1800s, and Markoff developed it in the early 1900s. Many test statistics for ANOVA and multivariate analysis of variance (MANOVA) have been proposed, primarily under independent disturbances of a MANOVA model. The early applications can be found in [10] and [14]. In addition, [3] and [4] obtained general theoretical results. They derived asymptotic expansions of the null and non-null distributions of the likelihood ratio test-statistics. [2] discussed higher-order approximations (Edgeworth expansions) and their validity. Furthermore, [8] developed higher-order asymptotic expansions of the null and non-null distributions of the likelihood ratio test statistic, Lawley-Hotelling test statistic, and Bartlett-Nanda-Pillai test statistic under high-dimensional and i.i.d. settings. Moreover, in a time-series analysis, [13] discussed the Edgeworth expansions for various statistics. Recently, under a high-dimensional time-series setting, [12] discussed the first-order asymptotics of Lawley-Hotelling test statistic, likelihood ratio test statistic, and Bartlett-Nanda-Pillai test statistic.

In the current era of big data, an analysis of high-dimensional time-series data is required in practical problems, such as those in economics, finance, and bioinformatics. Especially, the accuracy of statistical decisions for high-dimensional time-series data has become increasingly important. Many data analysts need accurate methods for the equivalence of the within-group means of big data, because this analysis is very basic. MANOVA will be useful for these needs. However, from the viewpoint of the numerical accuracy of approximations, higher-order asymptotics of ANOVA test statistics for high-dimensional data are not adequately developed. In the present study, we focus on Edgeworth expansions of distributions of Lawley-Hotelling test statistic, likelihood ratio test statistic, and Bartlett-Nanda-Pillai test statistic.

In this paper, we consider a one-way MANOVA model whose disturbance process is generated by a high-dimensional stationary process.

Herein, let δ_{ij} be Kronecker's delta, $\boldsymbol{I}_p$ be the p-dimensional identity matrix, $\boldsymbol{O}_{\mathrm{P}}(a_n)$ be an order of the probability that is, for a sequence of random variables $\{X_n\}$ and $\{a_n\}$, $0 < a_n \in \mathbb{R}$, $\{a_n^{-1}X_n\}$ is bounded in probability, and let $\boldsymbol{O}_{\mathrm{P}}^U(\cdot)$ be a $p \times p$ matrix whose elements are probability order $\boldsymbol{O}_{\mathrm{P}}(\cdot)$ with respect to all elements uniformly. In addition, let $|\cdot|$ be the determinant of $\cdot$, $\|\cdot\|$ be the Euclidean norm of $\cdot$, and $\mathbb{1}$ be the indicator function.

2 Problems and Preliminaries Throughout this paper, we consider the MANOVA model under which a q-tuple of p-dimensional time series $\boldsymbol{X}_{i1}, \cdots, \boldsymbol{X}_{in_i}$, $i = 1, \ldots, q$ satisfies

$$\boldsymbol{X}_{it} = \boldsymbol{\mu} + \boldsymbol{\alpha}_i + \boldsymbol{\epsilon}_{it}, \quad t = 1, \cdots, n_i, \quad i = 1, \cdots, q, \tag{1}$$

where $\mu \in \mathbb{R}^p$ is the global mean of the model (1), the disturbances $\boldsymbol{\epsilon}_i \equiv \{\boldsymbol{\epsilon}_{i1}, \cdots \boldsymbol{\epsilon}_{in_i}\}$ are kth-order stationary with mean $\mathbf{0}$, lag u autocovariance matrix $\boldsymbol{\Gamma}(u) = (\Gamma_{jk}(u))_{1 \le j,k \le p}$, $u \in \mathbb{Z}$, and n_i is the observation length of the ith group. Furthermore, the total observation length of all groups $n = \sum_{i=1}^q n_i$ and $\{\boldsymbol{\epsilon}_i\}$, $i = 1, \cdots, q$ are mutually independent. We impose a further standard assumption, which is called homoscedasticity (e.g., Ch. 8.9 of [1]). Now $\boldsymbol{\alpha}_i$ denotes the effect of the ith treatment, which measures the deviation from $\boldsymbol{\mu}$ satisfying $\sum_{i=1}^q \boldsymbol{\alpha}_i = \mathbf{0}$. Because the treatment effects sum to zero, we discuss the problem of testing:

$$H: \ \boldsymbol{\alpha}_1 = \cdots = \boldsymbol{\alpha}_q = \mathbf{0} \text{ vs. } A: \ \boldsymbol{\alpha}_i \neq \mathbf{0} \text{ for some } i. \tag{2}$$

The null hypothesis H implies that all effects are zero.

For our high-dimensional dependent observations, we use the Lawley-Hotelling test statistic $\tilde{T}_1$, likelihood ratio test statistic $\tilde{T}_2$, and Bartlett-Nanda-Pillai test statistic $\tilde{T}_3$:

$$\begin{aligned} \tilde{T}_1 &\equiv n\mathrm{tr}\hat{\mathcal{S}}_H\hat{\mathcal{S}}_E^{-1}, \\ \tilde{T}_2 &\equiv -n\log|\hat{\mathcal{S}}_E|/|\hat{\mathcal{S}}_E + \hat{\mathcal{S}}_H|, \\ \tilde{T}_3 &\equiv n\mathrm{tr}\hat{\mathcal{S}}_H(\hat{\mathcal{S}}_E + \hat{\mathcal{S}}_H)^{-1}, \end{aligned}$$

where

$$\hat{\mathcal{S}}_H \equiv \sum_{i=1}^q n_i(\hat{\boldsymbol{X}}_{i\cdot} - \hat{\boldsymbol{X}}_{\cdot\cdot})(\hat{\boldsymbol{X}}_{i\cdot} - \hat{\boldsymbol{X}}_{\cdot\cdot})' \text{ and } \hat{\mathcal{S}}_E \equiv \sum_{i=1}^q\sum_{t=1}^{n_i}(\boldsymbol{X}_{it} - \hat{\boldsymbol{X}}_{i\cdot})(\boldsymbol{X}_{it} - \hat{\boldsymbol{X}}_{i\cdot})' \ \text{ with}$$

$$\hat{\boldsymbol{X}}_{i\cdot} = \frac{1}{n_i}\sum_{t=1}^{n_i}\boldsymbol{X}_{it} \text{ and } \hat{\boldsymbol{X}}_{\cdot\cdot} = \frac{1}{n}\sum_{i=1}^q\sum_{t=1}^{n_i}\boldsymbol{X}_{it}.$$

Now, we call $\hat{\mathcal{S}}_H$ and $\hat{\mathcal{S}}_E$ the between-group sums of squares and products (SSP) and the within-group SSP, respectively. To derive the stochastic expansion of $n^{-1}\hat{\mathcal{S}}_E$ in Section 4, we introduce

$$\hat{\mathcal{S}}_i \equiv (n_i - 1)^{-1}\sum_{t=1}^{n_i}(\boldsymbol{X}_{it} - \hat{\boldsymbol{X}}_{i\cdot})(\boldsymbol{X}_{it} - \hat{\boldsymbol{X}}_{i\cdot})', \tag{3}$$

$$\boldsymbol{V} = \sum_{i=1}^q \sqrt{\frac{n_i}{n}}\boldsymbol{V}_i, \ \ \boldsymbol{V}_i = \sqrt{n_i}(\hat{\mathcal{S}}_i - \boldsymbol{I}_p). \tag{4}$$

In addition, to derive the Edgeworth expansion of distributions of the three test statistics under H, we impose the following assumptions:

Assumption 1

$$\frac{p^{3/2}}{\sqrt{n}} \to 0 \ \ as\ n, p \to \infty, \tag{5}$$

$$\frac{n_i}{n} \to \rho_i > 0 \ \ as\ n \to \infty, \tag{6}$$

where ρ_i is a positive constant which is independent of n and p for every i.

Here, the condition (6) implies the orders of n_i and n are asymptotically the same.

Assumption 2 *For the p-vectors $\boldsymbol{\epsilon}_{it} = (\epsilon_{it}^{(1)}, \cdots, \epsilon_{it}^{(p)})'$ given in (1), there exists an $\ell \geq 0$ with*

$$\sum_{t_1,\ldots,t_{k-1}=-\infty}^{\infty} \{1 + |t_j|\}^{\ell} |c^i_{a_1,\cdots,a_k}(t_1, \cdots, t_{k-1})| < \infty,$$

for $j = 1, \cdots, k-1$ and any k-tuple $a_1, \cdots, a_k \in \{1, \cdots, p\}$ and $i = 1, \cdots, q$, when $k = 2, 3, \cdots$. Here $c^i_{a_1,\cdots,a_k}(t_1, \cdots, t_{k-1}) = \mathrm{cum}\{\epsilon_{it_1}^{(a_1)}, \cdots, \epsilon_{it_k}^{(a_k)}\}$.

If $\epsilon_{it}^{(a_{m_1})}, \cdots, \epsilon_{it}^{(a_{m_h})}$ for any h-tuple $m_1, \cdots, m_h \in \{1, \cdots, k\}$ are independent of $\epsilon_{it}^{(a_{m_{h+1}})}, \cdots, \epsilon_{it}^{(a_{m_k})}$ for the remaining $(k-h)$-tuple $m_{h+1}, \cdots, m_k \in \{1, \cdots, k\}$, then $c^i_{a_{m_1},\cdots,a_{m_k}}(t_{m_1}, \cdots, t_{m_k-1}) = 0$ ([5], p. 19). Assumption 2 implies that if the time points of a group of $\epsilon_{it_l}^{(a_*)}$'s are well separated from the remaining time points of $\epsilon_{it_s}^{(a_*)}$'s, the values of $c^i_{a_1,\cdots,a_k}(t_1, \cdots, t_{k-1})$ become small (and hence summable) (see [5, p.19]).This property is natural for stochastic processes with short memory. We introduce a concrete example of the high-dimensional process ϵ_i's which satisfy Assumption 2. That is *DCC-GARCH*(p, q) model (9). [9] expressed a typical component of this model as

$$\sum_{l=0}^{\infty} \sum_{j_l < j_{l-1} < \cdots < j_1 < t} b_{t-j_1} \cdots b_{j_{l-1}-j_l} \eta_{j_1} \cdots \eta_{j_l} \tag{7}$$

where η_j's are i.i.d. with $E\eta_j^2 < \infty$. By (7), we can easily check this model satisfies Assumption 2.

Assumption 3

$$\boldsymbol{\Gamma}(j) = \mathbf{0} \ \ for\ all\ j \neq 0. \tag{8}$$

Assumption 3 means that the disturbance process $\{\boldsymbol{\epsilon}_i\}$ is an uncorrelated process. Now, note that the condition (8) is not very severe because of the very practical nonlinear time-series model

DCC-GARCH(q,r)

$$
\begin{aligned}
\boldsymbol{\epsilon}_{it} = {\boldsymbol{H}_{it}}^{1/2}\boldsymbol{\eta}_{it}, \quad \boldsymbol{\eta}_{it} \overset{i.i.d.}{\sim} (\mathbf{0}, \boldsymbol{I}_p)\,, \\
\boldsymbol{H}_{it} = \boldsymbol{D}_{it}\boldsymbol{R}_{it}\boldsymbol{D}_{it}, \quad \boldsymbol{D}_{it} = diag\left[\sqrt{\sigma_{it}^{(1)}}, \cdots, \sqrt{\sigma_{it}^{(p)}}\right], \\
\boldsymbol{\epsilon}_{it} = \begin{pmatrix} \epsilon_{it}^{(1)} \\ \vdots \\ \epsilon_{it}^{(p)} \end{pmatrix}, \quad \sigma_{it}^{(j)} = c_j + a_j \sum_{l=1}^{r} \left\{\epsilon_{i,t-l}^{(j)}\right\}^2 + b_j \sum_{l=1}^{q} \sigma_{i,t-l}^{(j)}, \\
\boldsymbol{R}_{it} = \left(diag\left[\boldsymbol{Q}_{it}\right]\right)^{-1/2} \boldsymbol{Q}_{it} \left(diag\left[\boldsymbol{Q}_{it}\right]\right)^{-1/2}, \\
\tilde{\boldsymbol{\epsilon}}_{it} = \begin{pmatrix} \tilde{\epsilon}_{it}^{(1)} \\ \vdots \\ \tilde{\epsilon}_{it}^{(p)} \end{pmatrix}, \quad \tilde{\epsilon}_{it}^{(j)} = \frac{\epsilon_{it}^{(j)}}{\sqrt{\sigma_{it}^{(j)}}}, \quad \boldsymbol{Q}_{it} = (1-\alpha-\beta)\tilde{\boldsymbol{Q}} + \alpha\tilde{\boldsymbol{\epsilon}}_{i,t-1}\tilde{\boldsymbol{\epsilon}}_{i,t-1}' + \beta\boldsymbol{Q}_{i,t-1},
\end{aligned}
\tag{9}
$$

(see [7]) satisfies (8). Here, $\tilde{\boldsymbol{Q}}$, the unconditional correlation matrix, is a constant positive semidefinite matrix, and $\boldsymbol{H}_{it}$'s are measurable with respect to $\boldsymbol{\eta}_{i,t-1}, \boldsymbol{\eta}_{i,t-2}, \cdots$.

3 Main Results

In what follows, without loss of generality, we assume $\boldsymbol{\Gamma}(0) = \boldsymbol{I}_p$, and $\boldsymbol{\mu} = \mathbf{0}$ because the three test statistics $\tilde{T}_1$, $\tilde{T}_2$, and $\tilde{T}_3$ are invariant under linear transformation, our discussion for $\boldsymbol{X}_{it}$ remains valid for the case where we apply a linear transformation $\{\boldsymbol{\Gamma}(0)\}^{-1/2}$ to $\boldsymbol{X}_{it}$. We derive the stochastic expansion of the standardized versions T_1, T_2, and T_3 of the three test statistics $\tilde{T}_1$ (Lawley-Hotelling test statistic), $\tilde{T}_2$ (likelihood ratio test statistic), and $\tilde{T}_3$ (Bartlett-Nanda-Pillai test statistic), respectively:

$$
T_1 \equiv \frac{1}{\sqrt{2(q-1)}}\left\{\frac{n}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H\hat{\mathcal{S}}_E^{-1} - \sqrt{p}(q-1)\right\}, \tag{10}
$$

$$
T_2 \equiv -\frac{1}{\sqrt{2(q-1)}}\left\{\frac{n}{\sqrt{p}}\log|\hat{\mathcal{S}}_E|/|\hat{\mathcal{S}}_E + \hat{\mathcal{S}}_H| + \sqrt{p}(q-1)\right\}, \tag{11}
$$

$$
T_3 \equiv \frac{1}{\sqrt{2(q-1)}}\left\{\frac{n}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H(\hat{\mathcal{S}}_E + \hat{\mathcal{S}}_H)^{-1} - \sqrt{p}(q-1)\right\}. \tag{12}
$$

This section provides their Edgeworth expansions. Lemmas and all proofs are provided in Section 4.

Theorem 1 *Suppose Assumptions 1-3. Then, under the null hypothesis H, we have the following Edgeworth expansions:*

$$
\begin{aligned}
P(T_i < z) &= \Phi(z) - \phi(z)\left\{p^{-1/2}\cdot\frac{c_3}{6}(z^2-1) + p^{-1}\cdot\frac{c_4}{24}(z^3-3z)\right\} \\
&\quad +\boldsymbol{o}\left(p^{-1}\right), \quad (i=1,2,3)
\end{aligned}
\tag{13}
$$

where

$$
\Phi(z) = \int_{-\infty}^{z}\phi(y)dy, \quad \phi(y) = (2\pi)^{-1/2}exp\left(-\frac{y^2}{2}\right),
$$

and

$$
\begin{aligned}
c_3 &= \left(\frac{2}{q-1}\right)^{3/2}\left\{q-3+3\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2-\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3\right\},\\
c_4 &= \left(\frac{2}{q-1}\right)^{2}\left\{q-4+6\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2-4\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3-\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^4\right\}.
\end{aligned}
$$

Remark 1 *This asymptotic result is an extended version of [8] and [12]. Our setting in Section 2 shows we can apply this result to not only high-dimensional i.i.d. data (that was discussed in [8]) but also high-dimensional time series data. Also, an approximation of the three test statistics T_i, $i = 1, 2, 3$ in Theorem 1 is more accurate than one of them in [12] because we investigated the higher order asymptotic structure of T_i, $i = 1, 2, 3$ by using Edgeworth expansion method.*

4 Asymptotic theory for main results In this section, we provide the lemmas and their proofs. In what follows, we use the same linear transformation as in Section 3. First, the stochastic expansion of $n^{-1}\hat{\mathcal{S}}_E$ and $\hat{\mathcal{S}}_H$ is given.

Lemma 1 *Suppose Assumptions 1-3. Then, under null hypothesis H, the following (14)-(16) hold true;*

$$\frac{1}{n}\hat{\mathcal{S}}_E = \boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{q}{n}\boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right), \tag{14}$$

$$\left\{\frac{1}{n}\hat{\mathcal{S}}_E\right\}^{-1} = \boldsymbol{I}_p - \frac{1}{\sqrt{n}}\boldsymbol{V} + \frac{1}{n}(\boldsymbol{V}^2 + q\boldsymbol{I}_p) + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right), \tag{15}$$

$$\hat{\mathcal{S}}_H = \boldsymbol{O}_{\mathrm{P}}^U(1). \tag{16}$$

Proof (Lemma 1) *By (4), write $n^{-1}\hat{\mathcal{S}}_E$ as*

$$
\begin{aligned}
\frac{1}{n}\hat{\mathcal{S}}_E &= \frac{1}{n}\sum_{i=1}^{q}(n_i-1)\hat{\mathcal{S}}_i\\
&= \frac{1}{n}\sum_{i=1}^{q}(n_i-1)\left(\boldsymbol{I}_p+\frac{1}{\sqrt{n_i}}\boldsymbol{V}_i\right)\\
&= \boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{q}{n}\boldsymbol{I}_p - \frac{1}{n}\sum_{i=1}^{q}\frac{1}{\sqrt{n_i}}\boldsymbol{V}_i. \qquad (17)
\end{aligned}
$$

In what follows, for each i, we will show $\boldsymbol{V}_i = \boldsymbol{O}_{\mathrm{P}}^U(1)$. By the null hypothesis H and $\boldsymbol{\mu} = \boldsymbol{0}$, we rewrite $\hat{\mathcal{S}}_i$ as follows:

$$
\begin{aligned}
\hat{\mathcal{S}}_i &= n_i(n_i-1)^{-1}\left(\frac{1}{n_i}\sum_{t=1}^{n_i}\boldsymbol{X}_{it}\boldsymbol{X}_{it}' - \hat{\boldsymbol{X}}_{i\cdot}\hat{\boldsymbol{X}}_{i\cdot}'\right)\\
&= n_i(n_i-1)^{-1}(\boldsymbol{A}-\boldsymbol{B})\ \textit{(say)}, \qquad (18)
\end{aligned}
$$

where $\boldsymbol{A} = 1/n_i \sum_{t=1}^{n_i} \boldsymbol{X}_{it}\boldsymbol{X}_{it}'$ *and* $\boldsymbol{B} = \hat{\boldsymbol{X}}_{i\cdot}\hat{\boldsymbol{X}}_{i\cdot}'$. *We observe*

$$\begin{aligned} &E\{\boldsymbol{A}\} = \boldsymbol{I}_p \quad \text{and} \\ &Cov\{A_{jk}, A_{lm}\} \\ &\quad = \frac{1}{n_i}\sum_{s=-n_i+1}^{n_i-1}\left(1 - \frac{|s|}{n_i}\right)\{c_{jl}(s)c_{km}(s) + c_{jm}(s)c_{kl}(s) + c^i_{jklm}(0,s,s)\} \\ &\quad = \boldsymbol{O}\left(\frac{1}{n_i}\right) = \boldsymbol{O}\left(\frac{1}{n}\right) \quad \text{uniformly in } j,k,l,m \text{ by Assumption 2.} \end{aligned} \tag{19}$$

Hence, $\boldsymbol{A} = \boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^U(1/\sqrt{n})$. *Next, we observe*

$$\begin{aligned} &E(\hat{\boldsymbol{X}}_{i\cdot}) = \boldsymbol{\alpha}_i \quad \text{and} \\ &Cov\{\hat{\boldsymbol{X}}_{i\cdot}, \hat{\boldsymbol{X}}_{i\cdot}\} \\ &\quad = \left\{\frac{1}{n_i}\sum_{s=-n_i+1}^{n_i-1}\left(1 - \frac{|s|}{n_i}\right)c_{jk}(s)\right\} \\ &\quad = \boldsymbol{O}^U\left(\frac{1}{n_i}\right). \end{aligned} \tag{20}$$

Thus,

$$\boldsymbol{B} = \boldsymbol{O}_{\mathrm{P}}^U\left(\frac{1}{n}\right). \tag{21}$$

Therefore,

$$\hat{\mathcal{S}}_i = \boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^U\left(\frac{1}{\sqrt{n}}\right),$$

and

$$\boldsymbol{V}_i = \boldsymbol{O}_{\mathrm{P}}^U(1). \tag{22}$$

By using (17) and (22), we can get

$$\frac{1}{n}\hat{\mathcal{S}}_E = \boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{q}{n}\boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right), \tag{14}$$

and

$$\left\{\frac{1}{n}\hat{\mathcal{S}}_E\right\}^{-1} = \left\{\boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{q}{n}\boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right)\right\}^{-1} = \{\boldsymbol{I}_p - \boldsymbol{M}_n\}^{-1} \text{ (say).}$$

It is known that

$$\{\boldsymbol{I}_p - \boldsymbol{M}_n\}^{-1} = \sum_{k=0}^{\infty}\boldsymbol{M}_n^k \tag{23}$$

(see p. 169 of [11]). From Assumption 1, it follows that

$$\begin{aligned}\boldsymbol{M}_n^0 &= \boldsymbol{I}_p,\\ \boldsymbol{M}_n &= -\frac{1}{\sqrt{n}}\boldsymbol{V} + \frac{q}{n}\boldsymbol{I}_p + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right),\\ \boldsymbol{M}_n^2 &= \frac{1}{n}\boldsymbol{V}^2 + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right),\\ \boldsymbol{M}_n^k &= \boldsymbol{O}_{\mathrm{P}}\left(n^{-\frac{k}{2}}\right)\boldsymbol{H}, \quad k \geq 3,\end{aligned}$$

where $\boldsymbol{H}$ *is a* $p \times p$*-matrix and* $\boldsymbol{H} = \boldsymbol{O}_{\mathrm{P}}^U(1)$*. Then, we obtain*

$$\left\{\frac{1}{n}\hat{\mathcal{S}}_E\right\}^{-1} = \boldsymbol{I}_p - \frac{1}{\sqrt{n}}\boldsymbol{V} + \frac{1}{n}(\boldsymbol{V}^2 + q\boldsymbol{I}_p) + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right). \tag{15}$$

Next, we show $\hat{\mathcal{S}}_H = \boldsymbol{O}_{\mathrm{P}}^U(1)$*. To this end, we recall*

$$\hat{\mathcal{S}}_H = \sum_{i=1}^{q} n_i(\hat{\boldsymbol{X}}_{i\cdot} - \hat{\boldsymbol{X}}_{\cdot\cdot})(\hat{\boldsymbol{X}}_{i\cdot} - \hat{\boldsymbol{X}}_{\cdot\cdot})'. \tag{24}$$

From (20), we observe that $\hat{\boldsymbol{X}}_{i\cdot} = \boldsymbol{\alpha}_i + \boldsymbol{O}_{\mathrm{P}}^U\left(1/\sqrt{n_i}\right)$, $\sum_{i=1}^q \boldsymbol{\alpha}_i = \boldsymbol{0}$*, and similarly,* $\hat{\boldsymbol{X}}_{\cdot\cdot} = \boldsymbol{O}_{\mathrm{P}}^U(1/\sqrt{n})$*. Thus, we have*

$$\hat{\mathcal{S}}_H = \boldsymbol{O}_{\mathrm{P}}^U(1). \tag{16}$$

Note that (14), (15), and (16) are derived for the multivariate i.i.d. case, e.g., [8, p.164].

Lemma 2 *Suppose Assumputions 1-3. Then, under null hypothesis* H*, it holds that*

$$\tilde{T}_i = U^{(0)} + \frac{1}{\sqrt{n}}U^{(1)} + \frac{1}{n}\left(U^{(2)} + \beta_i R^{(2)}\right) + \boldsymbol{O}_{\mathrm{P}}\left(\frac{p^{3/2}}{n}\right), \; i = 1, 2, 3, \tag{25}$$

where

$$\begin{aligned}U^{(0)} &= \mathrm{tr}\hat{\mathcal{S}}_H,\\ U^{(1)} &= -\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\},\\ U^{(2)} &= \mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\},\\ R^{(2)} &= \mathrm{tr}\{\hat{\mathcal{S}}_H^2\}, \text{ and}\\ (\beta_1, \beta_2, \beta_3) &= \left(0, -\frac{1}{2}, -1\right).\end{aligned}$$

Proof (Lemma 2) *From Lemma 1, it follows that*

$$\begin{aligned}\tilde{T}_1 &= \mathrm{tr}\left[\hat{\mathcal{S}}_H\left\{\frac{1}{n}\hat{\mathcal{S}}_E\right\}^{-1}\right]\\ &= \mathrm{tr}\left[\hat{\mathcal{S}}_H\left\{\boldsymbol{I}_p - \frac{1}{\sqrt{n}}\boldsymbol{V} + \frac{1}{n}(\boldsymbol{V}^2 + q\boldsymbol{I}_p) + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right)\right\}\right]\\ &= \mathrm{tr}\hat{\mathcal{S}}_H - \frac{1}{\sqrt{n}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\} + \frac{1}{n}\mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\} + \mathrm{tr}\left\{\hat{\mathcal{S}}_H \cdot \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right)\right\}.\end{aligned}$$

From (16),

$$\tilde{T}_1 = \mathrm{tr}\hat{\mathcal{S}}_H - \frac{1}{\sqrt{n}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\} + \frac{1}{n}\mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\} + \boldsymbol{O}_{\mathrm{P}}\left(\frac{p^{3/2}}{n}\right). \tag{25}$$

Next, to derive (25), first, note that for every matrix $\boldsymbol{F}$ *and the matrix differential operator* d

$$\begin{aligned} d\log|\boldsymbol{F}| &= \mathrm{tr}(\boldsymbol{F}^{-1}d\boldsymbol{F}), \\ d\boldsymbol{F}^{-1} &= -\boldsymbol{F}^{-1}(d\boldsymbol{F})\boldsymbol{F}^{-1}, \end{aligned}$$

and (23) (e.g., [11]). Then, a modification of Proposition 6.1.5 of [6] and Lemma 1 shows that for

$$f := n\log\left|\boldsymbol{I}_p + \frac{1}{n}\hat{\mathcal{S}}_H\left\{\frac{1}{n}\hat{\mathcal{S}}_E^{-1}\right\}\right|,$$

we have that

$$f = \sum_{m=0}^{\infty}\frac{1}{m!}d^m f.$$

where d^m*'s are* m*-th differentials of* f *which are calculated by*

$$\begin{aligned} d^0 f &= \mathrm{tr}\{\hat{\mathcal{S}}_H\} - \frac{1}{2n}\mathrm{tr}\{\hat{\mathcal{S}}_H^2\} + \boldsymbol{O}_{\mathrm{P}}\left(p\cdot n^{-2}\right), \\ d^1 f &= -\frac{1}{\sqrt{n}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\} + \frac{1}{n}\mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\} + \boldsymbol{O}_{\mathrm{P}}\left(p^2\cdot n^{-3/2}\right), \\ d^m f &= \boldsymbol{O}_{\mathrm{P}}\left(p\cdot n^{-2}\right), \quad m \geq 2. \end{aligned}$$

Thus, we obtain

$$\tilde{T}_2 = \mathrm{tr}\{\hat{\mathcal{S}}_H\} - \frac{1}{\sqrt{n}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\} + \frac{1}{n}\left[\mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\} - \frac{1}{2}\mathrm{tr}\{\hat{\mathcal{S}}_H^2\}\right] + \boldsymbol{O}_{\mathrm{P}}\left(p^2\cdot n^{-3/2}\right). \tag{25}$$

From Lemma 1 and (23), it follows that

$$\begin{aligned} \tilde{T}_3 &= \mathrm{tr}\left[\hat{\mathcal{S}}_H\left\{\frac{1}{n}\hat{\mathcal{S}}_E + \frac{1}{n}\hat{\mathcal{S}}_H\right\}^{-1}\right] \\ &= \mathrm{tr}\left[\hat{\mathcal{S}}_H\left\{\boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} + \frac{1}{n}(\hat{\mathcal{S}}_H - q\boldsymbol{I}_p) + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right)\right\}^{-1}\right] \\ &= \mathrm{tr}\left[\hat{\mathcal{S}}_H\sum_{k=0}^{\infty}\left\{-\frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{1}{n}(\hat{\mathcal{S}}_H - q\boldsymbol{I}_p) + \boldsymbol{O}_{\mathrm{P}}^U\left(n^{-3/2}\right)\right\}^k\right]. \end{aligned}$$

From (16),

$$\tilde{T}_3 = \mathrm{tr}\hat{\mathcal{S}}_H - \frac{1}{\sqrt{n}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\} + \frac{1}{n}[\mathrm{tr}\{\hat{\mathcal{S}}_H(\boldsymbol{V}^2 + q\boldsymbol{I}_p)\} - \mathrm{tr}\{\hat{\mathcal{S}}_H^2\}] + \boldsymbol{O}_{\mathrm{P}}\left(\frac{p^{3/2}}{n}\right) \tag{25}$$

(for the multivariate i.i.d. case, e.g., [8, p.164]).

Lemma 3 *Suppose Assumptions 1-3. Then, under the null hypothesis H, it holds that*

$$cum^{(J)}(\overbrace{\frac{1}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{K},\overbrace{-\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\},\cdots}^{L},\overbrace{\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}^2\},\cdots}^{M},\overbrace{\frac{q}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{M_0},\overbrace{\frac{\beta_i}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H^2,\cdots}^{N})$$

$$= \boldsymbol{O}\left(p^{1-J/2+N}\cdot n^{-2L-4M-M_0-N}\right) \tag{26}$$

$$= \boldsymbol{o}\left(p^{1-J/2-6L-12M-3M_0-2N}\right), \tag{27}$$

where $K, L, M, M_0, N \geq 0$, $J = K + L + M + +M_0 + N \geq 1$ and

$$(\beta_1, \beta_2, \beta_3) = \left(0, -\frac{1}{2}, -1\right).$$

Proof (Lemma 3) *First, under $\boldsymbol{\mu} = \boldsymbol{0}$ and null hypothesis H, we prepare S_{jk} and V_{jk} as (j,k)th components of $\hat{\mathcal{S}}_H$ and $\boldsymbol{V}$, respectively:*

$$S_{jk} = \sum_{i_1=1}^{q}\frac{1}{n_{i_1}}\sum_{r=1}^{n_{i_1}}\sum_{s=1}^{n_{i_1}}\epsilon_{i_1r}^{(j)}\epsilon_{i_1s}^{(k)} - \frac{1}{n}\sum_{i_2=1}^{q}\sum_{i_3=1}^{q}\sum_{t=1}^{n_{i_2}}\sum_{u=1}^{n_{i_3}}\epsilon_{i_2t}^{(j)}\epsilon_{i_3u}^{(k)}, \tag{28}$$

$$V_{jk} = \frac{1}{\sqrt{n}}\sum_{i_4=1}^{q}\frac{n_{i_4}}{n_{i_4}-1}\sum_{r=1}^{n_{i_4}}\epsilon_{i_4r}^{(j)}\epsilon_{i_4r}^{(k)} - \frac{1}{\sqrt{n}}\sum_{i_4=1}^{q}\frac{1}{n_{i_4}-1}\sum_{s=1}^{n_{i_4}}\sum_{t=1}^{n_{i_4}}\epsilon_{i_4s}^{(j)}\epsilon_{i_4t}^{(k)} - \sqrt{n}\delta_{jk}. \tag{29}$$

Here, we can write

$$\begin{aligned}
&cum^{(J)}(\overbrace{\frac{1}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{K},\overbrace{-\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\},\cdots}^{L},\overbrace{\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}^2\},\cdots}^{M},\overbrace{\frac{q}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{M_0},\overbrace{\frac{\beta_i}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H^2,\cdots}^{N})\\
= \; &(-1)^L q^{M_0}\beta_i^N\cdot p^{-J/2}n^{-L/2-M-M_0-N}\\
&\times\; cum^{(J)}(\overbrace{\mathrm{tr}\hat{\mathcal{S}}_H,\cdots}^{K+M_0},\overbrace{\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\},\cdots}^{L},\overbrace{\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}^2\},\cdots}^{M},\overbrace{\mathrm{tr}\hat{\mathcal{S}}_H^2,\cdots}^{N}).
\end{aligned} \tag{30}$$

By (28) and (29), a typical term of the cumulant in (30) is

$$\begin{aligned}
&\sum_{j_{1,1}}^{p}\cdots\sum_{j_{1,K+M_0}}^{p}\sum_{j_{2,1}}^{p}\cdots\sum_{j_{2,L}}^{p}\sum_{j_{3,1}}^{p}\cdots\sum_{j_{3,M}}^{p}\sum_{j_{4,1}}^{p}\sum_{k_{4,1}}^{p}\cdots\sum_{j_{4,N}}^{p}\sum_{k_{4,N}}^{p}\\
&\sum_{r_{1,1}}^{n_i}\sum_{s_{1,1}}^{n_i}\cdots\sum_{r_{1,K+M_0}}^{n_i}\sum_{s_{1,K+M_0}}^{n_i}\sum_{r_{2,1}}^{n_i}\sum_{s_{2,1}}^{n_i}\cdots\sum_{r_{2,L}}^{n_i}\sum_{s_{2,L}}^{n_i}\sum_{r_{3,1}}^{n_i}\sum_{s_{3,1}}^{n_i}\cdots\sum_{r_{3,M}}^{n_i}\sum_{s_{3,M}}^{n_i}\\
&\sum_{r_{4,1}}^{n_i}\sum_{s_{4,1}}^{n_i}\sum_{t_{4,1}}^{n_i}\sum_{u_{4,1}}^{n_i}\cdots\sum_{r_{4,N}}^{n_i}\sum_{s_{4,N}}^{n_i}\sum_{t_{4,N}}^{n_i}\sum_{u_{4,N}}^{n_i}\boldsymbol{O}\left(n^{-K-5L/2-4M-M_0-2N}\right)\\
&\times\; cum^{(J)}[\epsilon_{ir_{1,1}}^{(j_{1,1})}\epsilon_{is_{1,1}}^{(j_{1,1})},\cdots,\epsilon_{ir_{2,1}}^{(j_{2,1})}\epsilon_{is_{2,1}}^{(j_{2,1})},\cdots,\epsilon_{ir_{3,1}}^{(j_{3,1})}\epsilon_{is_{3,1}}^{(j_{3,1})},\cdots,\\
&\qquad\epsilon_{ir_{4,1}}^{(j_{4,1})}\epsilon_{is_{4,1}}^{(k_{4,1})}\epsilon_{it_{4,1}}^{(k_{4,1})}\epsilon_{iu_{4,1}}^{(j_{4,1})},\cdots].
\end{aligned} \tag{31}$$

By using the properties of the cumulant and Theorem 2.3.2 in [5, p.19-21], the cumulant appearing in (31) has a typical main-order term

$$
\begin{aligned}
& \boldsymbol{O}\left(n^{-K-5L/2-4M-M_0-2N}\right)\ n_i^{K+L+M+M_0+2N} \\
& \times \sum_{j_{1,1}}^{p} \cdots \sum_{j_{1,K+M_0}}^{p} \sum_{j_{2,1}}^{p} \cdots \sum_{j_{2,L}}^{p} \sum_{j_{3,1}}^{p} \cdots \sum_{j_{3,M}}^{p} \sum_{j_{4,1}}^{p} \cdots \sum_{j_{4,N}}^{p} c_{j_{1,1}j_{1,2}}(0) \cdots c_{j_{1,K+M_0}j_{2,1}}(0) \\
& \quad \times\ c_{j_{2,1}j_{2,2}}(0) \cdots c_{j_{2,L}j_{3,1}}(0)\ c_{j_{3,1}j_{3,2}}(0) \cdots c_{j_{3,M}j_{4,1}}(0)\ c_{j_{4,1}j_{4,2}}(0) \cdots c_{j_{4,N}j_{1,1}}(0) \\
& \times \sum_{k_{4,1}}^{p} \cdots \sum_{k_{4,N}}^{p} c_{k_{4,1}k_{4,1}}(0) \cdots c_{k_{4,N}k_{4,N}}(0) \\
= \ & \boldsymbol{O}\left(n^{-K-5L/2-4M-M_0-2N}\right)\ n_i^{K+L+M+M_0+2N} \qquad (\textit{By Assumption 3 and } \boldsymbol{\Gamma}(0) = \boldsymbol{I}_p) \\
& \times \sum_{j}^{p} c_{jj}(0) \cdots c_{jj}(0) \times \sum_{k_{4,1}}^{p} \cdots \sum_{k_{4,N}}^{p} c_{k_{4,1}k_{4,1}}(0) \cdots c_{k_{4,N}k_{4,N}}(0) \\
= \ & \boldsymbol{O}\left(p^{1+N} \cdot n^{-3L/2-3M}\right). \qquad (32)
\end{aligned}
$$

Thus, from (32), we rewrite a typical term of (30) as

$$
\begin{aligned}
& cum^{(J)}(\overbrace{\frac{1}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H, \cdots,}^{K} \overbrace{-\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H \boldsymbol{V}\}, \cdots,}^{L} \overbrace{\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H \boldsymbol{V}^2\}, \cdots,}^{M} \overbrace{\frac{q}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H, \cdots,}^{M_0} \overbrace{\frac{\beta_i}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H^2, \cdots}^{N}), \\
= \ & p^{-J/2} n^{-L/2-M-M_0-N} \boldsymbol{O}\left(p^{1+N} \cdot n^{-3L/2-3M}\right) \\
= \ & \boldsymbol{O}\left(p^{1-J/2+N} \cdot n^{-2L-4M-M_0-N}\right) \\
= \ & \boldsymbol{o}\left(p^{1-J/2-6L-12M-3M_0-2N}\right). \quad (\textit{By Assumption 1})
\end{aligned}
$$

Hence, we showed (26) and (27).

Lemma 4 *Suppose Assumputions 1-3. Define W_i for every $i = 1, 2, 3$ by*

$$
\begin{aligned}
W_i \ &= \ \frac{1}{\sqrt{2(q-1)}} \left\{ \frac{1}{\sqrt{p}} U^{(0)} + \frac{1}{\sqrt{pn}} U^{(1)} + \frac{1}{\sqrt{pn}} \left(U^{(2)} + \beta_i R^{(2)} \right) - \sqrt{p}(q-1) \right\} \qquad (33) \\
&= \ \frac{1}{\sqrt{2(q-1)}} \left\{ \frac{1}{\sqrt{p}} \mathrm{tr}\hat{\mathcal{S}}_H - \frac{1}{\sqrt{pn}} \mathrm{tr}\{\hat{\mathcal{S}}_H \boldsymbol{V}\} + \frac{1}{\sqrt{pn}} \mathrm{tr}\{\hat{\mathcal{S}}_H \boldsymbol{V}^2\} + \frac{q}{\sqrt{pn}} \mathrm{tr}\hat{\mathcal{S}}_H \right. \\
& \qquad \left. + \frac{\beta_i}{\sqrt{pn}} \mathrm{tr}\{\hat{\mathcal{S}}_H^2\} - \sqrt{p}(q-1) \right\}, \qquad (34) \\
(\beta_1, \beta_2, \beta_3) \ &= \ \left(0, -\frac{1}{2}, -1\right).
\end{aligned}
$$

Then, under the null hypothesis H, the following (35)-(39) hold that

$$cum(W_i) = 0 + \boldsymbol{o}\left(p^{-1/2}\right), \tag{35}$$

$$cum(W_i, W_i) = 1 + \boldsymbol{o}\left(p^{-1/2}\right), \tag{36}$$

$$cum(W_i, W_i, W_i) = p^{-1/2}\left(\frac{2}{q-1}\right)^{3/2} \times\left\{q - 3 + 3\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2 - \sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3\right\} + \boldsymbol{o}\left(p^{-1/2}\right), \tag{37}$$

$$cum^{(4)}(W_i, \cdots, W_i) = p^{-1}\left(\frac{2}{q-1}\right)^{2} \times\left\{q - 4 + 6\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2 - 4\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3 - \sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^4\right\} + \boldsymbol{o}\left(p^{-1}\right), \tag{38}$$

$$cum^{(J)}(W_i, \cdots, W_i) = \boldsymbol{O}\left(p^{1-J/2}\right), \quad (J \geq 5) \tag{39}$$

where (39) contains $K, L, M, M_0, N(\geq 0)$ of the first, second, third, fourth, and fifth terms of (34), respectively.

Proof (Lemma 4) *Now, from Lemma 3, we obtain from (33)*

$$cum(W_i) = \frac{1}{\sqrt{2(q-1)}}\left\{\frac{1}{\sqrt{p}}\{E[U^{(0)}] - p(q-1)\}\right\} + \boldsymbol{o}\left(p^{-1/2}\right).$$

Here, under Assumptions 2 and 3, from (28), we get

$$\begin{aligned} E[U^{(0)}] &= \sum_{j=1}^{p} E[S_{jj}] \\ &= \sum_{j=1}^{p}\sum_{i_1=1}^{q}\sum_{s=-n_{i_1}+1}^{n_{i_1}-1}\left(1 - \frac{|s|}{n_{i_1}}\right)c_{jj}(s) - \sum_{j=1}^{p}\sum_{i_2=1}^{q}\frac{n_{i_2}}{n}\sum_{r=-n_{i_2}+1}^{n_{i_2}-1}\left(1 - \frac{|r|}{n_{i_2}}\right)c_{jj}(r) \\ &= p(q-1). \end{aligned} \tag{40}$$

Then, we can obtain

$$cum(W_i) = 0 + \boldsymbol{o}\left(p^{-1/2}\right). \quad \textit{(By Assumption 1)} \tag{41}$$

Similarly, the main-order terms of $cum(W_i, W_i)$ and $cum(W_i, W_i, W_i)$ can be computed as follows. From (16) and (20),

$$\begin{aligned} cum(W_i, W_i) &= \frac{1}{2p(q-1)}cum(U^{(0)}, U^{(0)}) + \boldsymbol{o}\left(p^{-1/2}\right) \quad \textit{(By Lemma 3)} \\ &= \frac{1}{2p(q-1)}\sum_{j=1}^{p}\sum_{k=1}^{p}cum(S_{jj}, S_{kk}) + \boldsymbol{o}\left(p^{-1/2}\right) \\ &= 1 + \boldsymbol{o}\left(p^{-1/2}\right). \end{aligned} \tag{42}$$

In addition, we can obtain

$$cum(W_i, W_i, W_i) = \{2p(q-1)\}^{-3/2} cum(U^{(0)}, U^{(0)}, U^{(0)}) + \boldsymbol{o}\left(p^{-1/2}\right), \quad \textit{(By Lemma 3)}$$

and

$$\begin{aligned}
& cum(U^{(0)}, U^{(0)}, U^{(0)}) \\
= & \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} cum(S_{jj}, S_{kk}, S_{ll}) \\
= & \sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}[\\
& \sum_{i_1=1}^{q}\sum_{i_2=1}^{q}\sum_{i_3=1}^{q} \frac{1}{n_{i_1}}\frac{1}{n_{i_2}}\frac{1}{n_{i_3}} \sum_{r=1}^{n_{i_1}}\sum_{s=1}^{n_{i_1}}\sum_{t=1}^{n_{i_2}}\sum_{u=1}^{n_{i_2}}\sum_{v=1}^{n_{i_3}}\sum_{w=1}^{n_{i_3}} cum\{\epsilon_{i_1r}^{(j)}\epsilon_{i_2s}^{(j)}, \epsilon_{i_2t}^{(k)}\epsilon_{i_2u}^{(k)}, \epsilon_{i_3v}^{(l)}\epsilon_{i_3w}^{(l)}\} \\
& -3\frac{1}{n}\sum_{i_1=1}^{q}\sum_{i_2=1}^{q}\sum_{i_3=1}^{q} \frac{1}{n_{i_1}}\frac{1}{n_{i_2}} \sum_{r=1}^{n_{i_1}}\sum_{s=1}^{n_{i_1}}\sum_{t=1}^{n_{i_2}}\sum_{u=1}^{n_{i_2}}\sum_{v=1}^{n_{i_3}}\sum_{w=1}^{n_{i_3}} cum\{\epsilon_{i_1r}^{(j)}\epsilon_{i_2s}^{(j)}, \epsilon_{i_2t}^{(k)}\epsilon_{i_2u}^{(k)}, \epsilon_{i_3v}^{(l)}\epsilon_{i_3w}^{(l)}\} \\
& +3\frac{1}{n^2}\sum_{i_1=1}^{q}\sum_{i_2=1}^{q}\sum_{i_3=1}^{q} \frac{1}{n_{i_1}} \sum_{r=1}^{n_{i_1}}\sum_{s=1}^{n_{i_1}}\sum_{t=1}^{n_{i_2}}\sum_{u=1}^{n_{i_2}}\sum_{v=1}^{n_{i_3}}\sum_{w=1}^{n_{i_3}} cum\{\epsilon_{i_1r}^{(j)}\epsilon_{i_2s}^{(j)}, \epsilon_{i_2t}^{(k)}\epsilon_{i_2u}^{(k)}, \epsilon_{i_3v}^{(l)}\epsilon_{i_3w}^{(l)}\} \\
& -\frac{1}{n^3}\sum_{i_1=1}^{q}\sum_{i_2=1}^{q}\sum_{i_3=1}^{q} \sum_{r=1}^{n_{i_1}}\sum_{s=1}^{n_{i_1}}\sum_{t=1}^{n_{i_2}}\sum_{u=1}^{n_{i_2}}\sum_{v=1}^{n_{i_3}}\sum_{w=1}^{n_{i_3}} cum\{\epsilon_{i_1r}^{(j)}\epsilon_{i_2s}^{(j)}, \epsilon_{i_2t}^{(k)}\epsilon_{i_2u}^{(k)}, \epsilon_{i_3v}^{(l)}\epsilon_{i_3w}^{(l)}\} \\
&] \\
= & \sum_{j=1}^{p}\left\{\sum_{i_1=1}^{q} 8c_{jj}(0)c_{jj}(0)c_{jj}(0) - 3\sum_{i_1=1}^{q}\frac{n_{i_1}}{n}\cdot 8c_{jj}(0)c_{jj}(0)c_{jj}(0)\right. \\
& \left. +3\sum_{i_1=1}^{q}\left(\frac{n_{i_1}}{n}\right)^2 8c_{jj}(0)c_{jj}(0)c_{jj}(0) - \sum_{i_1=1}^{q}\left(\frac{n_{i_1}}{n}\right)^3\cdot 8c_{jj}(0)c_{jj}(0)c_{jj}(0)\right\} \\
& +\boldsymbol{O}\left(p\cdot n^{-1}\right) \\
= & 8p\left\{q-3+3\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2 - \sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3\right\} + \boldsymbol{O}\left(p\cdot n^{-1}\right). \qquad (43)
\end{aligned}$$

Therefore,

$$cum(W_i, W_i, W_i) = p^{-1/2}\left(\frac{2}{q-1}\right)^{3/2}\left\{q-3+3\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2 - \sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3\right\} + \boldsymbol{o}\left(p^{-1/2}\right). \qquad (44)$$

Similarly, we can compute

$$\begin{aligned}
cum^{(4)}(W_i, \cdots, W_i) &= \{2p(q-1)\}^{-1} cum^{(4)}(U^{(0)}, \cdots, U^{(0)}) + \boldsymbol{o}\left(p^{-2}\right) \\
&= \{2p(q-1)\}^{-1}\sum_{j_1=1}^{p}\sum_{j_2=1}^{p}\sum_{j_3=1}^{p}\sum_{j_4=1}^{p} cum(S_{j_1j_1}, \cdots, S_{j_4j_4}) + \boldsymbol{o}\left(p^{-1}\right)
\end{aligned}$$

$$= p^{-1}\left(\frac{2}{q-1}\right)^2\left\{q-4+6\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^2-4\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^3-\sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^4\right\}$$
$$+\boldsymbol{o}\left(p^{-1}\right). \tag{45}$$

Hence, (35), (36), (37), and (38) were shown (from (41), (42), (44), and (45)). Furthermore, we discuss the Jth order for $J \geq 5$ cumulant $cum^{(J)}(W_i, \cdots, W_i)$. From Lemma 3, we obtain

$$cum^{(J)}(W_i,\ldots,W_i) = \sum_{\substack{K,L,M,M_0,N;\\K+L+M+M_0+N=J}} \{2(q-1)\}^{-J/2}$$
$$\times\ cum^{(J)}(\overbrace{\frac{1}{\sqrt{p}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots,}^{K}\overbrace{-\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}\},\cdots,}^{L}\overbrace{\frac{1}{\sqrt{pn}}\mathrm{tr}\{\hat{\mathcal{S}}_H\boldsymbol{V}^2\},\cdots,}^{M}\overbrace{\frac{q}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H,\cdots,}^{M_0}\overbrace{\frac{\beta_i}{\sqrt{pn}}\mathrm{tr}\hat{\mathcal{S}}_H^2,\cdots}^{N})$$
$$= \sum_{\substack{K,L,M,M_0,N;\\K+L+M+M_0+N=J}} \boldsymbol{O}\left(p^{1-J/2+N}\cdot n^{-2L-4M-M_0-N}\right)$$
$$= \max_{K,L,M,M_0,N}\boldsymbol{O}\left(p^{1-J/2+N}\cdot n^{-2L-4M-M_0-N}\right)$$
$$= \boldsymbol{O}\left(p^{1-J/2}\right). \quad (L=M=M_0=N=0)$$

Then, (39) was shown.

Remark 2 *[12] also evaluated the high-order cumulants of T_i, $i = 1, 2, 3$ but there is a big difference between this paper and [12]. The order of the stochastic expansion in Lemma 2 is higher than that in [12], so we needed to derive asymptotics of W_i as in Lemmas 3 and 4.*

Proof (Theorem 1) *The Edgeworth expansion for a multivariate time series is derived by [13, p.168-170]. We extend it to the case of high-dimensional time series. First, by the Taylor expansion and Lemma 4, we write the characteristic function of W_i $(i = 1, 2, 3)$ in Lemma 4 as*

$$E[exp\{\mathrm{i}tW_i\}]$$
$$= exp\left\{cum(W_i)(\mathrm{i}t)+\frac{1}{2}cum(W_i,W_i)(\mathrm{i}t)^2+\frac{1}{6}cum(W_i,W_i,W_i)(\mathrm{i}t)^3\right.$$
$$\left.+\frac{1}{24}cum^{(4)}(W_i,\cdots,W_i)(\mathrm{i}t)^4+\cdots\right\}$$
$$= exp\left(-\frac{t^2}{2}\right)\times\left\{1+p^{-1/2}\cdot\frac{1}{6}cum(W_i,W_i,W_i)(\mathrm{i}t)^3+p^{-1}\cdot\frac{1}{24}cum^{(4)}(W_i,\cdots,W_i)(\mathrm{i}t)^4\right\}$$
$$+\boldsymbol{o}\left(p^{-1/2}\right).$$
$$= exp\left(-\frac{t^2}{2}\right)\times\left\{1+p^{-1/2}\cdot\frac{c_3}{6}(\mathrm{i}t)^3+p^{-1}\cdot\frac{c_4}{24}(\mathrm{i}t)^4\right\}+\boldsymbol{o}\left(p^{-1/2}\right). \tag{46}$$

Inverting (46) by the Fourier inverse transform, we have

$$P(W_i < z) \quad = \quad \Phi(z)-\phi(z)\left\{p^{-1/2}\cdot\frac{c_3}{6}(z^2-1)+p^{-1}\cdot\frac{c_4}{24}(z^3-3z)\right\}+\boldsymbol{o}\left(p^{-1/2}\right),$$

where

$$\Phi(z) = \int_{-\infty}^{z} \phi(y)dy, \quad \phi(y) = (2\pi)^{-1/2} exp\left(-\frac{y^2}{2}\right).$$

Here, from Lemma 2, we observe that

$$E[exp\{\mathrm{i}tT_i\}] = E[exp\{\mathrm{i}tW_i\}] + \boldsymbol{o}(1).$$

This implies (13), so we complete the proof.

5 Simulation to verify the finite sample performance We simulate the Edgeworth expansions of distributions of $T_i,\ i = 1, 2, 3$, which are given by Theorem 1. In this section, our purpose is to show that their Edgeworth expansions $P(T_i < z),\ i = 1, 2, 3$ in (13) are more numerically accurate than the first-order approximation, that is, $\Phi(z)$ in (13). Specifically, in the case of an uncorrelated disturbance that is assumed by Assumption 3, $DCC\text{-}GARCH(1, 1)$ is a typical example of that process (see [7]). Therefore, we introduce the following five simulation process steps.

1 Set $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_3 = \mathbf{0}$ for the null hypothesis H.

2 Generate 20-dimensional $\{\boldsymbol{X}_{1,1}, \dots, \boldsymbol{X}_{1,5000}\}, \{\boldsymbol{X}_{2,1}, \dots, \boldsymbol{X}_{2,5000}\}, \{\boldsymbol{X}_{3,1}, \dots, \boldsymbol{X}_{3,5000}\}$, with $DCC\text{-}GARCH(1, 1)$ disturbance.

3 Calculate the test statistics $T_i,\ i = 1, 2, 3$.

4 Repeat steps 2 and 3 1,000 times independently and obtain $\{T_i^{(1)}, \dots, T_i^{(1000)};\ i = 1, 2, 3\}$.

5 Calculate $\hat{F}_{i,n}(z),\ i = 1, 2, 3$, which is the empirical distribution of $\{T_i^{(1)}, \dots, T_i^{(1000)};\ i = 1, 2, 3\}$.

6 Write the plot of $|\hat{F}_{i,n}(z) - \Phi(z)|$ and $|\hat{F}_{i,n}(z) - P(T_i < z)|,\ i = 1, 2, 3$, which are plotted by dotted and thick lines, respectively, in Figures 1, 3, and 5.

7 Write the plot of $\{|\hat{F}_{i,n}(z) - \Phi(z)| - |\hat{F}_{i,n}(z) - P(T_i < z)|\},\ i = 1, 2, 3$, by a dotted line, in Figures 2, 4, and 6.

We set the 20-dimensional simulation from one-way MANOVA model (1) with a 20-dimensional vector $\boldsymbol{\mu}' = (1, \cdots, 1)'$ and generate the disturbance process $\{\boldsymbol{\epsilon}_{it}\}$ of observations $\{\boldsymbol{X}_{it}\}$ in (1) by using $DCC\text{-}GARCH(1, 1)$, whose innovation term is assumed to be Gaussian. The scenarios of $DCC\text{-}GARCH(q, r)$ (see (9)) in $\boldsymbol{\epsilon}_{it}$ are

$$\begin{gathered} p = 20,\ i = 1, 2, 3,\ t = 1, \cdots, 5000, \\ j = 1, \cdots, 20, \\ q = r = 1, \\ a_j = 0.2,\ b_j = 0.7,\ c_j = 0.002, \\ \alpha = 0.1,\ \beta = 0.8, \\ \tilde{Q}_{kl} = 0.7^{(|k-l|)}, \end{gathered}$$

where $\tilde{\boldsymbol{Q}}_{kl}$ is the (k, l)-element of $\tilde{\boldsymbol{Q}}$. We set the observation length $n_i = 5000,\ i = 1, 2, 3$, because Table 1 of Section 5.1 in [12] demonstrates that T_i are stable for $n_i = 2500$ or more uncorrelated observations ($i = 1, 2, 3$). The Mathematical code and the "ccgarch" package of R are used for this algorithm. We compare the numerical accuracy of $P(T_i < z)$ with $\Phi(z)$ based on $\hat{F}_n(z)$ by using $|\hat{F}_{i,n}(z) - \Phi(z)|$, $|\hat{F}_{i,n}(z) - P(T_i < z)|$ (see Figures 1, 3, and 5), and $\{|\hat{F}_{i,n}(z) - \Phi(z)| - |\hat{F}_{i,n}(z) - P(T_i < z)|\}$, $i = 1, 2, 3$ (see Figures 2, 4, and 6).

Figures 2, 4, and 6 indicate that the Edgeworth expansions $P(T_i < z)$ of T_i work better than the normal approximation $\Phi(z)$ from the perspective of numerical accuracy.

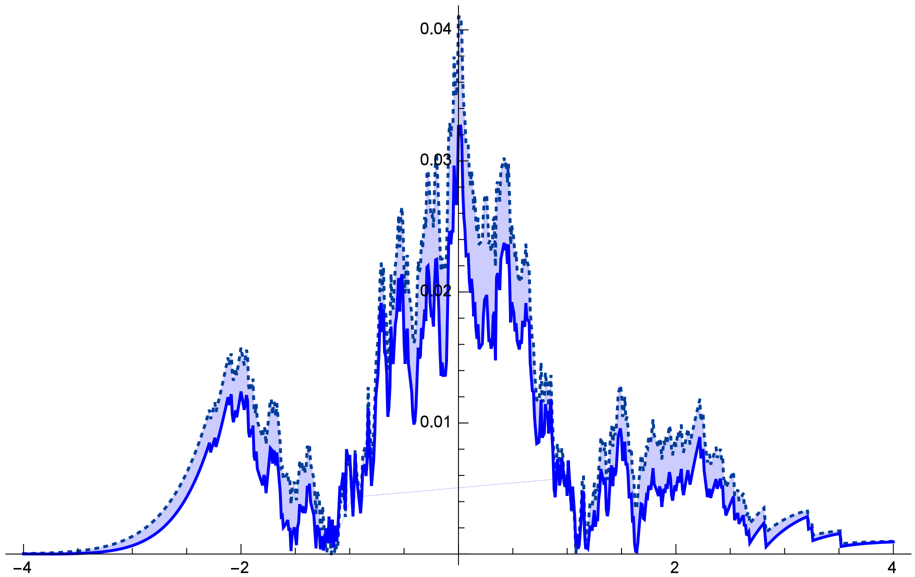

Figure 1: Plot of $|\hat{F}_{1,n}(z) - \Phi(z)|$ and $|\hat{F}_{1,n}(z) - P(T_1 < z)|$ by dotted and thick lines, respectively.

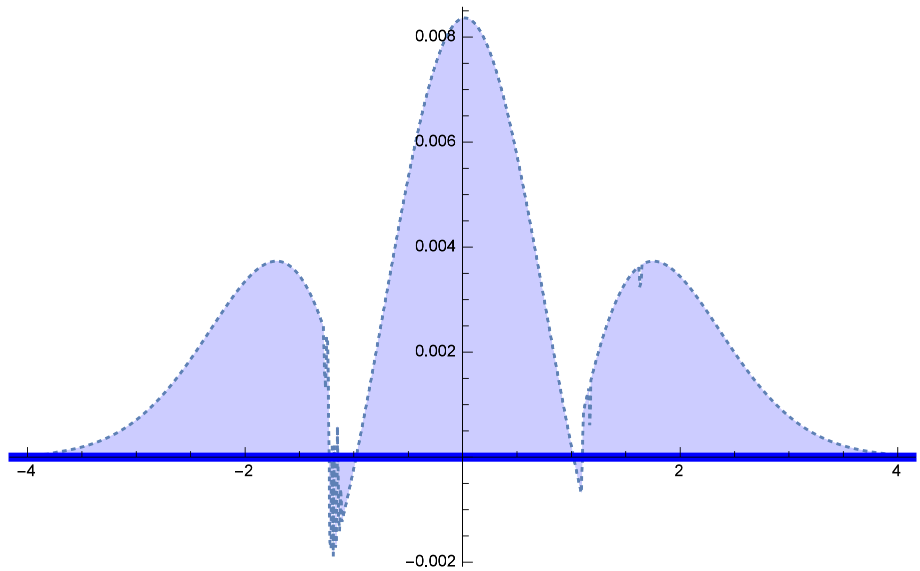

Figure 2: Plot of $\{|\hat{F}_{1,n}(z) - \Phi(z)| - |\hat{F}_{1,n}(z) - P(T_1 < z)|\}$ by a dotted line.

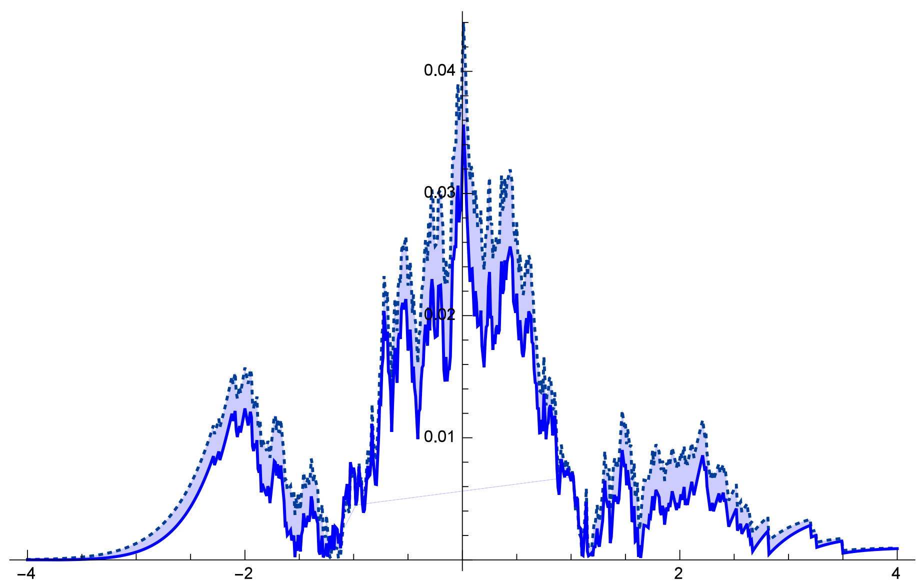

Figure 3: Plot of $|\hat{F}_{2,n}(z) - \Phi(z)|$ and $|\hat{F}_{2,n}(z) - P(T_2 < z)|$ by a dotted line and a thick one, respectively.

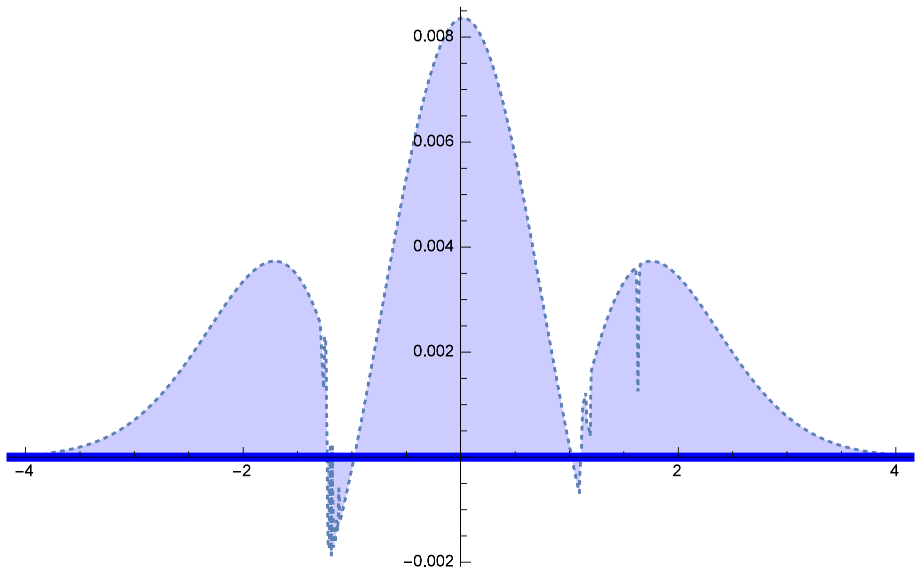

Figure 4: Plot of $\{|\hat{F}_{2,n}(z) - \Phi(z)| - |\hat{F}_{2,n}(z) - P(T_2 < z)|\}$ by a dotted line.

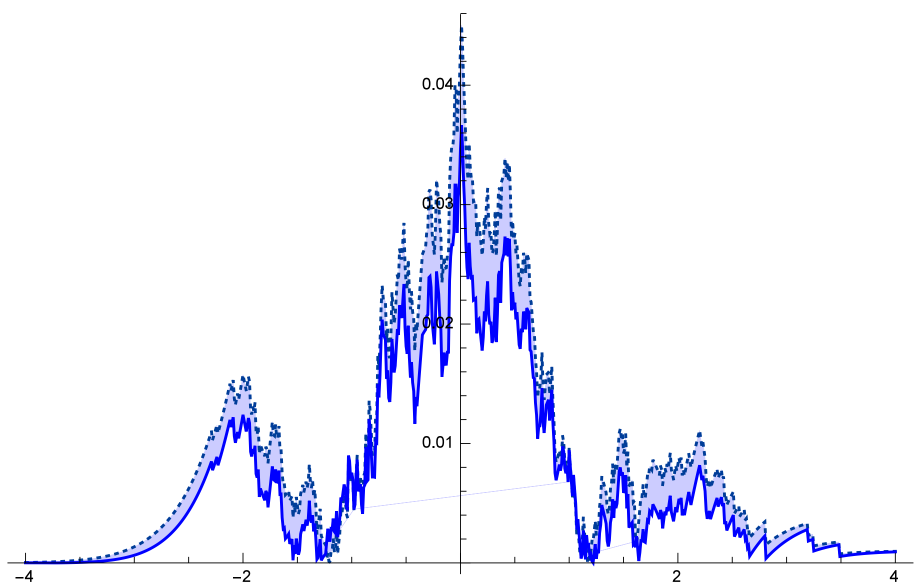

Figure 5: Plot of $|\hat{F}_{3,n}(z) - \Phi(z)|$ and $|\hat{F}_{3,n}(z) - P(T_3 < z)|$ by a dotted line and a thick one, respectively.

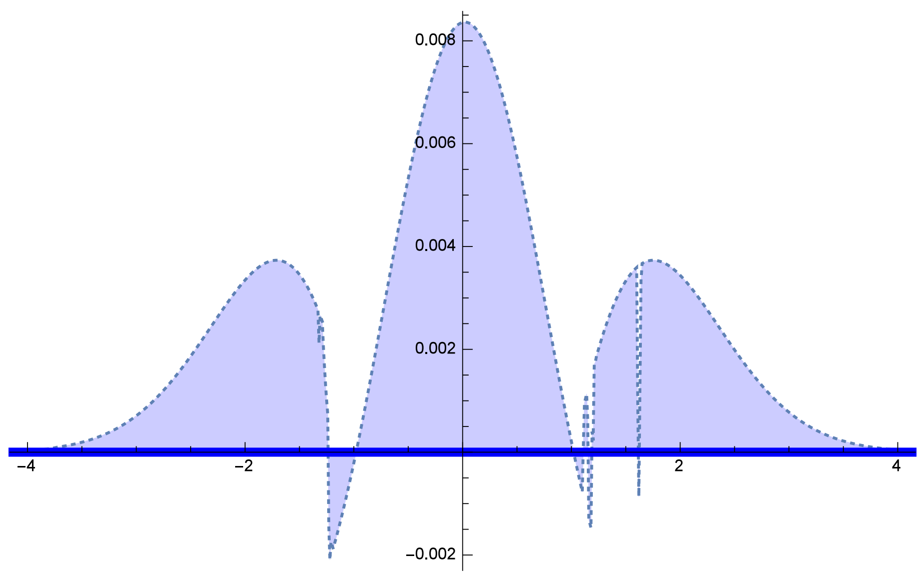

Figure 6: Plot of $\{|\hat{F}_{3,n}(z) - \Phi(z)| - |\hat{F}_{3,n}(z) - P(T_3 < z)|\}$ by a dotted line.

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