MIXED SCHWARZ INEQUALITIES VIA THE MATRIX GEOMETRIC MEAN

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Received June 18, 2018 ; revised September 8, 2018

ABSTRACT. In this paper, by using the Cauchy-Schwarz inequality for matrices via the matrix geometric mean due to J.I. Fujii, we show the following matrix version of a mixed Schwarz inequality for any square matrices: Let A be an n-square matrix. For any n-square matrices X, Y

$$|Y^*AX| \leq X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU$$

holds for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, where U is a unitary matrix in a polar decomposition of $Y^*AX = U|Y^*AX|$. As applications, we show matrix Parseval's equation, Lin's type extensions for a weighted version of a mixed Schwarz inequality, and a weighted version of the Wielandt inequality for matrices.

1 Introduction Let $\mathbb{M}_{m\times n} = \mathbb{M}_{m\times n}(\mathbb{C})$ be the space of $m \times n$ complex matrices and $\mathbb{M}_n = \mathbb{M}_{n\times n}(\mathbb{C})$, and denote the matrix absolute value of any $A \in \mathbb{M}_{m\times n}$ by $|A| = (A^*A)^{1/2}$. For $A \in \mathbb{M}_n$, we write $A \geq 0$ if A is positive semidefinite and $A > 0$ if A is positive definite; that is, $x^*Ax > 0$ for all nonzero column vectors $x \in \mathbb{C}^n$. For two Hermitian matrices A and B of the same size, we write $A \geq B$ if $A - B \geq 0$, and $A > B$ if $A - B > 0$. For $A \in \mathbb{M}_{m\times n}$, $\ker A$ and $\operatorname{ran} A$ mean the null space of A and the range of A, respectively.

The Cauchy-Schwarz inequality is one of the most useful and fundamental inequalities in functional analysis: For any complex n-dimensional column vectors x and y,

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle \tag{1.1}$$

and the equality holds if and only if x and y are linearly dependent. As an extension of (1.1), the following inequality holds: For any positive semidefinite matrix A in $\mathbb{M}_n$,

$$|\langle Ax, y\rangle|^2 \leq \langle Ax, x\rangle\langle Ay, y\rangle.$$

Even if A is an arbitrary matrix in $\mathbb{M}_n$, by virtue of the matrix absolute value of A, we have a mixed Schwarz inequality

$$|\langle Ax, y\rangle|^2 \leq \langle |A|x, x\rangle\langle |A^*|y, y\rangle, \tag{1.2}$$

also see [5]. In [3], Furuta showed the weighted version of (1.2) as follows: For any $A \in \mathbb{M}_n$

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\alpha}x, x\rangle\langle |A^*|^{2\beta}y, y\rangle \tag{1.3}$$

holds for any $x, y \in \mathbb{C}^n$ and any $\alpha, \beta \in [0,1]$ with $\alpha+\beta = 1$, and the equality in (1.3) holds if and only if $|A|^{2\alpha}x$ and A^*y are linearly dependent if and only if Ax and $|A^*|^{2\beta}y$ are linearly

2000 *Mathematics Subject Classification.* Primary 15A45; Secondary 47A63, 47A64.
Key words and phrases. weighted mixed Schwarz inequality, matrix geometric mean, Lin's type extension, Wielandt inequality.

dependent. In fact, Furuta has shown the operator version of (1.3). Moreover, Kittaneh extended (1.3) for two real valued continuous functions f and g under some conditions, also see [7]. We recall the matrix Cauchy-Schwarz inequality in terms of the matrix geometric mean due to [1], also see [2]: For any $X, Y \in \mathbb{M}_n$

$$|Y^*X| \le X^*X \sharp U^*Y^*YU \tag{1.4}$$

holds, where U is a unitary matrix in a polar decomposition of $Y^*X = U|Y^*X|$ and the matrix geometric mean $A \sharp B$ is defined by

$$A \sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

for any positive definite matrices A and B, also see [8].

In this paper, by virtue of the matrix Cauchy-Schwarz inequality (1.4) due to J.I.Fujii via the matrix geometric mean, we show the matrix version of a weighted mixed Schwarz inequality (1.3). As applications, we show matrix Parseval's equations, Lin's type extensions for a weighted version of a mixed Schwarz inequality, and a weighted version of the Wielandt inequality for matrices.

2 Weighted mixed Schwarz inequality In this section, we present a weighted version of the mixed Schwarz inequality (1.3) for matrices of the same size. As a preparation of our main assertion, we state the following matrix Cauchy-Schwarz inequality due to J.I.Fujii [2] via the matrix geometric mean:

Lemma 2.1 (Matrix Cauchy-Schwarz inequality)**.** *Let X and Y be matrices in $\mathbb{M}_n$, and $U \in \mathbb{M}_n$ a unitary matrix in a polar decomposition of $Y^*X = U|Y^*X|$. Then*

$$|Y^*X| \le X^*X \sharp U^*Y^*YU \tag{2.1}$$

and

$$|X^*Y| \le UX^*XU^* \sharp Y^*Y. \tag{2.2}$$

Under the assumption $\ker X \subseteq \ker YU$ (*resp.* $\ker Y \subseteq \ker XU^*$)*, the equality in* (2.1) (*resp. the equality in* (2.2)) *holds if and only if there exists* $W \in \mathbb{M}_n$ *such that* $YU = XW$ (*resp.* $XU^* = YW$)*.*

For any n-square matrix A, we denote the orthogonal projection on the column space of A by P_A. That is, P_A is the range projection of A. By Lemma 2.1, we have the following matrix version of the weighted Schwarz inequality (1.3) for matrices of the same size:

Theorem 2.2 (Weighted mixed Schwarz inequality)**.** *Let A, X and Y be matrices in $\mathbb{M}_n$ and $U \in \mathbb{M}_n$ a unitary matrix in a polar decomposition of $Y^*AX = U|Y^*AX|$, and $V \in \mathbb{M}_n$ a unitary matrix in a polar decomposition of $A = V|A|$. Then*

$$|Y^*AX| \le X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU \tag{2.3}$$

and

$$|X^*A^*Y| \le UX^*|A|^{2\alpha}XU^* \sharp Y^*|A^*|^{2\beta}Y \tag{2.4}$$

hold for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$. Under the assumption $\ker AX \subseteq \ker A^*YU$ (*resp.* $\ker A^*Y \subseteq \ker AXU^*$)*, the equality in* (2.3) (*resp. the equality in* (2.4)) *holds if and only if* *there exists* $W \in \mathbb{M}_n$ *such that* $A^*YU = |A|^{2\alpha}XW$ *(resp.* $AXU^* = |A^*|^{2\beta}YW$*) if and only if* $|A^*|^{2\beta}YU = AXW$ *(resp.* $|A|^{2\alpha}XU^* = A^*YW$*).*

In particular, for the case of $\alpha = 0$ *in* (2.3),

$$|Y^*AX| \le X^*P_{|A|}X \sharp U^*Y^*|A^*|^2YU. \tag{2.5}$$

Under the assumption $\ker P_{|A|}X \subseteq \ker |A|V^*YU$*, the equality in* (2.5) *holds if and only if there exists* $W \in \mathbb{M}_n$ *such that* $|A|V^*YU = P_{|A|}XW$*.*

For the case of $\alpha = 1$ *in* (2.3),

$$|Y^*AX| \le X^*|A|^2X \sharp U^*Y^*P_{|A^*|}YU. \tag{2.6}$$

Under the assumption $\ker |A|X \subseteq \ker P_{|A^*|}V^*YU$*, the equality in* (2.6) *holds if and only if there exists* $W \in \mathbb{M}_n$ *such that* $P_{|A|}V^*YU = |A|XW$*.*

Proof. Firstly, we show (2.3). For the case of $0 < \alpha < 1$, replacing X (resp. Y) by $|A|^\alpha X$ (resp. $|A|^\beta V^*Y$) in (2.1) of Lemma 2.1, then we obtain

$$|Y^*AX| = |(|A|^\beta V^*Y)^*|A|^\alpha X| \le X^*|A|^{2\alpha}X \sharp U^*Y^*V|A|^{2\beta}V^*YU.$$

It follows from [3] and [4, Theorem 4 in 2.2.2] that

$$V|A|^{2\beta}V^* = (V|A|V^*)^{2\beta} = (V|A||A|V^*)^\beta = (AA^*)^\beta = |A^*|^{2\beta}$$

and we can get the desired inequality (2.3):

$$|Y^*AX| \le X^*|A|^{2\alpha}X \sharp U^*Y^*V|A|^{2\beta}V^*YU = X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU.$$

For the case of $\alpha = 0$, since $|Y^*AX| = |Y^*V|A|P_{|A|}X| = |(|A|V^*Y)^*P_{|A|}X|$, by replacing X (resp. Y) by $P_{|A|}X$ (resp. $|A|V^*Y$) in (2.1) of Lemma 2.1, we obtain

$$|Y^*AX| \le X^*P_{|A|}X \sharp U^*Y^*V|A|^2V^*YU = X^*P_{|A|}X \sharp U^*Y^*|A^*|^2YU$$

and so we have (2.5). For the case of $\alpha = 1$, we have (2.6) similarly.

For the equality conditions, since $\ker AX \subseteq \ker A^*YU$ is equivalent to $\ker |A|^\alpha X \subseteq \ker |A|^\beta V^*YU$ for $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, it follows from Lemma 2.1 that under the assumption $\ker |A|^\alpha X \subseteq \ker |A|^\beta V^*YU$, the equality in (2.3) holds if and only if there exists $W \in \mathbb{M}_n$ such that $|A|^\beta V^*YU = |A|^\alpha XW$.

By a way similar to (2.3), we can get the inequality (2.4) and the equality condition of (2.4). □

Remark 2.3. Similarly, we can consider the case of $\alpha = 0, 1$ of (2.4) in Theorem 2.2.

For the case of $\alpha = 0$, then

$$|X^*A^*Y| \le UX^*P_{|A|}XU^* \sharp Y^*|A^*|^2Y.$$

Under the assumption $\ker |A^*|Y \subseteq \ker P_{|A^*|}VXU^*$, the equality holds if and only if there exists $W \in \mathbb{M}_n$ such that $P_{|A^*|}VXU^* = |A^*|YW$.

For the case of $\alpha = 1$, then

$$|X^*A^*Y| \le UX^*|A|^2XU^* \sharp Y^*P_{|A^*|}Y.$$

Under the assumption $\ker P_{|A^*|}Y \subseteq \ker |A^*|VXU^*$, the equality holds if and only if there exists $W \in \mathbb{M}_n$ such that $|A^*|VXU^* = P_{|A^*|}YW$.

3 Weighted mixed Schwarz inequality for an arbitrary matrix In this section, we present the weighted version of a mixed Schwarz inequality for matrices of any different sizes. For this, we need the following lemmas, see [6, p.449].

Lemma 3.1 (Polar decomposition)**.** *Let A be an $m \times n$ matrix in $\mathbb{M}_{m\times n}$.*

(i) *If $m > n$, then $A = U|A|$, in which $U \in \mathbb{M}_{m\times n}$ consists of orthonormal columns.*

(ii) *If $m = n$, then $A = U|A|$, in which $U \in \mathbb{M}_n$ is unitary.*

(iii) *If $m < n$, then $A = |A^*|U$, in which $U \in \mathbb{M}_{n\times m}$ consists of orthonormal rows.*

The following lemma is a matrix Cauchy-Schwarz inequality for an arbitrary matrix, also see [2, Corollary 2.7].

Lemma 3.2. *Let X be a matrix in $\mathbb{M}_{k\times m}$ and Y in $\mathbb{M}_{k\times n}$.*

(i) *If $m \le n$, then*

$$|Y^*X| \le X^*X \sharp U^*Y^*YU, \tag{3.1}$$

in which $U \in \mathbb{M}_{n\times m}$ consists of orthonormal columns and $Y^*X = U|Y^*X|$.

(ii) *If $m > n$, then*

$$|X^*Y| \le U^*X^*XU \sharp Y^*Y, \tag{3.2}$$

in which $U \in \mathbb{M}_{m\times n}$ consists of orthonormal columns and $X^*Y = U|X^*Y|$.

Under the assumption $\ker X \subseteq \ker YU$ (resp. $\ker Y \subseteq \ker XU$), the equality in (3.1) (resp. the equality in (3.2)) holds if and only if there exists $W \in \mathbb{M}_m$ (resp. $W \in \mathbb{M}_n$) such that $YU = XW$ (resp. $XU^* = YW$).

By using a polar decomposition for an arbitrary matrix, we have the following theorem, whose proof is similar to that of Theorem 2.2.

Theorem 3.3. *Let A be a matrix in $\mathbb{M}_{p\times m}$, X in $\mathbb{M}_{m\times n}$, Y in $\mathbb{M}_{p\times q}$. For all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, the following inequalities hold.*

(i) *If $q \ge n$, then*

$$|Y^*AX| \le X^*|A|^{2\alpha}X \sharp U_1^*Y^*|A^*|^{2\beta}YU_1, \tag{3.3}$$

in which $U_1 \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U_1|Y^*AX|$.

(ii) *If $q < n$, then*

$$|X^*A^*Y| \le U_2^*X^*|A|^{2\alpha}XU_2 \sharp Y^*|A^*|^{2\beta}Y, \tag{3.4}$$

in which $U_2 \in \mathbb{M}_{n\times q}$ consists of orthonormal columns and $X^*A^*Y = U_2|X^*A^*Y|$.

Under the assumption $\ker AX \subseteq \ker A^*YU_1$ (resp. $\ker A^*Y \subseteq \ker AXU_2$), the equality in (3.3) (resp. the equality in (3.4)) holds if and only if there exists $W \in \mathbb{M}_n$ (resp. $W \in \mathbb{M}_q$) such that $|A^*|^{2\beta}YU_1 = AXW$ (resp. $AXU_2 = |A^*|^{2\beta}YW$).

Proof. We show (3.3) only. If $p \geq m$, then by Lemma 3.1 we have $A = V_1|A|$, in which $V_1 \in \mathbb{M}_{p\times m}$ consists of orthonormal columns. In this case, we replace X (resp. Y) by $|A|^\alpha X$ (resp. $|A|^\beta V_1^* Y$) in (3.1) of Lemma 3.2, and we have $|A^*|^{2\beta} = V_1|A|^{2\beta}V_1^*$. If $p < m$, then we have $A = |A^*|V_2$, in which $V_2 \in \mathbb{M}_{m\times p}$ consists of orthonormal rows. In this case, we replace X (resp. Y) by $|A^*|^\alpha V_2 X$ (resp. $|A^*|^\beta Y$) in (3.1) of Lemma 3.2, and we have $|A|^{2\alpha} = V_2^*|A^*|^{2\alpha}V_2$. Hence we obtain (3.3) and the equality condition. □

Inspired by Kittaneh's result [7, Theorem 1], we show an extension of Theorem 3.3, which is a generalization of Schwarz inequality for two nonnegative functions f and g.

Theorem 3.4. *Let A be in $\mathbb{M}_{p\times m}$, X in $\mathbb{M}_{m\times n}$, Y in $\mathbb{M}_{p\times q}$ and f, g real valued continuous functions on $[0,\infty)$ which are nonnegative and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. If $q \geq n$ and $p \geq m$, then*

$$|Y^*AX| \leq X^*f(|A|)^2X \ \sharp\ U^*Y^*g(|A^*|)^2YU, \tag{3.5}$$

in which $U \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U|Y^*AX|$.

Under the assumption $\ker f(|A|)X \subseteq \ker g(|A|)V^*YU$ where $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$, the equality in (3.5) holds if and only if there exists $W \in \mathbb{M}_n$ such that $g(|A|)V^*YU = f(|A|)XW$.

Proof. Replacing X and Y by $f(|A|)X$ and $g(|A|)V^*Y$ respectively in (3.1) of Lemma 3.2, we obtain (3.5). In fact, we have $|A^*| = V|A|V^*$ and $VV^* \leq I$, and so $Vg(|A|)^2V^* \leq g(V|A|V^*)^2 = g(|A^*|)^2$. Therefore it follows that

$$\begin{aligned} |Y^*AX| &= |Y^*V|A|X| = |Y^*Vg(|A|)f(|A|)X| \\ &\leq X^*f(|A|)^2X \ \sharp\ U^*Y^*Vg(|A|)^2V^*YU \\ &\leq X^*f(|A|)^2X \ \sharp\ U^*Y^*g(V|A|V^*)^2YU \\ &= X^*f(|A|)^2X \ \sharp\ U^*Y^*g(|A^*|)^2YU \end{aligned}$$

and the equality condition holds. □

4 Lin's type extensions We consider further extensions of the weighted version of the mixed Schwarz inequality for matrices. Firstly, inspired by Lin [9], we show that some orthogonal conditions imply an improvement of the Cauchy-Schwarz inequality for matrices of any different sizes in Lemma 3.2. For this, we recall the result due to Lin [9], which is the sharpen (1.1) as follows: If $y, z \in \mathbb{C}^n$ and y is orthogonal to z, then

$$(|\langle x,y\rangle|^2 \leq) \quad |\langle x,y\rangle|^2 + \frac{\langle y,y\rangle|\langle x,z\rangle|^2}{\langle z,z\rangle} \leq \langle x,x\rangle\langle y,y\rangle \tag{4.1}$$

for all $x \in \mathbb{C}^n$. We show the matrix version of (4.1). For any matrix A, we denote by $P_A^\perp$ $(= I - P_A)$ the orthogonal projection on the orthogonal complement of the column space of A.

Lemma 4.1. *Let X be in $\mathbb{M}_{k\times m}$, Y in $\mathbb{M}_{k\times n}$, Z_X in $\mathbb{M}_{k\times l_X}$ and Z_Y in $\mathbb{M}_{k\times l_Y}$. Suppose that $X^*Z_X = 0$, $Y^*Z_Y = 0$ and $Z_Y^*Z_X = 0$.*

(i) *If $n \geq m$, then*

$$|Y^*X| \leq X^*P_{Z_Y}^\perp X \ \sharp\ U^*Y^*P_{Z_X}^\perp YU \quad (\leq X^*X \ \sharp\ U^*Y^*YU), \tag{4.2}$$

in which $U \in \mathbb{M}_{n\times m}$ consists of orthonormal columns and $Y^*X = U|Y^*X|$.

Under the assumption $\ker P_{Z_Y}^\perp X \subseteq \ker P_{Z_X}^\perp YU$, the equality in (4.2) holds if and only if there exists $W \in \mathbb{M}_m$ such that $P_{Z_X}^\perp YU = P_{Z_Y}^\perp XW$.

(ii) *If $n < m$, then*

$$|X^*Y| \le U^*X^*P_{Z_Y}^{\perp}XU \sharp Y^*P_{Z_X}^{\perp}Y \quad (\le U^*X^*XU \sharp Y^*Y),$$

in which $U \in \mathbb{M}_{m\times n}$ consists of orthonormal columns and $X^*Y = U|X^*Y|$.
Under the assumption $\ker P_{Z_X}^{\perp}Y \subseteq \ker P_{Z_Y}^{\perp}XU$, the equality holds if and only if there exists $W \in \mathbb{M}_n$ such that $P_{Z_Y}^{\perp}XU = P_{Z_X}^{\perp}YW$.

Proof. We only show (4.2). Put $X_1 = P_{Z_Y}^{\perp}X$ and $Y_1 = P_{Z_X}^{\perp}Y$. Since $X^*Z_X = Y^*Z_Y = Z_Y^*Z_X = 0$, we have $P_{Z_X}X = Y^*P_{Z_Y} = P_{Z_X}P_{Z_Y} = 0$ and it follows that

$$Y_1^*X_1 = Y^*P_{Z_X}^{\perp}P_{Z_Y}^{\perp}X = Y^*(I-P_{Z_X})(I-P_{Z_Y})X = Y^*X.$$

Hence it follows from Lemma 3.2 that

$$|Y^*X| = |Y_1^*X_1| \le X_1^*X_1 \sharp U^*Y_1^*Y_1U = X^*P_{Z_Y}^{\perp}X \sharp U^*Y^*P_{Z_X}^{\perp}YU,$$

and so we have the desired inequality (4.2) and the equality condition holds. □

Nextly, we focus on Parseval's equation: Let x, y be in $\mathbb{C}^k$ and $\{e_i\}_{i=1}^k$ a complete orthonormal system in $\mathbb{C}^k$. Then

$$\|x\|^2 = \sum_{i=1}^{k} |\langle x, e_i\rangle|^2 \tag{4.3}$$

and

$$\langle x, y\rangle = \sum_{i=1}^{k} \langle x, e_i\rangle\langle e_i, y\rangle. \tag{4.4}$$

The next result is a matrix generalization of Parseval's equation (4.3). It follows from a way similar to Gram-Schmidt orthogonalization.

Lemma 4.2. *Let X be in $\mathbb{M}_{k\times m}$, Y in $\mathbb{M}_{k\times n}$, $Z(X,1),\ldots,Z(X,x)$ in $\mathbb{M}_{k\times l_X}$ and $Z(Y,1),\ldots,Z(Y,y)$ in $\mathbb{M}_{k\times l_Y}$. Then*

$$X^*X = \sum_{j=0}^{y} S_j^*P_{Z(Y,j+1)}S_j \tag{4.5}$$

and

$$Y^*Y = \sum_{i=0}^{x} T_i^*P_{Z(X,i+1)}T_i,$$

where $S_0 = X$, $S_j = P_{Z(Y,j)}^{\perp}S_{j-1}$ for $j = 1,2,\ldots,y$, $T_0 = Y$, $T_i = P_{Z(X,i)}^{\perp}T_{i-1}$ for $i = 1,2,\ldots,x$ and $Z(Y,y+1)$ (resp. $Z(X,x+1)$) satisfies $\operatorname{ran} S_y \subseteq \operatorname{ran} Z(Y,y+1)$ (resp. $\operatorname{ran} T_x \subseteq \operatorname{ran} Z(X,x+1)$).

Proof. We only show (4.5). The following equation holds by induction:

$$\begin{aligned}
S_y^*S_y &= (S_{y-1}^* - S_{y-1}^*P_{Z(Y,y)})(S_{y-1} - P_{Z(Y,y)}S_{y-1}) \\
&= S_{y-1}^*S_{y-1} - S_{y-1}^*P_{Z(Y,y)}S_{y-1} \\
&\vdots \\
&= S_0^*S_0 - S_0^*P_{Z(Y,1)}S_0 - \cdots - S_{y-1}^*P_{Z(Y,y)}S_{y-1} \\
&= X^*X - \sum_{j=0}^{y-1} S_j^*P_{Z(Y,j+1)}S_j.
\end{aligned}$$

Since the assumption $\operatorname{ran} S_y \subseteq \operatorname{ran} Z(Y, y+1)$ implies $P_{Z(Y,y+1)}S_y = S_y$, we have

$$S_y^* P_{Z(Y,y+1)} S_y = X^*X - \sum_{j=0}^{y-1} S_j^* P_{Z(Y,j+1)} S_j$$

and so we have the desired equation (4.5). □

The following remak is a vector version of Lemma 4.2, see [9].

Remark 4.3. Let $x, z_1, \ldots, z_n \in \mathbb{C}^k$. Then we have a generalization of Parseval's equation (4.3):

$$\langle x, x\rangle = \sum_{i=0}^{n} \frac{|\langle u_i, z_{i+1}\rangle|^2}{\langle z_{i+1}, z_{i+1}\rangle},$$

where $u_0 = x$, $u_i = u_{i-1} - \dfrac{\langle u_{i-1}, z_i\rangle}{\langle z_i, z_i\rangle} z_i$ for $i = 1, 2, \ldots, n$ and $z_{n+1} = \dfrac{1}{\|u_n\|} u_n$. If $\{z_1, \ldots, z_k\}$ is a complete orthonormal system in $\mathbb{C}^k$, then we can just get Parseval's equation (4.3).

Under orthogonal conditions, we have the following matrix version of Parseval's equation (4.4).

Theorem 4.4. *Let X be in $\mathbb{M}_{k\times m}$, Y in $\mathbb{M}_{k\times n}$, $Z_1, \ldots, Z_p$ in $\mathbb{M}_{k\times l}$ and $Z_i^* Z_j = 0$ for all $i \neq j$, $i, j \in \{1, \ldots, p\}$. Then*

$$Y^*X = \sum_{q=0}^{p-1} Y^* P_{Z_{q+1}} X + T_p^* S_p \tag{4.6}$$

where $S_0 = X$, $S_j = P_{Z_j}^{\perp} S_{j-1}$ for $j = 1, \ldots, p$, $T_0 = Y$ and $T_i = P_{Z_i}^{\perp} T_{i-1}$ for $i = 1, \ldots, p$.

Proof. Since $Z_i^* Z_j = 0$, we have $P_{Z_i} P_{Z_j} = 0$ for all $i \neq j$, $i, j \in \{1, \ldots, p\}$ and it follows that

$$\begin{aligned} P_{Z_j} S_{j-1} &= P_{Z_j}(I - P_{Z_{j-1}}) S_{j-2} \\ &= P_{Z_j} S_{j-2} \\ &= \cdots = P_{Z_j} X \end{aligned}$$

and similarly we have

$$P_{Z_j} T_{j-1} = P_{Z_j} Y.$$

Hence it follows that

$$\begin{aligned} T_p^* S_p &= T_{p-1}^*(I - P_{Z_p})(I - P_{Z_p}) S_{p-1} \\ &= T_{p-1}^* S_{p-1} - T_{p-1}^* P_{Z_p} S_{p-1} \\ &\vdots \\ &= T_0^* S_0 - T_0^* P_{Z_1} S_0 - \cdots - T_{p-1}^* P_{Z_p} S_{p-1} \\ &= Y^*X - \sum_{q=0}^{p-1} T_q^* P_{Z_{q+1}} S_q \\ &= Y^*X - \sum_{q=0}^{p-1} Y^* P_{Z_{q+1}} X \end{aligned}$$

and hence we have the desired equality (4.6). □

Remark 4.5. We can consider a vector version of Theorem 4.4. Let $x, y, z_1, \dots, z_p$ be in $\mathbb{C}^k$ and $\langle z_i, z_j\rangle = 0$ for all $i \neq j$, $i, j \in \{1, \dots, p\}$. Then we have a generalziation of Parseval's equation (4.4):

$$\langle x, y\rangle = \sum_{i=0}^{p-1} \frac{\langle x, z_{i+1}\rangle\langle z_{i+1}, y\rangle}{\langle z_{i+1}, z_{i+1}\rangle} + \langle u_p, v_p\rangle,$$

where $u_0 = x$, $u_j = u_{j-1} - \dfrac{\langle u_{j-1}, z_j\rangle}{\langle z_j, z_j\rangle} z_j$ for $j = 1, \dots, p$, $v_0 = y$ and $v_i = v_{i-1} - \dfrac{\langle v_{i-1}, z_i\rangle}{\langle z_i, z_i\rangle} z_i$ for $i = 1, \dots, p$.

In [9], Lin showed the following refinement of a weighted mixed Schwarz inequality (1.3): Let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$ and $0 \neq y \in \mathcal{H}$. If A^*y is orthogonal to a vector $z \in \mathcal{H}$ with $Az \neq 0$, then

$$|\langle Ax, y\rangle|^2 + \frac{\langle |A^*|^{2\beta} y, y\rangle |\langle |A|^{2\alpha} x, z\rangle|^2}{\langle |A|^{2\alpha} z, z\rangle} \leq \langle |A|^{2\alpha} x, x\rangle\langle |A^*|^{2\beta} y, y\rangle \tag{4.7}$$

for all $x \in \mathcal{H}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. The next theorem is a matrix version of (4.7).

Theorem 4.6. *Let X be in $\mathbb{M}_{m\times n}$, Y in $\mathbb{M}_{p\times q}$, Z_X in $\mathbb{M}_{m\times l_X}$, Z_Y in $\mathbb{M}_{p\times l_Y}$ and A in $\mathbb{M}_{p\times m}$. Suppose that $X^*|A|^{2\alpha} Z_X = 0$, $Y^*|A^*|^{2\beta} Z_Y = 0$ and $Z_Y^* A Z_X = 0$ for given $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. If $q \geq n$ and $p \geq m$, then*

$$\begin{aligned} |Y^*AX| &\leq X^*|A|^{\alpha} P^{\perp}_{|A^*|Z_Y} |A|^{\alpha} X \sharp U^*Y^*V|A|^{\beta} P^{\perp}_{|A|Z_X} |A|^{\beta} V^*YU \\ (&\leq X^*|A|^{2\alpha} X \sharp U^*Y^*|A^*|^{2\beta} YU), \end{aligned}$$

in which $U \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U|Y^*AX|$, and $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$.

Under the assumption $\ker P^{\perp}_{|A^*|Z_Y} |A|^{\alpha} X \subseteq \ker P^{\perp}_{|A|Z_X} |A|^{\beta} V^*YU$, *the equality holds if and only if there exists $W \in \mathbb{M}_n$ such that $P^{\perp}_{|A|Z_X} |A|^{\beta} V^*YU = P^{\perp}_{|A^*|Z_Y} |A|^{\alpha} XW$.*

Proof. Replacing X by $|A|^{\alpha}X$, Y by $|A|^{\beta}V^*Y$, Z_X by $|A|^{\alpha}Z_X$ and Z_Y by $|A|^{\beta}V^*Z_Y$ in (4.2) of Lemma 4.1, then we obtain the desired inequality and the equality condition. □

The next result is a multivariate extension of Lemma 4.1, which is a refinement of matrix Cauchy-Schwarz inequality (2.1) of Lemma 2.1:

Lemma 4.7. *Let X be in $\mathbb{M}_{k\times m}$, Y in $\mathbb{M}_{k\times n}$, $Z(X, 1), \dots, Z(X, x)$ in $\mathbb{M}_{k\times l_X}$ and $Z(Y, 1)$, $\dots, Z(Y, y)$ in $\mathbb{M}_{k\times l_Y}$. Suppose that $X^*Z(X, i) = 0$, $Y^*Z(Y, j) = 0$ and $Z(Y, j)^*Z(X, i) = 0$ for $i = 1, 2, \dots, x$, $j = 1, 2, \dots, y$ If $n \geq m$, then*

$$\begin{aligned} |Y^*X| &\leq (X^*X - \sum_{j=0}^{y-1} S_j^* P_{Z(Y,j+1)} S_j) \sharp U^*(Y^*Y - \sum_{i=0}^{x-1} T_i^* P_{Z(X,i+1)} T_i) U \\ (&\leq X^*X \sharp U^*Y^*YU), \end{aligned}$$

in which $U \in \mathbb{M}_{n\times m}$ consists of orthonormal columns and $Y^*X = U|Y^*X|$, where $S_0 = X$, $S_j = P^{\perp}_{Z(Y,j)} S_{j-1}$ for $j = 1, 2, \dots, y$, $T_0 = Y$ and $T_i = P^{\perp}_{Z(X,i)} T_{i-1}$ for $i = 1, 2, \dots, x$.

Under the assumption $\ker(\prod_{b=1}^{y} P^{\perp}_{(Y,y-b+1)}) X \subseteq \ker(\prod_{a=1}^{x} P^{\perp}_{(X,x-a+1)}) YU$, *the equality holds if and only if there exists $W \in \mathbb{M}_m$ such that $(\prod_{a=1}^{x} P^{\perp}_{(X,x-a+1)}) YU = (\prod_{b=1}^{y} P^{\perp}_{(Y,y-b+1)}) XW$.*

Proof. By Lemma 4.2, the following equations hold:

$$S_y^*S_y = X^*X - \sum_{j=0}^{y-1} S_j^* P_{Z(Y,j+1)} S_j$$

and

$$T_x^*T_x = Y^*Y - \sum_{i=0}^{x-1} T_i^* P_{Z(X,i+1)} T_i.$$

Since $X^*Z(X,i) = 0$, $Y^*Z(Y,j) = 0$ and $Z(Y,j)^*Z(X,i) = 0$, we have $P_{Z(X,i)}X = Y^*P_{Z(Y,j)} = P_{Z(X,i)}P_{Z(Y,j)} = 0$ for $i = 1, 2, \ldots, x$ and $j = 1, 2, \ldots, y$. Then it follows that

$$\begin{aligned}
T_x^*S_y &= T_{x-1}^* P_{Z(X,x)}^{\perp} P_{Z(Y,y)}^{\perp} S_{y-1} \\
&= Y^* \left(I + \sum_{s=1}^{x} \left(\sum_{1\le c_1<\cdots<c_s\le x} \prod_{p=1}^{s} (-1)^s P_{Z(X,c_p)} \right) \right. \\
&\qquad \left. + \sum_{t=1}^{y} \left(\sum_{1\le d_1<\cdots<d_t\le y} \prod_{q=1}^{t} (-1)^t P_{Z(Y,d_{t+1-q})} \right) \right) X \\
&= Y^*X.
\end{aligned}$$

So, we can get the desired inequality by Lemma 3.2:

$$\begin{aligned}
|Y^*X| &= |T_x^*S_y| \\
&\le S_y^*S_y \,\sharp\, U^*T_x^*T_xU \\
&= (X^*X - \sum_{j=0}^{y-1} S_j^* P_{Z(Y,j+1)} S_j) \,\sharp\, U^*(Y^*Y - \sum_{i=0}^{x-1} T_i^* P_{Z(X,i+1)} T_i)U.
\end{aligned}$$

Since $S_y = (\prod_{b=1}^{y} P_{(Y,y-b+1)}^{\perp})X$ and $T_x = (\prod_{a=1}^{x} P_{(X,x-a+1)}^{\perp})Y$, we have the equality condition by Lemma 3.2. □

Moreover, Lin showed the following multivariate extension of (4.7): Under the hypotheses of (4.7), if A^*y is orthogonal to a set of vectors $\{z_1, \ldots, z_n\} \subseteq \mathcal{H}$ with $Az_i \neq 0$, $i = 1, \ldots, n$, then

$$|\langle Ax, y\rangle|^2 + \langle |A^*|^{2\beta} y, y\rangle \sum_{i=1}^{n} \frac{|\langle |A|^{2\alpha} u_{i-1}, z_i\rangle|^2}{\langle |A|^{2\alpha} z_i, z_i\rangle} \le \langle |A|^{2\alpha} x, x\rangle \langle |A^*|^{2\beta} y, y\rangle \tag{4.8}$$

for every $x \in \mathcal{H}$, where $u_i = u_{i-1} - \frac{\langle |A|^{2\alpha} u_{i-1}, z_i\rangle}{\langle |A|^{2\alpha} z_i, z_i\rangle} z_i$, $i = 1, \ldots, n$ with $u_0 = x$. The next result is a multivariate extension of Theorem 4.6 and a matrix version of (4.8).

Theorem 4.8. *Let X be in $\mathbb{M}_{m\times n}$, Y in $\mathbb{M}_{p\times q}$, $Z(X,1), \ldots, Z(X,x)$ in $\mathbb{M}_{m\times l_X}$, $Z(Y,1)$, $\ldots, Z(Y,y)$ in $\mathbb{M}_{p\times l_Y}$, and A in $\mathbb{M}_{p\times m}$. Suppose that $X^*|A|^{2\alpha}Z(X,i) = 0$, $Y^*|A^*|^{2\beta}Z(Y,j)$*

$= 0$, $Z(Y,j)^*AZ(X,i) = 0$ for $i = 1,2,\dots,x$, $j = 1,2,\dots,y$ for given $\alpha,\beta \in [0,1]$ with $\alpha+\beta = 1$. If $q \geq n$ and $p \geq m$, then

$$\begin{aligned}|Y^*AX| &\leq (X^*|A|^{2\alpha}X - \sum_{j=1}^{y-1} S_j^* P_{|A^*|Z(Y,j+1)} S_j) \sharp U^*(Y^*|A^*|^{2\beta}Y - \sum_{i=1}^{x-1} T_i^* P_{|A|Z(X,i+1)} T_i)U \\ &(\leq X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU),\end{aligned}$$

in which $U \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U|Y^*AX|$, and $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$, where $S_0 = |A|^\alpha X$, $S_j = P^{\perp}_{|A^*|Z(Y,j)} S_{j-1}$ for $j = 1,2,\dots,y$, $T_0 = |A|^\beta VY$ and $T_i = P^{\perp}_{|A|Z(X,i)} T_{i-1}$ for $i = 1,2,\dots,x$.

Under the assumption $\ker(\prod_{b=1}^{y} P^{\perp}_{|A^*|Z(Y,y-b+1)})|A|^\alpha X \subseteq \ker(\prod_{a=1}^{x} P^{\perp}_{|A|Z(X,x-a+1)})|A|^\beta V^*YU$, the equality holds if and only if there exists $W \in \mathbb{M}_n$ such that $(\prod_{a=1}^{x} P^{\perp}_{|A|Z(X,x-a+1)})|A|^\beta V^*YU$ $= (\prod_{b=1}^{y} P^{\perp}_{|A^*|Z(Y,y-b+1)})|A|^\alpha XW$, where $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$.

Proof. Replacing X by $|A|^\alpha X$, Y by $|A|^\beta V^*Y$, $Z(X,i)$ by $|A|^\alpha Z(X,i)$ and $Z(Y,j)$ by $|A|^\beta V^*Z(Y,j)$ in Lemma 4.7 for all $i = 1,2,\dots,x$ and $j = 1,2,\dots,y$, then we obtain the desired inequality and the equality condition. □

We note that the vector version of Theorem 4.8 is a matrix version of Theorem 4 in [9]: Let x be in $\mathbb{C}^m$, y in $\mathbb{C}^p$, $z(x,1),\dots,z(x,a)$ in $\mathbb{C}^m$, $z(y,1),\dots,z(y,b)$ in $\mathbb{C}^p$, and A in $\mathbb{M}_{p\times m}$. Suppose that $\langle |A|^{2\alpha} z(x,i), x\rangle = 0$, $\langle |A^*|^{2\beta} z(y,j), y\rangle = 0$, $\langle Az(x,i), z(y,j)\rangle = 0$ for $i = 1,2,\dots,a$, $j = 1,2,\dots,b$ for given $\alpha,\beta \in [0,1]$ with $\alpha+\beta = 1$. If $p \geq m$, then

$$\begin{aligned}|\langle Ax,y\rangle|^2 &\leq \left(\langle |A|^{2\alpha}x,x\rangle - \sum_{j=1}^{b-1} \frac{|\langle |A^*|z(y,j+1), s_j\rangle|^2}{\langle |A^*|^2 z(y,j+1), z(y,j+1)\rangle}\right) \\ &\quad\times \left(\langle |A^*|^{2\beta}y,y\rangle - \sum_{i=1}^{a-1} \frac{|\langle |A|z(x,i+1), t_i\rangle|^2}{\langle |A|^2 z(x,i+1), z(x,i+1)\rangle}\right),\end{aligned}$$

in which $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$, $s_0 = |A|^\alpha x$, $s_j = P^{\perp}_{|A^*|z(y,j)} s_{j-1}$ for $j = 1,2,\dots,b$, $t_0 = |A|^\beta Vy$ and $t_i = P^{\perp}_{|A|z(x,i)} t_{i-1}$ for $i = 1,2,\dots,a$.

5 Weighted Wielandt inequality We consider a different way of a refinement of a weighted Schwarz inequality in §4. We show a weighted version of matrix Wielandt inequality. We proved a matrix version of Wielandt inequality, see [2]: Let A be a positive semidefinite matrix in $\mathbb{M}_k$, with $\mathrm{rank}(A) = r$, $\lambda_1 \geq \cdots \geq \lambda_r > 0$ eigenvalues of A, and X,Y in $\mathbb{M}_{k\times n}$ such that $Y^*P_AX = 0$ where P_A is the orthogonal projection on the column space of A. Then

$$|Y^*AX| \leq \left(\frac{\lambda_1-\lambda_r}{\lambda_1+\lambda_r}\right)(X^*AX \sharp U^*Y^*AYU), \tag{5.1}$$

in which $U \in \mathbb{M}_n$ is a unitary matrix such that $Y^*AX = U|Y^*AX|$. The following theorem is a weighted version of (5.1).

Theorem 5.1. *Let A be a matrix in $\mathbb{M}_{p\times m}$, with rank(A)=r, $\sigma_1 \geq \cdots \geq \sigma_r > 0$ singular values of A, $X \in \mathbb{M}_{m\times n}$ and $Y \in \mathbb{M}_{p\times q}$. For all $\alpha, \beta \in [0,1]$ with $\alpha+\beta = 1$, if $p \geq m$, $q \geq n$ and $Y^*VP_{|A|}X = 0$, then*

$$|Y^*AX| \leq \left(\frac{\sigma_1-\sigma_r}{\sigma_1+\sigma_r}\right)(X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU),$$

in which $U \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U|Y^*AX|$, and $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$.

Proof. Let $c = \dfrac{2\sigma_1\sigma_r}{\sigma_1+\sigma_r}$. Since $\sigma_1 P_{|A|} - |A|$ and $|A| - \sigma_r P_{|A|}$ are positive semidefinite and they commute, it follows that $(\sigma_1 P_{|A|} - |A|)(|A| - \sigma_r P_{|A|}) \geq 0$ and hence

$$(P_{|A|} - c|A|^\dagger)^2 \leq \left(\frac{\sigma_1-\sigma_r}{\sigma_1+\sigma_r}\right)^2 I, \tag{5.2}$$

where $|A|^\dagger$ means the Moore-Penrose generalized inverse of $|A|$. So, we can get the desired inequality:

$$\begin{aligned}
|Y^*AX| &= |Y^*AX - cY^*VP_{|A|}X| = |Y^*V|A|^\beta(P_{|A|} - c|A|^\dagger)|A|^\alpha X| \\
&= |(P_{|A|} - c|A|^\dagger)|A|^\beta V^*Y)^*(|A|^\alpha X)| \\
&\leq X^*|A|^{2\alpha}X \sharp U^*Y^*V|A|^\beta(P_{|A|} - c|A|^\dagger)^2|A|^\beta V^*YU \quad \text{by Lemma 3.2} \\
&\leq X^*|A|^{2\alpha}X \sharp U^*Y^*V|A|^\beta \left(\frac{\sigma_1-\sigma_r}{\sigma_1+\sigma_r}\right)^2 |A|^\beta V^*YU \quad \text{by (5.2)} \\
&= \left(\frac{\sigma_1-\sigma_r}{\sigma_1+\sigma_r}\right)(X^*|A|^{2\alpha}X \sharp U^*Y^*|A^*|^{2\beta}YU).
\end{aligned}$$

□

Lastly, we consider a Wielandt version of Theorem 3.4 by a way similar to the proof of Theorem 5.1.

Theorem 5.2. *Let A be a matrix in $\mathbb{M}_{p\times m}$, with rank(A)=r, $\sigma_1 \geq \cdots \geq \sigma_r > 0$ singular values of A, $X \in \mathbb{M}_{m\times n}$, $Y \in \mathbb{M}_{p\times q}$ and f, g complex functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$ For all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, if $p \geq m$, $q \geq n$ and $Y^*VP_{|A|}X = 0$, then*

$$|Y^*AX| \leq \left(\frac{\sigma_1-\sigma_r}{\sigma_1+\sigma_r}\right)(X^*|f(|A|)|^2X \sharp U^*Y^*|g(|A^*|)|^2YU),$$

in which $U \in \mathbb{M}_{q\times n}$ consists of orthonormal columns and $Y^*AX = U|Y^*AX|$, and $V \in \mathbb{M}_{p\times m}$ consists of orthonormal columns and $A = V|A|$.

Acknowledgement. The second author is partially supported by JSPS KAKENHI Grant Number JP 16K05253.

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Communicated by *Masatoshi Fujii*

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