ON A DUALITY BETWEEN THE OPERATORS AND THE SPACE OF SEQUENCES

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Abstract. On a space of sequences, the multiplication operator and the Hankel operator are defined and investigated. On the other hand, the concept of a space of sequences is basic, but its properties are not well known nevertheless. In this paper, we prove some properties of the space of sequences, and by means of this, we show certain modification of H^1-$BMOA$ duality and L^1-L^∞ duality (Theorem 5.2) from the viewpoint of theory of these operators.

1 Introduction. The multiplication operator is naturally defined on the Lebesgue space L^p as well as on the space ℓ^p. The Hankel operator is also defined on the Hardy space H^p as well as on the space ℓ^p_+. These operators are well investigated, but properties of a space of sequences are not well known nevertheless. In this paper, we shall prove some properties of the space obtained from these operators (Section 3), of the space of sequences (Section 4), and show certain modification of H^1-$BMOA$ duality and L^1-L^∞ duality (Theorem 5.2) from the viewpoint of theory of these operators.

Let T be the unit circle in the complex plane and L^p be the L^p space of functions on T with respect to Lebesgue measure. We denote by H^p the Hardy space defined by

$$H^p := \{f \in L^p \mid (f)_n = 0 \quad \text{for } n < 0\},$$

where $(f)_n$ means the n-th Fourier coefficient of f. We also denote by H^p_0 the space of functions in H^p whose zeroth Fourier coefficient is zero, and by $BMOA$ the set of all analytic functions of bounded mean oscillation on T.

Let $1 < p < \infty$. It is known that for a in L^1, a function a is in L^∞ if and only if the multiplication operator $M(a)$ is defined on L^p, and L^∞ is isomorphic to $(L^1)^*$. It is also known that for a in H^2, a function a is in $BMOA$ if the Hankel operator $H(\chi_1 a)$ is defined on H^p, where $\chi_j(\theta) := e^{\sqrt{-1}j\theta}$ $(0 \le \theta \le 2\pi)$, and $BMOA$ is isomorphic to $(H^1)^*$ (cf. [1], [5]).

Now we consider the discrete versions of these topics. Let ℓ^p be the Banach space of sequences of complex numbers defined by

$$\ell^p := \left\{ \varphi = \{\varphi_n\}_{n\in\mathbb{Z}} \mid \|\varphi\|_{\ell^p} := (\sum_{n\in\mathbb{Z}} |\varphi_n|^p)^{\frac{1}{p}} < \infty \right\},$$

and ℓ^p_+ be the space defined by

$$\ell^p_+ := \{\varphi \in \ell^p \mid \varphi_n = 0 \quad \text{for } n < 0\}.$$

Let $1 \le p < \infty$. For $a \in L^1$, a function a is in a subspace $M^p \subset L^1$ given in Section 2 if and only if the multiplication operator $M(a)$ is defined on ℓ^p. For $a \in H^2$, a function a

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is in a subspace $M_+^p \cap H_0^2$ given in Section 3 if and only if the Hankel operator $H(\chi_1 a)$ is defined on ℓ_+^p.

Therefore it is a natural question whether there are normed spaces V^p and V_+^p such that M^p and $M_+^p \cap H_0^2$ are isomorphic to $(V^p)^*$ and $(V_+^p)^*$, respectively. We will show such spaces V^p and V_+^p exist by construction. These are certain modification of H^1-$BMOA$ duality and L^1-L^∞ duality.

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2 Preliminaries. In this section, we shall give some basic facts on the multiplication operators and the Hankel operators.

We denote by $\mathfrak{B}(X)$ the set of all bounded linear operators on a Banach space X to itself, and by $(a)_n$ the n-th Fourier coefficient of a. Let $e_j := \{\delta_{j,n}\}_{n\in\mathbb{Z}}$ (δ : Kronecker's delta).

For $1 < p < \infty$ and $a \in L^\infty$, the multiplication operator $M(a)$ on L^p is defined by

$$M(a) \ : \ L^p \longrightarrow L^p \ : \ f \longmapsto a \cdot f,$$

and it is easy to see that $\|a\|_{L^\infty} = \|M(a)\|_{\mathfrak{B}(L^p)}$. Note that the j-th Fourier coefficient $(a \cdot f)_j$ of $a \cdot f$ is equal to $\sum_{k\in\mathbb{Z}} (a)_{j-k}(f)_k$ for all $j \in \mathbb{Z}$.

For a function a in L^1 and a sequence φ, we put

$$a * \varphi := \left\{ \sum_{k\in\mathbb{Z}} (a)_{j-k}\varphi_k \right\}_{j\in\mathbb{Z}},$$

whenever the sequence $a * \varphi$ can be defined. For $1 \le p < \infty$, a vector space M^p is defined by

$$M^p := \left\{ a \in L^1 \mid \|a\|_{M^p} := \sup\left\{ \|a * \varphi\|_{\ell^p} \mid \|\varphi\|_{\ell^p} \le 1 \right\} < \infty \right\}.$$

It is obvious that $\|\cdot\|_{M^p}$ is a norm on M^p. For $a \in M^p$, the multiplication operator $M(a)$ on ℓ^p is defined by

$$M(a) \ : \ \ell^p \longrightarrow \ell^p \ : \ \varphi \longmapsto a * \varphi,$$

and $\|a\|_{M^p} = \|M(a)\|_{\mathfrak{B}(\ell^p)}$.

The following properties of M^p are basic to our argument (cf. [1]).

Proposition 2.1. *1. For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\|\cdot\|_{M^p} = \|\cdot\|_{M^q}$ and $M^p = M^q$.*

2. $\|\cdot\|_{M^2} = \|\cdot\|_{L^\infty}$ and $M^2 = L^\infty$.

3. $M^1 = \left\{ a \in L^1 \mid \sum_{n\in\mathbb{Z}} |(a)_n| < \infty \right\}$ and $\|a\|_{M^1} = \sum_{n\in\mathbb{Z}} |(a)_n|$.

4. For $1 \le p \le r \le 2$, $\|\cdot\|_{M^2} \le \|\cdot\|_{M^r} \le \|\cdot\|_{M^p} \le \|\cdot\|_{M^1}$ and $M^1 \subset M^p \subset M^r \subset M^2$.

5. For $1 \le p < \infty$, M^p is a Banach algebra with respect to $\|\cdot\|_{M^p}$.

Now we define the Hankel operators. Let $1 < p < \infty$. The flip operator J on L^p is defined by

$$J \ : \ L^p \longrightarrow L^p \ : \ \sum_{n\in\mathbb{Z}} (f)_n \chi_n \longmapsto \sum_{n\in\mathbb{Z}} (f)_n \chi_{-n-1},$$

the Riesz projection P is defined by

$$P \,:\, L^p \longrightarrow H^p \,:\, \sum_{n\in\mathbb{Z}}(f)_n\chi_n \longmapsto \sum_{n\geq 0}(f)_n\chi_n,$$

and it is well known that

$$c_p := \sup\{\|P(f)\|_p \mid \|f\|_p \leq 1\} < \infty$$

by the M. Riesz theorem (cf. [2]). Let I be the identity operator on L^p, and $Q := I - P$. For $a \in L^\infty$, the Hankel operator $H(a)$ on H^p is defined by

$$H(a) \,:\, H^p \longrightarrow H^p \,:\, f \longmapsto PM(a)QJf.$$

The discrete versions of these operators are similarly defined. Let $1 \leq p < \infty$. The flip operator J on ℓ^p is given by

$$J \,:\, \ell^p \longrightarrow \ell^p \,:\, \{\varphi_n\}_{n\in\mathbb{Z}} \longmapsto \{\varphi_{-n-1}\}_{n\in\mathbb{Z}},$$

the Riesz projection P is

$$P \,:\, \ell^p \longrightarrow \ell^p_+ \,:\, \{\varphi_n\}_{n\in\mathbb{Z}} \longmapsto \{\varphi_n\}_{n\geq 0},$$

and $Q := I - P$ (I : the identity operator on ℓ^p). For $a \in M^p$, the Hankel operator $H(a)$ on ℓ^p_+ is defined by

$$H(a) \,:\, \ell^p_+ \longrightarrow \ell^p_+ \,:\, \varphi \longmapsto PM(a)QJ\varphi.$$

3 New classes M^p_+ and N^p_+. Note that $H(a)\{\varphi_j\}_{j\geq 0}$ is equal to $\left\{\sum_{k\geq 0}(a)_{j+k+1}\varphi_k\right\}_{j\geq 0}$. We consider a new class M^p_+ to extend the domain of the Hankel operator. For a function a in L^1 and a sequence φ, we define a sequence $a \odot \varphi$ by

$$a \odot \varphi := \left\{\sum_{k\geq 0}(a)_{j+k+1}\varphi_k\right\}_{j\geq 0},$$

whenever the sequence $a \odot \varphi$ can be defined.

Definition 3.1. *For $1 \leq p < \infty$, we define a vector space M^p_+ as*

$$M^p_+ := \left\{a \in L^1 \mid \|a\|_{M^p_+} := \sup\left\{\|a \odot \varphi\|_{\ell^p_+} \mid \|\varphi\|_{\ell^p_+} \leq 1\right\} < \infty\right\}.$$

For $a \in M^p_+$, we define the Hankel operator $H(a)$ on ℓ^p_+ as

$$H(a) \,:\, \ell^p_+ \longrightarrow \ell^p_+ \,:\, \varphi \longmapsto a \odot \varphi,$$

and $\|a\|_{M^p_+} = \|H(a)\|_{\mathfrak{B}(\ell^p_+)}$.

It is easy to see that $\|\cdot\|_{M^p_+} \leq \|\cdot\|_{M^p}$ and $M^p \subset M^p_+$. Indeed, let ℓ^p_- be the space defined by

$$\ell^p_- := \{\varphi \in \ell^p \mid \varphi_n = 0 \quad \text{for } n \geq 0\},$$

and

$$\begin{aligned}
\|a\|_{M_+^p} &= \sup\left\{\left(\sum_{j\geq 0}|\sum_{k\leq -1}(a)_{j-k}\varphi_k|^p\right)^{\frac{1}{p}} \mid \|\varphi\|_{\ell_-^p}\leq 1\right\}\\
&= \sup\left\{\|P(a*\varphi)\|_{\ell_+^p}\mid \|\varphi\|_{\ell_-^p}\leq 1\right\}\\
&\leq \sup\left\{\|a*\varphi\|_{\ell^p}\mid \|\varphi\|_{\ell_-^p}\leq 1\right\}\\
&\leq \sup\{\|a*\varphi\|_{\ell^p}\mid \|\varphi\|_{\ell^p}\leq 1\}\\
&= \|a\|_{M^p}.
\end{aligned}$$

Hence, we extended the domain of the Hankel operator to M_+^p.

Here, $\|\cdot\|_{M_+^p}$ is actually a norm on $M_+^p\cap H_0^2$. In fact, it is a semi-norm and we see that

$$\begin{aligned}
\left(\sum_{n\geq 1}|(a)_n|^p\right)^{\frac{1}{p}} &= \left(\sum_{n\geq 0}|(a)_{n+1}|^p\right)^{\frac{1}{p}}\\
&\leq \sup\left\{\left(\sum_{j\geq 0}|\sum_{k\geq 0}(a)_{j+k+1}\varphi_k|^p\right)^{\frac{1}{p}} \mid \|\varphi\|_{\ell_+^p}\leq 1\right\}\\
&= \|a\|_{M_+^p}
\end{aligned}$$

for $a\in M_+^p$. Thus $\|\cdot\|_{M_+^p}$ is a norm on $M_+^p\cap H_0^2$. The real inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is defined by

$$\langle\{a_n\}_{n\in\mathbb{Z}},\{b_n\}_{n\in\mathbb{Z}}\rangle_{\mathbb{R}} := \sum_{n\in\mathbb{Z}} a_n b_n.$$

The space M_+^p has the following properties like M^p.

Proposition 3.2. *1.* $M_+^1\cap H_0^2 = M^1\cap H_0^2$.

2. For $1<p<\infty$ *and* $\frac{1}{p}+\frac{1}{q}=1$, $\|\cdot\|_{M_+^p}=\|\cdot\|_{M_+^q}$ *and* $M_+^p=M_+^q$.

3. For $1<p\leq r\leq 2$, $\|\cdot\|_{M_+^2}\leq\|\cdot\|_{M_+^r}\leq\|\cdot\|_{M_+^p}$ *and* $M_+^p\subset M_+^r\subset M_+^2$.

Proof. 1. We show

$$M_+^1\cap L^2 = \left\{a\in L^2 \mid \sum_{n\geq 1}|(a)_n|<\infty\right\}.$$

It is already proved that $\sum_{n\geq 1}|(a)_n|\leq\|a\|_{M_+^1}$ for $a\in M_+^1$. For $a\in L^2$ with $\sum_{n\geq 1}|(a)_n|<\infty$, we have

$$H(a) = H(\sum_{n\in\mathbb{Z}}(a)_n\chi_n) = H(\sum_{n\geq 1}(a)_n\chi_n) = \sum_{n\geq 1}(a)_n H(\chi_n).$$

Therefore

$$\|a\|_{M_+^1}\leq\sum_{n\geq 1}|(a)_n|\|\chi_n\|_{M_+^1}\leq\sum_{n\geq 1}|(a)_n|,$$

and we conclude

$$M^1_+ \cap L^2 = \left\{ a \in L^2 \mid \sum_{n\geq 1} |(a)_n| < \infty \right\}.$$

This implies the conclusion.

2. Let $a \in M^p_+$. Since $\langle e_j, H(a)e_k\rangle_{\mathbb{R}} = (a)_{j+k+1} = \langle H(a)e_j, e_k\rangle_{\mathbb{R}}$ holds for $j,k \geq 0$, $\langle \varphi, H(a)\psi\rangle_{\mathbb{R}} = \langle H(a)\varphi, \psi\rangle_{\mathbb{R}}$ holds for $\varphi, \psi \in \ell^0_+$. Thus

$$\begin{aligned}\|a\|_{M^p_+} &= \sup\left\{ \|H(a)\varphi\|_{\ell^p_+} \mid \|\varphi\|_{\ell^p_+} \leq 1 \right\} \\ &= \sup\left\{ \langle H(a)\varphi, \psi\rangle_{\mathbb{R}} \mid \|\varphi\|_{\ell^p_+}, \|\psi\|_{\ell^q_+} \leq 1 \right\} \\ &= \sup\left\{ \langle \varphi, H(a)\psi\rangle_{\mathbb{R}} \mid \|\varphi\|_{\ell^p_+}, \|\psi\|_{\ell^q_+} \leq 1 \right\} \\ &= \sup\left\{ \|H(a)\psi\|_{\ell^q_+} \mid \|\psi\|_{\ell^q_+} \leq 1 \right\} \\ &= \|a\|_{M^q_+}\end{aligned}$$

and $a \in M^q_+$. This implies the conclusion.

3. Let $a \in M^p_+$ and $1 < p \leq r \leq 2$. Since $H(a) \in \mathfrak{B}(\ell^p_+) \cap \mathfrak{B}(\ell^q_+)$, $\|H(a)\varphi\|_{\ell^r_+} \leq \|H(a)\|^{1-t}_{\mathfrak{B}(\ell^p_+)} \|H(a)\varphi\|^t_{\mathfrak{B}(\ell^q_+)}$ $(0 \leq t \leq 1)$ by the Riesz-Thorin interpolation theorem. Hence $\|a\|_{M^2_+} \leq \|a\|_{M^r_+} \leq \|a\|_{M^p_+}$. □

We also consider another new class N^p_+ to extend the domain of the Hankel operator on the Hardy space. Note that the j-th Fourier coefficient $(H(a)f)_j$ of $H(a)f$ is equal to $\sum_{k\geq 0} (a)_{j+k+1}(f)_k$ for all $j \geq 0$.

Definition 3.3. *For $1 < p < \infty$, we define a vector space N^p_+ as*

$$N^p_+ := \left\{ a \in L^1 \mid \|a\|_{N^p_+} := \sup\left\{ \|\sum_{j\geq 0}\sum_{k\geq 0} (a)_{j+k+1}(f)_k \chi_j\|_{H^p} \mid \|f\|_{H^p} \leq 1 \right\} < \infty \right\}.$$

For $a \in N^p_+$, we define the Hankel operator $H(a)$ on H^p as

$$H(a) \ : \ H^p \ \longrightarrow \ H^p \ : \ f \ \longmapsto \ \sum_{j\geq 0}\sum_{k\geq 0} (a)_{j+k+1}(f)_k \chi_j,$$

and $\|a\|_{N^p_+} = \|H(a)\|_{\mathfrak{B}(H^p)}$.

It is easy to see that $\|\cdot\|_{N^p_+} \leq c_p^2 \|\cdot\|_{L^\infty}$ and $L^\infty \subset N^p_+$. Indeed,

$$\begin{aligned}\|a\|_{N^p_+} &= \sup\{\|P(a\cdot(QJf))\|_{H^p} \mid \|f\|_{H^p} \leq 1\} \\ &\leq \sup\{c_p\|a\|_{L^\infty} c_q \|f\|_{H^p} \mid \|f\|_{H^p} \leq 1\} \\ &\leq c_p^2 \|a\|_{L^\infty}.\end{aligned}$$

Hence, the domain of the Hankel operator is extended to N^p_+.

Here, $\|\cdot\|_{N^p_+}$ is actually a norm on $N^p_+ \cap H^2_0$. In fact, it is a semi-norm and we see that

$$\begin{aligned}\|a\|_{H^p} &= \|P(a\cdot\chi_{-1})\|_{H^p} \\ &= \|P(a\cdot(QJ\chi_0))\|_{H^p} \\ &\leq \|a\|_{N^p_+}\end{aligned}$$

for $a \in H_0^2$. Thus $\|\cdot\|_{N_+^p}$ is a norm on $N_+^p \cap H_0^2$.

The space N_+^p has the following properties like M^p too.

Proposition 3.4. *1. For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\|\cdot\|_{N_+^p} = \|\cdot\|_{N_+^q}$ and $N_+^p = N_+^q$.*

2. For $1 < p \leq r \leq 2$, $\|\cdot\|_{N_+^2} \leq \|\cdot\|_{N_+^r} \leq \|\cdot\|_{N_+^p}$ and $N_+^p \subset N_+^r \subset N_+^2$.

3. For $1 < p < \infty$, $N_+^p \cap H_0^2$ is isomorphic to the subspace of $BMOA$ as normed spaces, via the isomorphism :

$$a \longmapsto \chi_{-1} a$$

.

4. $\|\cdot\|_{M_+^2} = \|\cdot\|_{N_+^2}$ and $M_+^2 = N_+^2$.

5. $M_+^2 \cap H_0^2$ is isomorphic to the subspace of $BMOA$ as normed spaces.

6. For $1 \leq p < \infty$, $M_+^p \cap H_0^2$ is a Banach space with respect to $\|\cdot\|_{M_+^p}$.

Proof. 1. The proof is the same as that of Proposition 3.2.2.

2. The proof is the same as that of Proposition 3.2.3.

3. The statement is the known fact by the proof of the Nehari Theorem (cf. [1], [5]).

4. A unitary operator

$$U \ : \ H^2 \longrightarrow \ell_+^2 \ : \ \sum_{n \geq 0} \varphi_n \chi_n \longmapsto \{\varphi_n\}_{n \geq 0}$$

implies the conclusion.

5. 3 and 4 imply the conclusion.

6. For $1 < p < \infty$, 3, 4 and Proposition 3.2.3 show the statement. For $p = 1$, Proposition 2.1.5 and Proposition 3.2.1 show the statement too. ☐

In Section 5, we will prove that M^p and $M_+^p \cap H_0^2$ are not only Banach spaces but also dual spaces of some spaces.

Let c^0 and c_+^0 be subspaces of ℓ^∞ given by

$$c^0 := \left\{ \{\varphi_n\}_{n \in \mathbb{Z}} \mid \|\varphi\|_{c^0} := \sup_{n \in \mathbb{Z}} |\varphi_n| < \infty, \quad \lim_{n \to \pm\infty} \varphi_n = 0 \right\}$$

and

$$c_+^0 := \left\{ \varphi \in c^0 \mid \varphi_n = 0 \quad \text{for } n < 0 \right\},$$

respectively. For two sequences φ and ψ, we define a sequence $\varphi * \psi$ by

$$\varphi * \psi := \left\{ \sum_{k \in \mathbb{Z}} \varphi_{j-k} \psi_k \right\}_{j \in \mathbb{Z}},$$

whenever the sequence $\varphi * \psi$ can be defined.

We show the following norm estimates of $M_+^p \cap H_0^2$ and M^p.

Proposition 3.5. *Let $1 \leq p < \infty$, and $a \in L^1$.*

1. $\|a\|_{M_+^p} = \sup \left\{ |\langle \{(a)_{n+1}\}_{n \geq 0}, \varphi * \psi \rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell_+^p}, \ \|\psi\|_{\ell_+^q} \leq 1 \right\}$ holds. When $p = 1$, we replace ℓ_+^q with c_+^0 in the right-hand side.

2. $\|a\|_{M^p} = \sup\left\{|\langle \{(a)_n\}_{n\in\mathbb{Z}}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \leq 1\right\}$ *holds. When* $p = 1$*, we replace* ℓ^q *with* c^0 *in the right-hand side.*

Proof. 1. Let $1 < p \leq 2$. For $\varphi \in \ell^p_+$ with $\|\varphi\|_{\ell^p_+} \leq 1$, a linear mapping

$$D_\varphi \ :\ \ell^q_+ \longrightarrow \mathbb{C} \ :\ \psi \longmapsto \langle \psi, a \odot \varphi\rangle_{\mathbb{R}}$$

satisfies

$$\begin{aligned}\|a \odot \varphi\|_{\ell^p_+} &= \|D_\varphi\|_{(\ell^q_+)^*}\\ &= \sup\left\{|\langle \psi, a\odot\varphi\rangle_{\mathbb{R}}| \mid \|\psi\|_{\ell^q_+} \leq 1\right\}\\ &\leq \sup\left\{|\langle a\odot\varphi, \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p_+},\ \|\psi\|_{\ell^q_+} \leq 1\right\}\\ &= \sup\left\{|\langle \{(a)_{n+1}\}_{n\geq 0}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p_+},\ \|\psi\|_{\ell^q_+} \leq 1\right\}.\end{aligned}$$

It implies

$$\begin{aligned}\|a\|_{M^p_+} &= \sup\left\{\|a\odot\varphi\|_{\ell^p_+} \mid \|\varphi\|_{\ell^p_+} \leq 1\right\}\\ &\leq \sup\left\{|\langle \{(a)_{n+1}\}_{n\geq 0}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p_+},\ \|\psi\|_{\ell^q_+} \leq 1\right\}.\end{aligned}$$

Conversely,

$$\begin{aligned}\|a\|_{M^p_+} &= \|H(a)\|_{\mathfrak{B}(\ell^p_+)}\\ &\geq \sup\left\{|G(H(a)\varphi)| \mid \|\varphi\|_{\ell^p_+},\ \|G\|_{(\ell^p_+)^*} \leq 1\right\}\\ &= \sup\left\{|\langle H(a)\varphi, g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p_+},\ \|g\|_{\ell^q_+} \leq 1\right\}\\ &= \sup\left\{|\langle \{(a)_{n+1}\}_{n\geq 0}, \varphi * g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p_+},\ \|g\|_{\ell^q_+} \leq 1\right\}.\end{aligned}$$

Now let $p = 1$. $\|a\|_{M^1_+} \leq \sup\left\{|\langle \{(a)_{n+1}\}_{n\geq 0}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1_+},\ \|\psi\|_{c^0_+} \leq 1\right\}$ holds from as above. Conversely,

$$\begin{aligned}\|a\|_{M^1_+} &= \|H(a)\|_{\mathfrak{B}(\ell^1_+)}\\ &\geq \sup\left\{|G(H(a)\varphi)| \mid \|\varphi\|_{\ell^1_+},\ \|G\|_{(\ell^1_+)^*} \leq 1\right\}\\ &= \sup\left\{|\langle H(a)\varphi, g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1_+},\ \|g\|_{\ell^\infty_+} \leq 1\right\}\\ &\geq \sup\left\{|\langle \{(a)_{n+1}\}_{n\geq 0}, \varphi * g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1_+},\ \|g\|_{c^0_+} \leq 1\right\}.\end{aligned}$$

When $2 < p < \infty$, Proposition 3.2.2 leads the conclusion.

2. For $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}} \in \ell^\infty$, we define φ^b as $\varphi^b := \{\varphi_{-n}\}_{n\in\mathbb{Z}}$.

Let $1 < p \leq 2$. For $\varphi \in \ell^p$ with $\|\varphi\|_{\ell^p} \leq 1$, a linear mapping

$$D_\varphi \ :\ \ell^q \longrightarrow \mathbb{C} \ :\ \psi \longmapsto \langle \psi, a * \varphi\rangle_{\mathbb{R}}$$

satisfies

$$\begin{aligned}\|a * \varphi\|_{\ell^p} &= \|D_\varphi\|_{(\ell^q)^*}\\ &= \sup\left\{|\langle \psi, a*\varphi\rangle_{\mathbb{R}}| \mid \|\psi\|_{\ell^q} \leq 1\right\}\\ &\leq \sup\left\{|\langle a*\varphi, \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \leq 1\right\}\\ &= \sup\left\{|\langle \{(a)_n\}_{n\in\mathbb{Z}}, \varphi^b * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \leq 1\right\}\\ &= \sup\left\{|\langle \{(a)_n\}_{n\in\mathbb{Z}}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \leq 1\right\}.\end{aligned}$$

It implies

$$\begin{aligned}\|a\|_{M^p} &= \sup\{\|a*\varphi\|_{\ell^p} \mid \|\varphi\|_{\ell^p} \le 1\} \\ &\le \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi*\psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \le 1\right\}.\end{aligned}$$

Conversely,

$$\begin{aligned}\|a\|_{M^p} &= \|M(a)\|_{\mathfrak{B}(\ell^p)} \\ &\ge \sup\left\{|G(M(a)\varphi)| \mid \|\varphi\|_{\ell^p},\ \|G\|_{(\ell^p)^*} \le 1\right\} \\ &= \sup\{|\langle M(a)\varphi, g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|g\|_{\ell^q} \le 1\} \\ &= \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi*g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|g\|_{\ell^q} \le 1\right\}.\end{aligned}$$

Hence $\|a\|_{M^p} = \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi*\psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \le 1\right\}$.

Now let $p=1$. $\|a\|_{M^1} \le \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi*\psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1},\ \|\psi\|_{c^0} \le 1\right\}$ from as above. Conversely,

$$\begin{aligned}\|a\|_{M^1} &= \|M(a)\|_{\mathfrak{B}(\ell^1)} \\ &\ge \sup\left\{|G(M(a)\varphi)| \mid \|\varphi\|_{\ell^1},\ \|G\|_{(\ell^1)^*} \le 1\right\} \\ &= \sup\{|\langle M(a)\varphi, g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1},\ \|g\|_{\ell^\infty} \le 1\} \\ &\ge \sup\left\{|\langle\{(a)_n\}_{n\in\mathbb{Z}}, \varphi*g\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell^1},\ \|g\|_{c^0} \le 1\right\}.\end{aligned}$$

When $2 < p < \infty$, Proposition 2.1.1 leads the conclusion. □

4 Some spaces of sequences. In this section, we show some properties of spaces which are linearization of sets of all $\varphi*\psi$. Namely,

Definition 4.1. *For $1 \le p \le \infty$, we define V^p_+ and V^p as*

$$V^p_+ := \left\{\varphi^1*\psi^1 + \cdots + \varphi^n*\psi^n \mid n\in\mathbb{N},\ \varphi^1,\cdots,\varphi^n \in \ell^p_+,\ \psi^1,\cdots,\psi^n \in \ell^q_+\right\},$$

and

$$V^p := \left\{\varphi^1*\psi^1 + \cdots + \varphi^n*\psi^n \mid n\in\mathbb{N},\ \varphi^1,\cdots,\varphi^n \in \ell^p,\ \psi^1,\cdots,\psi^n \in \ell^q\right\}.$$

When $p=1$, we replace ℓ^q_+ and ℓ^q with c^0_+ and c^0 in the right-hand side, respectively. When $p=\infty$, we also replace ℓ^p_+ and ℓ^p with c^0_+ and c^0 in the right-hand side, respectively.

If $f = \varphi^1*\psi^1 + \cdots + \varphi^n*\psi^n$ and $g = \Phi^1*\Psi^1 + \cdots + \Phi^m*\Psi^m$ belong to each of above sets, then

$$f+g = \varphi^1*\psi^1 + \cdots + \varphi^n*\psi^n + \Phi^1*\Psi^1 + \cdots + \Phi^m*\Psi^m$$

belongs to the same sets. Thus it is easy to see that V^p_+ and V^p are vector spaces, respectively.

Definition 4.2. *For $f = \varphi^1*\psi^1 + \cdots + \varphi^n*\psi^n \in V^p_+$, we define $\|f\|_{V^p_+}$ as*

$$\|f\|_{V^p_+} := \inf\left\{\sum_{1\le j\le n} \|\varphi^j\|_{\ell^p_+}\|\psi^j\|_{\ell^q_+} \mid \text{representations of } f \text{ in } V^p_+\right\}.$$

For $f = \varphi^1*\psi^1 + \cdots + \varphi^n*\psi^n \in V^p$, we define $\|f\|_{V^p}$ as

$$\|f\|_{V^p} := \inf\left\{\sum_{1\le j\le n} \|\varphi^j\|_{\ell^p}\|\psi^j\|_{\ell^q} \mid \text{representations of } f \text{ in } V^p\right\}.$$

These spaces have the following properties.

Proposition 4.3. *1. For $\varphi \in \ell^p_+$ and $\psi \in \ell^q_+$, there are $\Phi \in \ell^p_+$ and $\Psi \in \ell^q_+$ with $\varphi * \psi = \Phi * \Psi$ and $\|\Phi\|_{\ell^p_+} = \|\Psi\|_{\ell^q_+}$.*

2. For $f \in V^p_+$, $\sup_{j\geq 0}|f_j| \leq \|f\|_{V^p_+}$.

3. $V^p_+ \subset c^0_+$.

4. For $f \in \ell^q_+$, $\|f\|_{V^p_+} \leq \|f\|_{\ell^q_+}$.

5. $\|\cdot\|_{V^1_+} = \|\cdot\|_{c^0_+}$ and $V^1_+ = c^0_+$.

6. $\|\cdot\|_{V^p_+} = \|\cdot\|_{V^q_+}$ and $V^p_+ = V^q_+$.

7. V^2_+ and H^1 are isometrically isomorphic via the isomorphism :

$$\{(f)_j\}_{j\geq 0} \longleftrightarrow f \text{ whose Fourier coefficients are } \{(f)_j\}_{j\geq 0}.$$

Proof. 1. If we set $\Phi := \sqrt{\frac{\|\psi\|_q}{\|\varphi\|_p}}\varphi$ and $\Psi := \sqrt{\frac{\|\varphi\|_p}{\|\psi\|_q}}\psi$, then $\varphi * \psi = \Phi * \Psi$ and $\|\Phi\|_p = \sqrt{\|\varphi\|_p\|\psi\|_q} = \|\Psi\|_q$.

2. Take $f = \varphi^1 * \psi^1 + \cdots + \varphi^n * \psi^n \in V^p_+$. By the Hölder's inequality,

$$\begin{aligned} |f_j| &\leq |(\varphi^1 * \psi^1)_j| + \cdots + |(\varphi^n * \psi^n)_j| \\ &\leq \|\varphi^1\|_{\ell^p_+}\|\psi^1\|_{\ell^q_+} + \cdots + \|\varphi^n\|_{\ell^p_+}\|\psi^n\|_{\ell^q_+} \end{aligned}$$

for $j \geq 0$. Thus $\sup_{j\geq 0}|f_j| \leq \|f\|_{V^p_+}$.

3. For any $\epsilon > 0$, $\varphi = \{\varphi_j\}_{j\geq 0} \in \ell^p_+$, $\psi = \{\psi_j\}_{j\geq 0} \in \ell^q_+$, there is an $N \in \mathbb{N}$ such that $\|\{\varphi_j\}_{j\geq N}\|_p < \frac{\epsilon}{2\|\psi\|_q}$ and $\|\{\psi_j\}_{j\geq N}\|_q < \frac{\epsilon}{2\|\varphi\|_p}$. By the Hölder's inequality,

$$\begin{aligned} |(\varphi * \psi)_j| &= |\sum_{k\geq 0}\varphi_{j-N+k}\psi_{N-k} + \sum_{k\geq 1}\varphi_{j-N-k}\psi_{N+k}| \\ &\leq |\sum_{k\geq 0}\varphi_{j-N+k}\psi_{N-k}| + |\sum_{k\geq 1}\varphi_{j-N-k}\psi_{N+k}| \\ &\leq \|\{\varphi_{j-N+k}\}_{k\geq 0}\|_p\|\{\psi_{N-k}\}_{k\geq 0}\|_q + \|\{\varphi_{j-N-k}\}_{k\geq 1}\|_p\|\{\psi_{N+k}\}_{k\geq 1}\|_q \\ &< \frac{\epsilon}{2\|\psi\|_q}\|\psi\|_q + \|\varphi\|_p\frac{\epsilon}{2\|\varphi\|_p} = \epsilon \end{aligned}$$

hold for any $j \geq 2N$.

Thus, for any $\epsilon > 0$, $f = \varphi^1*\psi^1+\cdots+\varphi^m*\psi^m \in V^p_+$, if we put $\epsilon_n := \frac{\epsilon}{(n+1)^2}$ $(1 \leq n \leq m)$, then there exist $N_1, \cdots, N_m \in \mathbb{N}$ such that

$$\begin{aligned} |f_j| &\leq |(\varphi^1 * \psi^1)_j| + \cdots + |(\varphi^n * \psi^n)_j| \\ &< \epsilon_1 + \cdots + \epsilon_m < \epsilon \end{aligned}$$

for any $j \geq 2\max_{1\leq n\leq m} N_n$. This and 2 mean the conclusion.

4. Let $f \in \ell^q_+$. We regard f as $e_0 * f$, and therefore $\|f\|_{V^p_+} \leq \|e_0\|_{\ell^p_+}\|f\|_{\ell^q_+} = \|f\|_{\ell^q_+}$.

5. By 3 and 4, it is immediately seen.

6. By $\varphi * \psi = \psi * \varphi$, it is easy to see.

7. Since ℓ^2_+ and H^2 are isometrically isomorphic via the isomorphism : $\{(\varphi)_j\}_{j\geq 0} \longleftrightarrow \varphi = \sum_{j\geq 0}(\varphi)_j\chi_j$, we can see easily that V^2_+ and

$$\left\{\varphi^1\psi^1 + \cdots + \varphi^n\psi^n \mid n \in \mathbb{N},\ \varphi^1, \cdots, \varphi^n \in H^2,\ \psi^1, \cdots, \psi^n \in H^2\right\}$$

are isometrically isomorphic, whenever a norm of the space of the right-hand side is defined by

$$\|f\| := \inf\left\{\|\varphi^1\|_{H^2}\|\psi^1\|_{H^2} + \cdots + \|\varphi^n\|_{H^2}\|\psi^n\|_{H^2} \mid \text{representations of } f\right\}.$$

We show that this normed space is equal to H^1. By the Hölder's inequality,

$$\begin{aligned}\int |\varphi^1\psi^1 + \cdots + \varphi^n\psi^n|\frac{d\theta}{2\pi} &\leq \int |\varphi^1\psi^1|\frac{d\theta}{2\pi} + \cdots + \int |\varphi^n\psi^n|\frac{d\theta}{2\pi} \\ &\leq \|\varphi^1\|_{H^2}\|\psi^1\|_{H^2} + \cdots + \|\varphi^n\|_{H^2}\|\psi^n\|_{H^2},\end{aligned}$$

therefore $\|\cdot\|_{H^1} \leq \|\cdot\|$. Conversely, let $f \in H^1$. By the inner-outer factorization theorem, there are an inner function $g \in H^\infty$ and an outer function $h \in H^1$ satisfying $f = gh$. If we set $\varphi := gh^{\frac{1}{2}} \in H^2$ and $\psi := h^{\frac{1}{2}} \in H^2$, then $f = gh = \varphi\psi$ and $\|f\| \leq \|\varphi\|_{H^2}\|\psi\|_{H^2} = \|f\|_{H^1}$. Consequently, V^2_+ and H^1 are isometrically isomorphic. □

Proposition 4.4. *1. For $\varphi \in \ell^p$ and $\psi \in \ell^q$, there are $\Phi \in \ell^p$ and $\Psi \in \ell^q$ with $\varphi * \psi = \Phi * \Psi$ and $\|\Phi\|_{\ell^p} = \|\Psi\|_{\ell^q}$.*

2. *For $f \in V^p$, $\sup_{j\in\mathbb{Z}}|f_j| \leq \|f\|_{V^p}$.*
3. *$V^p \subset c^0$.*
4. *For $f \in \ell^q$, $\|f\|_{V^p} \leq \|f\|_{\ell^q}$.*
5. *$\|\cdot\|_{V^1} = \|\cdot\|_{c^0}$ and $V^1 = c^0$.*
6. *$\|\cdot\|_{V^p} = \|\cdot\|_{V^q}$ and $V^p = V^q$.*
7. *V^2 and L^1 are isometrically isomorphic via the isomorphism :*

$$\{(f)_j\}_{j\in\mathbb{Z}} \longleftrightarrow f \text{ whose Fourier coefficients are } \{(f)_j\}_{j\in\mathbb{Z}}.$$

Proof. The proof is the same as that of Proposition 4.3. □

Remark 4.5. *By Proposition 4.3.2 and Proposition 4.4.2, it is seen that $\|\cdot\|_{V^p_+}$ and $\|\cdot\|_{V^p}$ are norms on V^p_+ and V^p, respectively.*

Now, we consider what representations of an element of these spaces we can take. In general, it doesn't say that a representation $\varphi^1 * \psi^1 + \cdots + \varphi^n * \psi^n$ of an element f satisfies

$$\|\varphi^1\|_p\|\psi^1\|_q \risingdotseq \cdots \risingdotseq \|\varphi^n\|_p\|\psi^n\|_q.$$

However, the following result says that there is such a representation for all f.

Theorem 4.6. *Let $\epsilon, \epsilon' > 0$.*

1. *For $f \in V^p_+$, there is a representation of f, $f = \Phi^1 * \Psi^1 + \cdots + \Phi^N * \Psi^N$ such that $\frac{\|f\|_{V^p_+} - \epsilon'}{N} < \|\Phi^j\|^2_{\ell^p_+} < \frac{\|f\|_{V^p_+} + \epsilon}{N}$ and $\|\Phi^j\|_{\ell^p_+} = \|\Psi^j\|_{\ell^q_+}$ for $1 \leq j \leq N$.*

2. For $f \in V^p$, there is a representation of f, $f = \Phi^1 * \Psi^1 + \cdots + \Phi^N * \Psi^N$ such that $\frac{\|f\|_{V^p} - \epsilon'}{N} < \|\Phi^j\|_{\ell^p}^2 < \frac{\|f\|_{V^p} + \epsilon}{N}$ and $\|\Phi^j\|_{\ell^p} = \|\Psi^j\|_{\ell^q}$ for $1 \le j \le N$.

Proof. 1. If we take an $f \in V^p_+$, then by Proposition 4.3.1, there is a representation of f, $f = \varphi^1 * \psi^1 + \cdots + \varphi^n * \psi^n$ such that

$$\|f\|_{V^p_+} \le \|\varphi^1\|_p^2 + \cdots + \|\varphi^n\|_p^2 < \|f\|_{V^p_+} + \epsilon$$

and $\|\varphi^j\|_p = \|\psi^j\|_q$ for $1 \le j \le n$.

Assume $n \ge 2$ and $\|\varphi^1\|_p^2 \le \cdots \le \|\varphi^n\|_p^2$.

We show that it may assume $\|\varphi^1\|_p^2 > 0$ without loss of generality. Indeed, assume $\|\varphi^n\|_p^2 > 0$ and $\|\varphi^1\|_p^2 = 0$. We take $2 \le k \le n$ with $\|\varphi^k\|_p^2 > 0$ and $\|\varphi^{k-1}\|_p^2 = 0$. If we set $\Phi^i := \begin{cases} \frac{\varphi^k}{\sqrt{k}}, & 1 \le i \le k \\ \varphi^i, & k+1 \le i \le n \end{cases}$ and $\Psi^i := \begin{cases} \frac{\psi^k}{\sqrt{k}}, & 1 \le i \le k \\ \psi^i, & k+1 \le i \le n, \end{cases}$
then $f = \varphi^1 * \psi^1 + \cdots + \varphi^n * \psi^n = \Phi^1 * \Psi^1 + \cdots + \Phi^n * \Psi^n$ and $0 < \|\Phi^1\|_p^2 \le \cdots \le \|\Phi^n\|_p^2$.

When $\|f\|_{V^p_+} > \epsilon'$, if we take an ϵ'' with $0 < \epsilon'' \le \frac{\|f\|_{V^p_+}}{\|f\|_{V^p_+} - \epsilon'} - 1$, then we can take $r_j \in \mathbb{Q}$ satisfying

$$\|\varphi^j\|_p^2 \le r_j < \min\left\{ \frac{\|\varphi^j\|_p^2(\|f\|_{V^p_+} + \epsilon)}{\|\varphi^1\|_p^2 + \cdots + \|\varphi^n\|_p^2}, \|\varphi^j\|_p^2(1 + \epsilon'') \right\}$$

for $1 \le j \le n$, and take $k_j \in \mathbb{N}$ satisfying the ratio

$$r_1 : \cdots : r_n = k_1 : \cdots : k_n$$

for $1 \le j \le n$.

Let $k_0 := 0$ and $N := \sum_{0 \le \ell \le n} k_\ell$. If we set $\Phi^j := \frac{\varphi^i}{\sqrt{k_i}}$ and $\Psi^j := \frac{\psi^i}{\sqrt{k_i}}$ for $1 \le i \le n$ and $\sum_{0 \le \ell \le i-1} k_\ell + 1 \le j \le \sum_{0 \le \ell \le i-1} k_\ell + k_i$, then we see that

$$f = \varphi^1 * \psi^1 + \cdots + \varphi^n * \psi^n = \Phi^1 * \Psi^1 + \cdots + \Phi^N * \Psi^N$$

and $\frac{\|f\|_{V^p_+} - \epsilon'}{N} < \|\Phi^j\|_p^2 = \|\Psi^j\|_q^2 < \frac{\|f\|_{V^p_+} + \epsilon}{N}$ for $1 \le j \le N$. Indeed, for $1 \le i \le n$ and $\sum_{0 \le \ell \le i-1} k_\ell + 1 \le j \le \sum_{0 \le \ell \le i-1} k_\ell + k_i$,

$$\|\Phi^j\|_p^2 = \|\frac{\varphi^i}{\sqrt{k_i}}\|_p^2 = \frac{\|\varphi^i\|_p^2}{r_i} \cdot \frac{r_i}{k_i} = \frac{\|\varphi^i\|_p^2}{r_i} \cdot \frac{r_1 + \cdots + r_n}{k_1 + \cdots + k_n} = \frac{\|\varphi^i\|_p^2}{r_i} \cdot \frac{r_1 + \cdots + r_n}{N},$$

and

$$\begin{aligned} \frac{\|f\|_{V^p_+} - \epsilon'}{N} &\le \frac{\|f\|_{V^p_+}}{N(1+\epsilon'')} \\ &\le \frac{\|\varphi^1\|_p^2 + \cdots + \|\varphi^n\|_p^2}{N(1+\epsilon'')} \\ &\le \frac{r_1 + \cdots + r_n}{N(1+\epsilon'')} \\ &< \frac{\|\varphi^i\|_p^2}{r_i} \cdot \frac{r_1 + \cdots + r_n}{N} \\ &\le \frac{r_1 + \cdots + r_n}{N} \\ &< \frac{\|f\|_{V^p_+} + \epsilon}{N} \end{aligned}$$

hold.

When $\|f\|_{V_+^p} \leq \epsilon'$, we can take $r_j \in \mathbb{Q}$ satisfying

$$\|\varphi^j\|_p^2 \leq r_j < \frac{\|\varphi^j\|_p^2(\|f\|_{V_+^p} + \epsilon)}{\|\varphi^1\|_p^2 + \cdots + \|\varphi^n\|_p^2}$$

for $1 \leq j \leq n$, and take $k_j \in \mathbb{N}$ satisfying the ratio

$$r_1 : \cdots : r_n = k_1 : \cdots : k_n$$

for $1 \leq j \leq n$.

Let $k_0 := 0$ and $N := \sum_{0\leq \ell \leq n} k_\ell$. If we set $\Phi^j := \frac{\varphi^i}{\sqrt{k_i}}$ and $\Psi^j := \frac{\psi^i}{\sqrt{k_i}}$, then we see that

$$\begin{aligned} \frac{\|f\|_{V_+^p} - \epsilon'}{N} &\leq 0 \\ &< \|\Phi^j\|_p^2 \\ &= \frac{\|\varphi^i\|_p^2}{r_i} \cdot \frac{r_1 + \cdots + r_n}{N} \\ &\leq \frac{r_1 + \cdots + r_n}{N} \\ &< \frac{\|f\|_{V_+^p} + \epsilon}{N} \end{aligned}$$

hold for $1 \leq i \leq n$ and $\sum_{0\leq \ell \leq i-1} k_\ell + 1 \leq j \leq \sum_{0 \leq \ell \leq i-1} k_\ell + k_i$. This leads the conclusion.

2. The proof is the same as that of 1. □

Remark 4.7. *The auther generalized Theorem 4.6 to Theorem 2.2 of [4] after this paper submitted.*

5 Duality theorems.

Here, we extend Proposition 3.5 to V_+^p and V^p.

Proposition 5.1. *Let $1 \leq p < \infty$, and $a \in L^1$.*

1. $\|a\|_{M_+^p} = \sup\left\{|\langle\{(a)_{k+1}\}_{k\geq 0}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V_+^p} \leq 1\right\}$ *holds.*
2. $\|a\|_{M^p} = \sup\left\{|\langle\{(a)_k\}_{k\in\mathbb{Z}}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V^p} \leq 1\right\}$ *holds.*

Proof. 1. Since $\left\{\varphi * \psi \mid \|\varphi\|_{\ell_+^p},\ \|\psi\|_{\ell_+^q} \leq 1\right\} \subset \left\{f \in V_+^p \mid \|f\|_{V_+^p} \leq 1\right\}$,

$$\begin{aligned} \|a\|_{M_+^p} &= \sup\left\{|\langle\{(a)_{k+1}\}_{k\geq 0}, \varphi * \psi\rangle_{\mathbb{R}}| \mid \|\varphi\|_{\ell_+^p},\ \|\psi\|_{\ell_+^q} \leq 1\right\} \\ &\leq \sup\left\{|\langle\{(a)_{k+1}\}_{k\geq 0}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V_+^p} \leq 1\right\} \end{aligned}$$

hold by Proposition 3.5.1.

Conversely, for any $\epsilon, \epsilon' > 0$, $f \in V_+^p$, there is a representation of f in V_+^p, $f = \sum_{1\leq j\leq n} \phi^j * \psi^j$ such that $\frac{\|f\|_{V_+^p} - \epsilon'}{n} < \|\phi^j\|_{\ell^p}^2 = \|\psi^j\|_{\ell^q}^2 < \frac{\|f\|_{V_+^p} + \epsilon}{n}$ for all $1 \leq j \leq n$ by

Theorem 4.6.1. Thus

$$\begin{aligned}
&|\langle\{(a)_{k+1}\}_{k\geq 0}\,, f\rangle_{\mathbb{R}}| \leq \sum_{1\leq j\leq n} |\langle\{(a)_{k+1}\}_{k\geq 0}\,, \varphi^j * \psi^j\rangle_{\mathbb{R}}| \\
&\leq \sum_{1\leq j\leq n} \sup\left\{|\langle\{(a)_{n+1}\}_{n\geq 0}\,, \varphi * \psi\rangle_{\mathbb{R}}| \mid (\varphi,\psi)\in E_+^p, \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \leq \frac{\|f\|_{V_+^p}+\epsilon}{n}\right\} \\
&= n\cdot\frac{\|f\|_{V_+^p}+\epsilon}{n}\sup\left\{|\langle\{(a)_{n+1}\}_{n\geq 0}\,, \varphi * \psi\rangle_{\mathbb{R}}| \mid (\varphi,\psi)\in E_+^p, \|\varphi\|_{\ell^p},\ \|\psi\|_{\ell^q} \leq 1\right\} \\
&= (\|f\|_{V_+^p}+\epsilon)\|a\|_{M_+^p}
\end{aligned}$$

hold by Proposition 3.5.1. Hence,

$$\sup\left\{|\langle\{(a)_{k+1}\}_{k\geq 0}\,, f\rangle_{\mathbb{R}}| \mid \|f\|_{V_+^p}\leq 1\right\} \leq \|a\|_{M_+^p}.$$

2. The proof is the same as that of 1. □

Finally, we show the main results.

Theorem 5.2. *Let* $1\leq p<\infty$.

1. $M_+^p\cap H_0^2$ *and* $(V_+^p)^*$ *are isometrically isomorphic as normed spaces.*
2. M^p *and* $(V^p)^*$ *are isometrically isomorphic as normed spaces.*

Proof. 1. Let $a\in M_+^p\cap H_0^2$. By Proposition 5.1.1,

$$D\ :\ V_+^p \longrightarrow \mathbb{C}\ :\ f\longmapsto \langle f, \{(a)_{n+1}\}_{n\geq 0}\rangle_{\mathbb{R}}$$

satisfies $\|D\|_{(V_+^p)^*} = \sup\left\{|\langle\{(a)_{k+1}\}_{k\geq 0}\,, f\rangle_{\mathbb{R}}| \mid \|f\|_{V_+^p}\leq 1\right\} = \|a\|_{M_+^p}$ and $D\in (V_+^p)^*$.

Coversely, let $D\in (V_+^p)^*$ and let $a_{n+1} := D(e_n)$ $n\geq 0$.

When $1<p\leq 2$, since

$$D(f) = \sum_{n\geq 0} f_n D(e_n) = \sum_{n\geq 0} f_n a_{n+1} = \langle f, \{a_{n+1}\}_{n\geq 0}\rangle_{\mathbb{R}}$$

holds for $f=\sum_{n\geq 0} f_n e_n\in V_+^p$,

$$\begin{aligned}
\infty > \|D\|_{(V_+^p)^*} &= \sup\left\{|\langle f, \{a_{n+1}\}_{n\geq 0}\rangle_{\mathbb{R}}| \mid \|f\|_{V_+^p}\leq 1\right\} \\
&\geq \sup\left\{|\langle f, \{a_{n+1}\}_{n\geq 0}\rangle_{\mathbb{R}}| \mid \|f\|_{\ell_+^q}\leq 1\right\}.
\end{aligned}$$

This implies $\{a_{n+1}\}_{n\geq 0}\in (\ell_+^q)^*\cong \ell_+^p\subset \ell_+^2\cong H^2$ and $a:=\sum_{n\geq 1} a_n\chi_n\in H_0^2$. Thus, by Proposition 5.1.1, $\|a\|_{M_+^p} = \sup\left\{|\langle\{(a)_{k+1}\}_{k\geq 0}\,, f\rangle_{\mathbb{R}}| \mid \|f\|_{V_+^p}\leq 1\right\} = \|D\|_{(V_+^p)^*}$ and $a\in M_+^p$.

When $p=1$, we replace ℓ_+^q with c_+^0. When $2<p<\infty$, it is soon from $M_+^p\cap H_0^2 = M_+^q\cap H_0^2$ and $V_+^p = V_+^q$.

2. Let $a\in M^p$. By Proposition 5.1.2,

$$D\ :\ V^p \longrightarrow \mathbb{C}\ :\ f\longmapsto \langle f, \{(a)_n\}_{n\in\mathbb{Z}}\rangle_{\mathbb{R}}$$

satisfies $\|D\|_{(V^p)^*} = \sup\left\{|\langle\{(a)_k\}_{k\in\mathbb{Z}}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V^p} \leq 1\right\} = \|a\|_{M^p}$ and $D \in (V^p)^*$.

Coversely, let $D \in (V^p)^*$ and let $a_n := D(e_n)$ $n \in \mathbb{Z}$.

When $1 < p \leq 2$, since

$$D(f) = \sum_{n\in\mathbb{Z}} f_n D(e_n) = \sum_{n\in\mathbb{Z}} f_n a_n = \langle f, \{a_n\}_{n\in\mathbb{Z}}\rangle_{\mathbb{R}}$$

holds for $f = \sum_{n\in\mathbb{Z}} f_n e_n \in V^p$,

$$\begin{aligned}\infty > \|D\|_{(V^p)^*} &= \sup\left\{|\langle f, \{a_n\}_{n\in\mathbb{Z}}\rangle_{\mathbb{R}}| \mid \|f\|_{V^p} \leq 1\right\} \\ &\geq \sup\left\{|\langle f, \{a_n\}_{n\in\mathbb{Z}}\rangle_{\mathbb{R}}| \mid \|f\|_{\ell^q} \leq 1\right\}.\end{aligned}$$

This implies $\{a_n\}_{n\in\mathbb{Z}} \in (\ell^q)^* \cong \ell^p \subset \ell^2 \cong L^2$ and $a := \sum_{n\in\mathbb{Z}} a_n \chi_n \in L^2$. Thus, by Proposition 5.1.2, $\|a\|_{M^p} = \sup\left\{|\langle\{(a)_k\}_{k\in\mathbb{Z}}, f\rangle_{\mathbb{R}}| \mid \|f\|_{V^p} \leq 1\right\} = \|D\|_{(V^p)^*}$ and $a \in M^p$.

When $p = 1$, we replace ℓ^q with c^0. When $2 < p < \infty$, it is soon from $M^p = M^q$ and $V^p = V^q$. □

Remark 5.3. *Theorem 5.2.1 is the modification of the H^1-BMOA duality because of Proposition 3.4.5 and Proposition 4.3.7. Also Theorem 5.2.2 is the modification of the classical L^1-L^∞ duality because of Proposition 2.1.2 and Proposition 4.4.7.*

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