OCTAGONAL FUZZY CHOQUET INTEGRAL OPERATOR FOR MULTI-ATTRIBUTE DECISION MAKING

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Abstract. This paper introduces two types of aggregations, namely the octagonal fuzzy weighted averaging(OFWA) operator for non-interactive aggregation and octagonal fuzzy Choquet integral(OFCI) operator for interactive aggregation. The paper emphasis the use of octagonal fuzzy number as a general case of some well known linear fuzzy numbers. Procedure for solving multi-attribute decision making(MADM) problem using OFWA and OFCI operators are described and algorithms for the same are presented to handle large data. Finally, an illustrative example is provided to demonstrate the application of the OFCI operator in MADM problem.

Keyword: Octagonal fuzzy number, Choquet integral, aggregation, MADM, algorithm

1 Introduction Multi-attribute Decision Making (MADM) problems involve aggregating information from various decision makers, aggregating the interactive criteria and then the final selection through ranking the alternatives. In real situations, quantifying the quality of the alternative may not be precise[2]. Zadeh[33] suggested employing the fuzzy set theory as a modeling tool that can help overcome the situation. However, the presence of fuzziness in decision making increases the computational difficulty in aggregating and ranking the alternatives, which has been handled by various authors including us. To cite a few [1, 3, 4, 7, 8, 17, 20, 24].

The Choquet integral based aggregation finds its use in cases where individual criteria importance and group importance are required. The Choquet integral is related to a fuzzy measure which considers the interaction among the criteria to be aggregated [16, 21, 25]. For this reason, Choquet integral is more suited to deal with fuzzy MCDM problems and in recent years, many scholars have done a lot of good research in this field. Yang et. al. [31, 32] studied the real and fuzzy Choquet integrals for fuzzy integrand. Tan [23], Xu [30], Wei et.al. [28], Wu et. al. [29] used Choquet integral to propose some intuitionistic fuzzy aggregation operators. Tan [22], Qin et.al.[18], Meng et. al. [15] studied and used Choquet integral to determine attribute weight and applied it in decision making problems under interval intuitionistic environment. Rebille [19] used decision making over necessity measures through Choquet integral.

In this paper, we introduce two types of aggregations on octagonal fuzzy numbers [14], namely octagonal fuzzy weighted averaging(OFWA) operator and octagonal fuzzy Choquet integral(OFCI) operator. OFWA deals with non-interactive aggregation to aggregate the evaluations of different decision makers, OFCI operator deals with interactive aggregation that aggregates the different criteria for the same alternative.

The paper is organized as follows. Section 2 discusses some of the properties of octagonal fuzzy numbers which are used to describe the linguistic terms for expert evaluations. In the Section 3, we recall the concept of fuzzy measure, introduce octagonal fuzzy Choquet integral($OFCI$) and then investigate the aggregation properties of $OFCI$. In Section 4, we present the procedure for solving MADM problem using $OFCI$ operator, also algorithms are provided so as to apply it to the real life situations which usually comes with large number of alternatives and criteria. The application of the proposed method is given in Section 5 and conclusion is presented in Section 6.

2 Octagonal Fuzzy Numbers

Definition 2.1 [14] *A fuzzy number $\tilde{A}$ is said to be an octagonal fuzzy number denoted by $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k, w)$ with membership function*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}k & \text{if } a_1 \le x \le a_2 \\ k & \text{if } a_2 \le x \le a_3 \\ \dfrac{k(a_4-x)+w(x-a_3)}{a_4-a_3} & \text{if } a_3 \le x \le a_4 \\ w & \text{if } a_4 \le x \le a_5 \\ \dfrac{k(x-a_5)+w(a_6-x)}{a_6-a_5} & \text{if } a_5 \le x \le a_6 \\ k & \text{if } a_6 \le x \le a_7 \\ \dfrac{a_8-x}{a_8-a_7}k & \text{if } a_7 \le x \le a_8 \\ 0 & \textit{otherwise} \end{cases} \tag{2.1}$$

where $0 < k < w, w = height(\tilde{A}), w > k$.

Remark 2.1 *The fuzzy number defined in [14] is piecewise and made up of 8 linear curves and therefore named as 'octagonal'. Note that it satisfies the properties of fuzzy number in accordance with the definition by Klir in [13].*

Remark 2.2 *The above defined octagonal fuzzy number is a generalised form of some of the popular linear fuzzy numbers like, crisp, rectangular, triangular and trapezoidal fuzzy numbers. As all these numbers can be represented as an octagonal fuzzy number, the operations defined for octagonal fuzzy numbers will hold good for them. The equivalent forms are as follows:*

Fuzzy Numbers	*Equivalent Octagonal Fuzzy Numbers*
Crisp Numbers a	$(a, a, a, a, a, a, a, a; k, w)$
Interval Numbers $[a_1, a_2]$	$(a_1, a_1, a_1, a_1, a_2, a_2, a_2, a_2; k, w)$
Triangular Fuzzy Numbers (a_1, a_2, a_3)	$\left(a_1, \dfrac{ka_2-ka_1+wa_1}{w}, \dfrac{ka_2-ka_1+wa_1}{w}, a_2, a_2, \dfrac{-ka_3+ka_2+wa_3}{w}, \dfrac{-ka_3+ka_2+wa_3}{w}, a_3; k, w\right)$
Trapezoidal Fuzzy Number (a_1, a_2, a_3, a_4)	$\left(a_1, \dfrac{ka_2-ka_1+wa_1}{w}, \dfrac{ka_2-ka_1+wa_1}{w}, a_2, a_3, \dfrac{-ka_4+ka_3+wa_4}{w}, \dfrac{-ka_4+ka_3+wa_4}{w}, a_4; k, w\right)$

Remark 2.3 *The fuzzy numbers that are piece-wise linear and are made of less than 8 line segments can be directly expressed as octagonal fuzzy number as pointed out in Remark 2.2. Fuzzy numbers which may constitute more than 8 linear segments or those which are piece-wise non-linear are not exactly octagonal fuzzy numbers but can be approximated to octagonal fuzzy numbers in a particular sense (Theorem 2.4.1 in [5]).*

Definition 2.2 *Let $\tilde{A} = (a_1, a_2, ..., a_8; k, w)$ and $\tilde{B} = (b_1, b_2, ..., b_8, k, w)$ be two octagonal fuzzy numbers, then*
(i) $\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, ..., a_8 + b_8; k, w)$
(ii) $c\tilde{A} = \quad (ca_1, ca_2, ..., ca_8; k, w), \quad$ *for* $c \ge 0$

Remark 2.4 *In [9], it is verified that the sum and scalar multiplication obtained from definition 2.2 is as that using $\alpha-$ cut approach.*

Remark 2.5 *It is clear that $\tilde{A} + \tilde{B}$ and $c\tilde{A}$ are also octagonal fuzzy numbers.*

Proposition 2.1 *Let $\tilde{A} = (a_1, a_2, ..., a_8; k, w)$, $\tilde{B} = (b_1, b_2, ..., b_8, k, w)$ be two octagonal fuzzy numbers and let $c_1, c_2 > 0$, then we have*
(i) $\tilde{A} + \tilde{B} = \tilde{B} + \tilde{A}$
(ii) $c_1(\tilde{A} + \tilde{B}) = c_1\tilde{A} + c_1\tilde{B}$
(iii) $(c_1 + c_2)\tilde{A} = c_1\tilde{A} + c_2\tilde{A}$

Definition 2.3 *An octagonal fuzzy weighted averaging operator on a collection of n octagonal fuzzy numbers is defined as*

$$OFWA_{wv}(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = wv_1\tilde{A}_1 + wv_2\tilde{A}_2 + ... + wv_n\tilde{A}_n \tag{2.2}$$

where $wv = (wv_1, wv_2, ..., wv_n)^T$ is the weight vector of $\tilde{A}_i (i = 1, 2, ..., n)$ with $wv_i \in [0, 1]$ and $\sum_{i=1}^{n} wv_i = 1$.

Definition 2.4 *Ranking using Radius of Gyration:[6] Area between the radius of gyration point $(r_x^{\tilde{A}}, r_y^{\tilde{A}})$ of the octagonal fuzzy number $\tilde{A}$ and the origin $(0, 0)$ is given by*

$$\mathcal{R}(\tilde{A}) = r_x^{\tilde{A}} r_y^{\tilde{A}}$$

where $r_x^{\tilde{A}} = \sqrt{\frac{I_x(\tilde{A})}{Area(\tilde{A})}}$ and $r_y^{\tilde{A}} = \sqrt{\frac{I_y(\tilde{A})}{Area(\tilde{A})}}$, $I_x(\tilde{A}), I_y(\tilde{A})$ are respectively the moment of inertia with respect to the x-axis and y-axis and $Area(\tilde{A})$ the area of the octagonal fuzzy number $\tilde{A}$.

Remark 2.6 *Ranking using radius of gyration is used in the procedure for defuzzification, whereas to compare the octagonal fuzzy numbers, we use the ranking algorithm introduced by us in Section 3.5 of the paper [6] . The ranking algorithm compares any two octagonal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ in 10 steps and we have proved that the algorithm returns either $\tilde{A} \prec \tilde{B}$, $\tilde{B} \prec \tilde{A}$ or the two octagonal fuzzy numbers are equal(not just equivalent). Thus any two octagonal fuzzy numbers are comparable and the ordering is anti-symmetric.*

3 Fuzzy Measure and Choquet Integral For the sake of completion, we recall the concept of fuzzy measure [12]. Using this, we define octagonal fuzzy Choquet integral operator which is then verified for fundamental properties of aggregation operator, like idempotency, monotonicity, boundedness and symmetry.

Definition 3.1 *[13] A fuzzy measure on X is a set function $m : \mathcal{P}(X) \to [0, 1]$ such that*
(i) $m(\phi) = 0,\ m(X) = 1$
(ii) $A, B \in \mathcal{P}(X),\ A \subseteq B \Rightarrow m(A) \leq m(B)$.

Considering the MADM problems, the number $m(A)$ can be interpreted as the importance of the subset A, and the monotonicity condition (ii) in Definition 3.1 of the fuzzy measure means that the importance of a subset of criteria cannot decrease when new criteria are added to it [26].
Let $E_j = \{x_j, x_{j+1}, ..., x_n\} (1 \leq j \leq n)$ be a criteria set. The interaction among the criteria in E_j can be described by employing $m(E_j)$to express the degree of importance of E_j. That is, the degree of importance of E_j is evaluated by simultaneously considering $x_j, x_{j+1}, ..., x_n$. Hence, m can be called an importance measure [27].
In order to determine such fuzzy measure, we generally need to find $2^n - 2$ values for n criteria, where $m(\phi) = 0$ and $m(X) = 1$ always. So the evaluation model obtained becomes quite complex, and the structure is difficult to grasp. To avoid the problems with computational complexity and practical estimations, $\lambda-$ fuzzy measure m, a special kind of fuzzy measure, was proposed by Sugeno, which satisfies the following additional property:

$$m(A \cup B) = m(A) + m(B) + \lambda m(A)m(B), \tag{3.1}$$

for all $A, B \in \mathcal{P}(X)$ and $A \cap B = \phi$ where $\lambda > -1$.

Definition 3.2 *[26] If X is a finite set, then $\cup_{i=1}^{n}\{x_i\} = X$. The $\lambda-$ fuzzy measure $m : \mathcal{P}(X) \to [0, 1]$ for every subset $A \in \mathcal{P}(X)$, satisfies*

$$m(A) = \begin{cases} \frac{1}{\lambda}\left(\prod\limits_{x_i \in A} [1 + \lambda m(\{x_i\})] - 1\right) & \text{if } \lambda \neq 0 \\ \sum\limits_{x_i \in A} m(\{x_i\}) & \text{if } \lambda = 0 \end{cases}$$

Remark 3.1 *[26] Based on the above definition of $m(A)$ and using the fact that $m(X) = 1$, we can uniquely solve λ which is equivalent to solving*

$$\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda m(\{x_i\})) \tag{3.2}$$

and

$\sum_{i=1}^{n} m(\{x_i\})$	*Range of* λ	*Type of the* $\lambda-$ *fuzzy measure*
$= 1$	$\lambda = 0$	*Additive*
< 1	$\lambda > 0$	*Super-additive*
> 1	$-1 < \lambda < 0$	*Sub-additive*

Definition 3.3 *Let $\tilde{A}_i = (a_1^i, a_2^i, ..., a_8^i; k, w)(i = 1, 2, ..., n)$ be a collection of n octagonal fuzzy numbers on X and m be a $\lambda-$ fuzzy measure on X. The octagonal fuzzy Choquet integral of $\tilde{A}_i$ with respect to m is defined by*

$$OFCI(\tilde{A}_1, ..., \tilde{A}_n) = \sum_{i=1}^{n}(m(E_{(i)}) - m(E_{(i+1)})\tilde{A}_{(i)} \tag{3.3}$$

where $(\cdot)$ indicates a permutation on X such that $\tilde{A}_{(1)} \preceq \tilde{A}_{(2)} \preceq ... \preceq \tilde{A}_{(n)}$ and $E_{(i)} = \{x_i, ..., x_n\}$, $E_{(n+1)} = \phi$.

Proposition 3.1 *Let $\tilde{A}_i = (a_1^i, a_2^i, ..., a_8^i; k, w)(i = 1, 2, ..., n)$ be a collection of n octagonal fuzzy numbers on X and m be a $\lambda-$ fuzzy measure on X, then their aggregated value $OFCI(\tilde{A}_1, ..., \tilde{A}_n)$ is also an octagonal fuzzy number.*

Proof: The result follows immediately from Definition 2.2□

Proposition 3.2 *Let $\tilde{A}_i = (a_1^i, a_2^i, ..., a_8^i; k, w)(i = 1, 2, ..., n)$ be a collection of n octagonal fuzzy numbers on X, such that $\sum_{i=1}^{n} m(\{x_i\}) = 1$. Then the octagonal fuzzy choquet integral coincides with the octagonal fuzzy weighted average.*

Proof: From Remark 3.1 we see that $\lambda = 0$ here. According to Definition 3.2 the $\lambda-$fuzzy measure is given by $m(E_{(i)}) = \sum_{j=i}^{n} m(\{x_j\})$. Thus

$$\begin{aligned} OFCI(\tilde{A}_1, ..., \tilde{A}_n) &= \sum_{i=1}^{n}(m(E_{(i)}) - m(E_{(i+1)})\tilde{A}_{(i)} \\ &= \sum_{i=1}^{n}\left[\sum_{j=i}^{n} m(\{x_j\}) - \sum_{j=i+1}^{n} m(\{x_j\})\right]\tilde{A}_{(i)} \\ &= \sum_{i=1}^{n} m(\{x_i\})\tilde{A}_{(i)} \\ &= OFWA(\tilde{A}_1, ..., \tilde{A}_n) \end{aligned}$$

Here $(m(\{x_1\}), m(\{x_2\}), ..., m(\{x_n\}))^T$ is the weight vector satisfying $\sum_{i=1}^{n} m(\{x_i\}) = 1$.□

Proposition 3.3

$$OFCI(\tilde{A}, ..., \tilde{A}) = \tilde{A}$$

Proof: From equation 3.3, we have

$$\begin{aligned} OFCI(\tilde{A},...,\tilde{A}) &= \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{A} \\ &= \tilde{A}\sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)}) \\ &= \tilde{A}\left(m(E_{(1)}-m(E_{(n+1)})\right) \\ &= \tilde{A}\left(m(X)-m(\phi)\right) \\ &= \tilde{A} \ \square \end{aligned}$$

Proposition 3.4 *Let $\tilde{A}_i = (a_1^i, a_2^i, ..., a_8^i; k, w)$ and $\tilde{B}_i = (b_1^i, b_2^i, ..., b_8^i; k, w)$ $(i = 1, 2, ..., n)$ be a collection of $2n$ octagonal fuzzy numbers on X such that $\tilde{A}_i \preceq \tilde{B}_i$ $(i = 1, 2, ..., n)$ but there exists no j and k such that $\tilde{A}_i \preceq \tilde{A}_j \preceq \tilde{B}_k \preceq \tilde{B}_i$ for any $j, k (\neq i) \in \{1, 2, ..., n\}$ and m be a $\lambda-$ fuzzy measure on X, then $OFCI(\tilde{A}_1, ..., \tilde{A}_n) \leq OFCI(\tilde{B}_1, ..., \tilde{B}_n)$.*

Proof: Since $E_{(i+1)} \subseteq E_{(i)}$, we have $m(E_{(i+1)}) \leq m(E_{(i)})$. Thus $m(E(i)) - m(E_{(i+1)}) \geq 0$ for all i. Suppose after rearranging in ascending order, $\tilde{A}_i$ is moved to $\tilde{A}_{(j)}$ and $\tilde{B}_i$ is moved to $\tilde{B}_{(k)}$, then $\tilde{A}_{(j)} \preceq \tilde{B}_{(k)}$ and no $\tilde{A}_.$ or $\tilde{B}_.$ comes in between. Also, we have n such inequalities. Thus, $j = k$. i.e. $\tilde{A}_{(i)} \preceq \tilde{B}_{(i)}$ for $i = 1, 2, ..., n$ Now,

$$\begin{aligned} OFCI(\tilde{A}_1,...,\tilde{A}_n) &= \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{A}_{(i)} \\ &\preceq \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{B}_{(i)} \\ &= OFCI(\tilde{B}_1,...,\tilde{B}_n) \ \square \end{aligned}$$

Proposition 3.5 *Let $\tilde{A}_i = (a_1^i, a_2^i, ..., a_8^i; k, w)$ $(i = 1, 2, ..., n)$ be a collection of n octagonal fuzzy numbers on X and m be a $\lambda-$ fuzzy measure on X, then $OFCI(\tilde{A}_1, ..., \tilde{A}_n)$ is bounded.*

Proof: From the definition of $OFCI$,

$$OFCI(\tilde{A}_1,...,\tilde{A}_n) = \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{A}_{(i)}$$

where $(\cdot)$ indicates a permutation on X such that $\tilde{A}_{(1)} \preceq \tilde{A}_{(2)} \preceq ... \preceq \tilde{A}_{(n)}$. Thus

$$\begin{aligned} OFCI(\tilde{A}_1,...,\tilde{A}_n) &= \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{A}_{(i)} \\ &\succeq \tilde{A}_{(1)}\sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)}) \\ &\succeq \tilde{A}_{(1)}\left(m(E_{(1)})-m(E_{(n+1)})\right) \\ &\succeq \tilde{A}_{(1)}\left(m(X)-m(\phi)\right) \\ &\succeq \tilde{A}_{(1)} \end{aligned}$$

Also

$$\begin{aligned} OFCI(\tilde{A}_1,...,\tilde{A}_n) &= \sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)})\tilde{A}_{(i)} \\ &\preceq \tilde{A}_{(n)}\sum_{i=1}^{n}(m(E_{(i)})-m(E_{(i+1)}) \\ &\preceq \tilde{A}_{(n)} \ \square \end{aligned}$$

From Definition 3.3, the following property can easily be obtained.

Proposition 3.6 *Let $\tilde{A}_i = (a_1^i, a_2^i, ..., a_8^i; k, w)(i = 1, 2, ..., n)$ be a collection of n octagonal fuzzy numbers on X and m be a $\lambda-$ fuzzy measure on X. If $(\tilde{A}'_1, \tilde{A}'_2, ..., \tilde{A}'_n)$ is any permutation of $(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$, then $OFCI(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = OFCI(\tilde{A}'_1, \tilde{A}'_2, ..., \tilde{A}'_n)$.*

Proof: The proof is obvious, as whatever the permutation the $OFCI$ first orders the given collections of octagonal fuzzy numbers and then aggregates. $\square$

4 Multi-Attribute Decision Making with OFCI Operator Consider the MADM problem handled in [7] with k decision makers $D_1, D_2, ..., D_k$. evaluating the importance of n criteria $c_1, c_2, ..., c_n$ and m alternatives $A_1, A_2, ..., A_m$ based on each of the n criteria. The problem is considered in octagonal fuzzy environment.

4.1 Abstract Algorithm for solving the MCDM problem using OFCI operator:

Step 1: Aggregate the evaluations of the decision makers:
Use $OFWA$ operator for this step, so that the problem now has a vector C of size n, which gives the importance of the n criteria and an $m \times n$ matrix, which is the evaluations of the m alternatives based on n criteria. All the entries in the vector and the matrix are octagonal fuzzy numbers

Step 2: Find the $\lambda-$fuzzy measure of the power set of the criteria set:

(i) Compute the $\lambda-$ fuzzy measure for individual criteria as

$$g_\lambda(C_i) = \frac{\mathcal{R}(C_i)}{2 \times max(\mathcal{R}(C_i))}, \ i = 1, 2, ..., n$$

where $\mathcal{R}$ is the radius of gyration as given in Definition 2.4

(ii) Solve the equation $\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda g_\lambda(C_i))$ for λ and

$\lambda = 0$ if $\sum_{i=1}^{m} g_\lambda(C_i) = 1$

$\lambda < 0$ if $\sum_{i=1}^{m} g_\lambda(C_i) > 1$

$\lambda > 0$ if $\sum_{i=1}^{m} g_\lambda(C_i) < 1$

(iii) $g_\lambda(A)$ is obtained using Definition 3.2, where $A \in \mathcal{P}(\{c_1, c_2, ..., c_n\})$

Step 3: Aggregate the criterias for the alternatives:
Use the octagonal fuzzy Choquet integral operator to aggregate the n evaluations for each alternative, to obtain an octagonal fuzzy number.

Step 4: Order the alternatives:
Sort the alternatives.

4.2 Algorithms for solving the MCDM problem using OFCI operator: In the above abstract algorithm, Step 1 is direct as it is the weighted average which involves addition and scalar multiplication only. The result of this Step is the matrix DM with m rows and n columns with each entry (i, j) the aggregation of the decision makers' evaluation of i^{th} alternative versus j^{th} criteria. Also Step 2 (i) and (ii) are direct calculations. Step 3 is tricky as we have to identify the subsets of the criteria set and then the corresponding $\lambda-$measure. Hence we present an algorithm to find $g_\lambda(A)$, where A is the subset of the criteria set. In this algorithm, we will obtain matrix M with two columns and 2^n rows, the first column gives the binary equivalent of the numbers $1, 2, ..., 2^n$ and the second column gives the g_λ measure of the subset of the criteria set, which is identified using the corresponding first column entry. For example, the binary number "10110" will represent the subset $\{c_2, c_3, c_5\}$ i.e from right to left the entries denote $c_1, c_2, ..., c_n$ with each binary digit acting like a characteristic function of the subset.

Algorithm 4.1 Subset of the Criteria set and its Measure

Require: $g_\lambda(C_i), (i = 1, 2, ...n)$, n - number of criteria
  for $r \leftarrow 1$ to 2^n **do**
    $M_{r,1} = ""$
    for $i \leftarrow 1$ to n **do**
      $t_i \leftarrow floor(mod(\frac{r-1}{2^{i-1}}, 2))$
      $M_{r,1} = Concatenate(M_{r,1}, t_i)$
                ▷ First column of M identifies the subsets of the criteria set

end for
for $i \leftarrow 1$ to n **do**
$s_i \leftarrow floor(mod(\frac{r-1}{2^{i-1}}, 2)) * (1 + \lambda g_\lambda(C_i))$
end for
$prod \leftarrow 1$
for $j \leftarrow 1$ to n **do**
if $s_i \neq 0$ **then**
$prod \leftarrow prod * s_i$
end if
end for
$M_{r,2} = \frac{prod - 1}{\lambda}$
▷ Second column gives the measure of the set identified in the corresponding first column
end for

Algorithm 4.2 Octagonal Fuzzy Choquet Integral to aggregate the criteria

Require: the order of the decision matrix
for $i \leftarrow 1, m$ **do** ▷ Identifying the set $E_{(i)}$
for $l \leftarrow 1, n$ **do**
$OB_{i,l} \leftarrow$ ""
$t_{i,l} \leftarrow 1$
end for
for $p \leftarrow n, 1$ step -1 **do**
$OB_{i,l} \leftarrow$concatenate$(OB_{i,l}, t_{i,p})$
end for
end for
for $j \leftarrow 2, n$ **do**
for $i \leftarrow 1, m$ **do**
for $l \leftarrow 1, n$ **do**
if $s_{i,j-1} = l$ **then**
$t_{i,l} \leftarrow 0$
end if
end for
for $p \leftarrow n, 1$ step -1 **do**
$OB_{i,j} \leftarrow$concatenate$(OB_{i,l}, t_{i,p})$ ▷ $OB_{i,j}$ denote the set $E_{(j)}$
end for ▷ for the alternative i
end for
end for
for $u \leftarrow 1, m$ **do**
for $r \leftarrow 1, 2^n$ **do**
for $j \leftarrow 1, n$ **do**
if $M_{r,1} = OB_{u,j}$ **then**
$a_j \leftarrow M_{r,2}$ ▷ a_j is the measure of the set $E_{(j)}$
end if
end for
end for
$a_{n+1} \leftarrow 0$
$CI_u \leftarrow \sum_{s=1}^{n} DM_{u,s_{u,s}} * (a_s - a_{s+1})$
▷ CI is a vector of size n with CI_u is the aggregated evaluation for alternative u
end for

To end the procedure, the vector CI is sorted using the ranking method, radius of gyration and the alternative with maximum $\mathcal{R}(CI_u)$ is the best alternative.

5 Illustration Consider an hypothetical problem of selecting a supplier among four suppliers. They determine five attributes, namely capacity, quality, cost, distance and delivery time. By the help of three experts, they evaluate all the suppliers, also the experts determine the fuzzy weights of the criteria. Assume that the experts are equally important. The evaluations are as follows:

Importance of criteria matrix

$$DC = \begin{pmatrix} VH & H & H & VH & M \\ VH & H & MH & H & MH \\ VH & H & MH & VH & M \end{pmatrix}$$

Evaluation matrix of Expert 1

$$DM1 = \begin{pmatrix} VG & VG & VG & VG & VG \\ G & VG & VG & VG & MG \\ VG & MG & G & G & G \\ G & M & M & G & MG \end{pmatrix}$$

Evaluation matrix of Expert 2

$$DM2 = \begin{pmatrix} G & MG & G & G & VG \\ G & VG & VG & VG & MG \\ G & G & MG & VG & G \\ VG & M & MG & M & G \end{pmatrix}$$

Evaluation matrix of Expert 3

$$DM3 = \begin{pmatrix} MG & MG & G & VG & VG \\ MG & MG & G & MG & G \\ VG & VG & VG & VG & MG \\ MG & VG & MG & VG & M \end{pmatrix}$$

where the corresponding octagonal fuzzy numbers for the above used linguistic term set are as given in the following table:

Linguistic term set for attributes	Linguistic term set for Weights	Corresponding octagonal fuzzy number
VP	VL	$(0, 10, 20, 30, 40, 50, 60, 70; \frac{1}{2}, 1)$
P	L	$(10, 20, 30, 40, 50, 60, 70, 80; \frac{1}{2}, 1)$
MP	ML	$(20, 30, 40, 50, 60, 70, 80, 90; \frac{1}{2}, 1)$
M	M	$(30, 40, 50, 60, 70, 80, 90, 100; \frac{1}{2}, 1)$
MG	MH	$(40, 50, 60, 70, 80, 90, 100, 100; \frac{1}{2}, 1)$
G	H	$(50, 60, 70, 80, 90, 100, 100, 100; \frac{1}{2}, 1)$
VG	VH	$(60, 70, 80, 90, 100, 100, 100, 100; \frac{1}{2}, 1)$

As the experts are considered equal, their weight vector will be $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

The first step to the problem is to aggregate the evaluations of the three experts and then to obtain the λ- fuzzy measure of the singleton sets $\{C_i\}, (i = 1, 2, ..., 5)$ which is as $0.5, 0.467, 0.43, 0.489, 0.378$ respectively.

Solving the equation

$$(1 + 0.5\lambda)(1 + 0.467\lambda)(1 + 0.43\lambda)(1 + 0.489\lambda)(1 + 0.378\lambda) - \lambda - 1 = 0$$

we get the λ- values to be $0, -0.93772, -5.19866, -2.51050 + 2.76915i, -2.51050 - 2.76915i$ and considering the cases in Remark 3.1, we let $\lambda = -0.938$

Following the algorithms, we aggregate all the information and obtain a octagonal fuzzy number for each alternative follows:

Alternative 1 $(56.206, 66.204, 76.202, 86.201, 96.199, 99.291, 99.983, 99.983; \frac{1}{2}, 1)$

Alternative 2 $(53.788, 63.786, 73.785, 83.783, 93.781, 97.962, 99.983, 99.983; \frac{1}{2}, 1)$

Alternative 3 $(54.933, 64.932, 74.93, 84.928, 94.927, 99.182, 99.983, 99.983; \frac{1}{2}, 1)$

Alternative 4 $(46.574, 56.572, 66.571, 76.569, 86.567, 93.745, 98.028, 99.983; \frac{1}{2}, 1)$

The order of the alternatives is $A_1 \succeq A_3 \succeq A_2 \succeq A_4$.

Remark 5.1 *The method proposed seems to be helpful in many cases provided the situation in any practical example can be described in terms of ideas in fuzzy sets on which the method is based.*

6 Conclusion In this paper, we introduced two aggregation operators, which are used to aggregate two types of information, namely, interactive and non-interactive. The aggregation for non-interactive information is verified to be a particular case of $OFCI$ operator. The fundamental aggregation properties are verified for $OFCI$ operator and a procedure for solving MADM problem involving the two types of aggregation is considered. An illustrative example is given to demonstrate the same. We note that algorithms are presented for complicated steps in the procedure, so that computer programs can be written to handle the real life problems which comes with large number of alternatives and criterias'(as pointed out with a concrete example in the second authors' thesis [5]). Also from Remark 2.1, we see that the problem with any other linear fuzzy numbers, like crisp, interval, triangular or trapezoidal fuzzy numbers, can be used, by considering their equivalent octagonal fuzzy numbers.

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Figure 1: MathCAD 14 programs for Algorithm 2.1

```
Order(M) := | for p ∈ 1..rows(M)
            |   B ← ((M^T)^<p>)^T
            |   for i ∈ 1..cols(M)
            |     for j ∈ i..cols(M)
            |       if Rank(B_{1,i}, B_{1,j}) = 0
            |         m ← B_{1,j}
            |         B_{1,j} ← B_{1,i}
            |         B_{1,i} ← m
            |   D ← ((M^T)^<p>)^T
            |   for i ∈ 1..cols(M)
            |     sc_{p,i} ← 0
            |     for j ∈ 1..cols(M)
            |       if (D_{1,j} = B_{1,i})
            |         n_{p,i} ← j
            |         sc_{p,i} ← sc_{p,i} + 1
            |   for i ∈ 1..cols(M)
            |     if sc_{p,i} > 1
            |       jj ← 1
            |       for k ∈ 1..cols(M)
            |         if n_{p,i} = n_{p,k}
            |           s_{jj} ← k
            |           jj ← jj + 1
            |       j ← 1
            |       for k ∈ 1..cols(M)
            |         if B_{1,i} = D_{1,k}
            |           r_j ← k
            |           j ← j + 1
            |       for j ∈ 1..sc_{p,i}
            |         n_{p,(s_j)} ← r_j
            |         sc_{p,(s_j)} ← 1
            | n
```

Figure 2: MathCAD 14 programs for Algorithm 4.1

```
r := 1..2^n

M[r,1] := | t ← ""
          | for i ∈ 1..n
          |   t ← concat(num2str(floor(mod((r − 1)/2^(i−1), 2))), t)
          | t

M[r,2] := | for i ∈ 1..n
          |   t[i] ← floor(mod((r − 1)/2^(i−1), 2))·(1 + λ·CI[i])
          | pro ← 1
          | for i ∈ 1..n
          |   pro ← pro·t[i]  if t[i] ≠ 0
          | (pro − 1)/λ
```

Figure 3: MathCAD 14 programs for Algorithm 4.2

```
Order_Binary(M) := | Order ← OrderManyRows(M)
                   | for i ∈ 1..rows(M)
                   |   | for l ∈ 1..cols(M)
                   |   |   t[i,l] ← 1
                   |   | s[i,1] ← ""
                   |   | for x ∈ 1..5
                   |   |   s[i,1] ← concat(num2str(t[i,x]), s[i,1])
                   | for j ∈ 2..cols(M)
                   |   for i ∈ 1..rows(M)
                   |     | for l ∈ 1..cols(M)
                   |     |   t[i,l] ← 0  if Order[i,j−1] = l
                   |     | s[i,j] ← ""
                   |     | for x ∈ 1..5
                   |     |   s[i,j] ← concat(num2str(t[i,x]), s[i,j])
                   | s
```

Choquet Integral Value for the Alternatives:

```
CI_A(M) := | for u ∈ 1..rows(M)
           |   | for r ∈ 1..2^cols(M)
           |   |   for i ∈ 1..cols(M)
           |   |     a[i] ← Mea[r,3]  if Mea[r,2] = Order_Binary(M)[u,i]
           |   | a[cols(M)+1] ← 0
           |   | f[u] ← Σ_{s=1}^{cols(M)} DM[u, OrderManyRows(M)[u,s]]·(a[s] − a[s+1])
           | f
```

Figure 4: Illustration

Linguistic Term Set For Attributes and Weights:

$$VP := \begin{pmatrix} 0 \\ 10 \\ 20 \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \end{pmatrix} \quad P := \begin{pmatrix} 10 \\ 20 \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \end{pmatrix} \quad MP := \begin{pmatrix} 20 \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \end{pmatrix} \quad M := \begin{pmatrix} 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{pmatrix} \quad MG := \begin{pmatrix} 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \\ 100 \end{pmatrix} \quad G := \begin{pmatrix} 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \\ 100 \\ 100 \end{pmatrix} \quad VG := \begin{pmatrix} 60 \\ 70 \\ 80 \\ 90 \\ 100 \\ 100 \\ 100 \\ 100 \end{pmatrix}$$

$VL := VP$ $\quad L := P$ $\quad ML := MP$ $\quad MH := MG$ $\quad H := G$ $\quad VH := VG$

Number of Decision Makers: $q := 3$

Number of Alternatives: $m := 4$ $\quad i := 1..m$ $\quad k := \frac{1}{2}$ $\quad w := 1$

Number of Attributes: $n := 5$ $\quad j := 1..n$

Importance of Attributes Matrix:

$$D\_C := \begin{pmatrix} VH & H & H & VH & M \\ VH & H & MH & H & MH \\ VH & H & MH & VH & M \end{pmatrix}$$

Evaluation matrix of Decision Maker 1:

$$DM1 := \begin{pmatrix} VG & VG & VG & VG & VG \\ G & VG & VG & VG & MG \\ VG & MG & G & G & G \\ G & M & M & G & MG \end{pmatrix}$$

Evaluation matrix of Decision Maker 2:

$$DM2 := \begin{pmatrix} G & MG & G & G & VG \\ G & VG & VG & VG & MG \\ G & G & MG & VG & G \\ VG & M & MG & M & G \end{pmatrix}$$

Evaluation matrix of Decision Maker 3:

$$DM3 := \begin{pmatrix} MG & MG & G & VG & VG \\ MG & MG & G & MG & G \\ VG & VG & VG & VG & MG \\ MG & VG & MG & VG & M \end{pmatrix}$$

The $\lambda-$ fuzzy measure for individual criteria is $g_\lambda(C_i) = \begin{pmatrix} 0.5 \\ 0.467 \\ 0.43 \\ 0.489 \\ 0.378 \end{pmatrix}$

Solving the equation $\lambda + 1 = (1 + 0.5\lambda)(1 + 0.467\lambda)(1 + 0.43\lambda)(1 + 0.489\lambda)(1 + 0.378\lambda)$, we get $\lambda = \begin{pmatrix} 0 \\ -5.1986 \\ -2.5105 - 2.7691i \\ -2.5105 + 2.7691i \\ -0.9377 \end{pmatrix}$ The $\lambda-$ fuzzy measure of the power set of the criteria set:

M=

$$\begin{pmatrix} \text{"00000"} & 0 \\ \text{"00001"} & 0.5 \\ \text{"00010"} & 0.467 \\ \text{"00011"} & 0.748 \\ \text{"00100"} & 0.43 \\ \text{"00101"} & 0.728 \\ \text{"00110"} & 0.709 \\ \text{"00111"} & 0.876 \\ \text{"01000"} & 0.489 \\ \text{"01001"} & 0.76 \\ \text{"01010"} & 0.742 \\ \text{"01011"} & 0.894 \\ \text{"01100"} & 0.722 \\ \text{"01101"} & 0.883 \\ \text{"01110"} & 0.873 \\ \text{"01111"} & 0.963 \\ \text{"10000"} & 0.378 \\ \text{"10001"} & 0.701 \\ \text{"10010"} & 0.68 \\ \text{"10011"} & 0.861 \\ \text{"10100"} & 0.656 \\ \text{"10101"} & 0.848 \\ \text{"10110"} & 0.836 \\ \text{"10111"} & 0.944 \\ \text{"11000"} & 0.694 \\ \text{"11001"} & 0.868 \\ \text{"11010"} & 0.857 \\ \text{"11011"} & 0.955 \\ \text{"11100"} & 0.844 \\ \text{"11101"} & 0.948 \\ \text{"11110"} & 0.941 \\ \text{"11111"} & 1 \end{pmatrix}$$

$$\text{Order}(DM) = \begin{pmatrix} 2 & 1 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \\ 5 & 2 & 3 & 1 & 4 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

$$OB(DM) = \begin{pmatrix} \text{"11111"} & \text{"11101"} & \text{"11100"} & \text{"11000"} & \text{"10000"} \\ \text{"11111"} & \text{"01111"} & \text{"01110"} & \text{"01100"} & \text{"00100"} \\ \text{"11111"} & \text{"01111"} & \text{"01101"} & \text{"01001"} & \text{"01000"} \\ \text{"11111"} & \text{"11011"} & \text{"11001"} & \text{"01001"} & \text{"00001"} \end{pmatrix}$$

$$CI(DM)^T = \left[\begin{pmatrix} 56.206 \\ 66.204 \\ 76.202 \\ 86.201 \\ 96.199 \\ 99.291 \\ 99.983 \\ 99.983 \end{pmatrix} \begin{pmatrix} 53.788 \\ 63.786 \\ 73.785 \\ 83.783 \\ 93.781 \\ 97.962 \\ 99.983 \\ 99.983 \end{pmatrix} \begin{pmatrix} 54.933 \\ 64.932 \\ 74.93 \\ 84.928 \\ 94.927 \\ 99.182 \\ 99.983 \\ 99.983 \end{pmatrix} \begin{pmatrix} 46.574 \\ 56.572 \\ 66.571 \\ 76.569 \\ 86.567 \\ 93.745 \\ 98.028 \\ 99.983 \end{pmatrix} \right]$$

$$\text{Order}\left(CI(DM)^T\right) = (4 \ \ 2 \ \ 3 \ \ 1)$$