STRUCTURE STUDY OF SYMMETRIC FUZZY NUMBERS

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Received September 27, 2017

Abstract. In many practical situations, intervals or fuzzy numbers are used to model imprecise observations derived from uncertain measurements or linguistic assessments. When using fuzzy numbers the shape of the membership function is important in modelling. In this paper, we consider the fuzzy numbers whose membership function is symmetric with respect to a vertical axis. For $\alpha \in (0,1]$ the $\alpha-$ cuts of such fuzzy numbers will have a constant mid-point and the upper end of the interval will be a non-increasing function of α, the lower end will be the image of this function. Hence these symmetric fuzzy numbers can be fully described by a constant and a non-increasing function. Based on this description, we define the arithmetic operations and a ranking technique to order the symmetric fuzzy numbers. We also discuss various properties of interest. Using Radstorm embedding theorem[5], we conduct a structure study on symmetric fuzzy numbers.

1 Introduction The operations on the set of fuzzy numbers are usually obtained by the Zadeh extension principle [7], [8], [6]. These definitions can have some disadvantages for the applications, both by an algebraic point of view and by logical and practical aspects. In particular, the shape of fuzzy numbers is not preserved by multiplication, the indeterminateness of the sum and product is often too increasing.

Dong Qiu et.al. [1] studied the algebraic properties of fuzzy numbers using equivalence classes on fuzzy numbers and identified the group structure for addition. In this paper, we are studying a special class of fuzzy numbers, namely the symmetric fuzzy numbers, whose membership function is symmetric with respect to a vertical axis, define various arithmetic operations anew to suit our need. Also applying Radstorm embedding theorem[5] we are identifying the vector space structure. We define the arithmetic operations, such as addition, subtraction, scalar multiplication, product, inverse on symmetric fuzzy numbers in a way that the resultants are also symmetric fuzzy numbers.

Section 2 introduces symmetric fuzzy numbers, the arithmetic operations and the ranking technique on them. We also verify various properties of the arithmetic operations in this section. Based on the properties verified, section 3 gives an embedding of the class of symmetric fuzzy numbers into a collection of equivalence classes of symmetric fuzzy numbers which forms a group and a vector space.

2 Symmetric Fuzzy Numbers

Definition 2.1 The characteristic function χ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function

$\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e. $\mu_{\tilde{A}} : X \to [0,1]$. The assigned value indicates the membership grade of the element in the set A. The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is called a fuzzy set.

Definition 2.2 A fuzzy set $\tilde{A}$, defined on the universal set of real numbers $\mathbb{R}$, is said to be a fuzzy number if its membership function has the following characteristics:
i. $\tilde{A}$ is convex i.e. $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \ \forall \ x_1, x_2 \in \mathbb{R}, \ \forall \ \lambda \in [0,1]$
ii. $\tilde{A}$ is normal i.e. $\exists x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$
iii. $\mu_{\tilde{A}}$ is piecewise continuous

The height of a fuzzy set $A \in \mathcal{F}(X)$, is the value $hgt(A) = \sup_{x \in X} \mu_A(x)$. From the definition of a fuzzy set it is immediate that $hgt(A) \leq 1$. If there exists $x_0 \in X$ such that $hgt(A) = \mu_A(x_0) = 1$, then the fuzzy set A is called normal.
The core of a fuzzy set $A \in \mathcal{F}(X)$ is denoted with $core(A)$ and it is given by $core(A) = \{x \in X \mid \mu_A(x) = 1\}$. The support of a fuzzy set $A \in \mathcal{F}(X)$ is denoted with $supp(A)$ and represents the set of all elements of X with a nonzero degree of membership, that is $supp(A) = \{x \in X \mid \mu_A(x) > 0\}$
For $\alpha \in [0,1]$, the α-cut of a fuzzy set $A \in \mathcal{F}(X)$ denoted by $[A]_\alpha$ and is given by $[A]_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$. It is clear that $[A]_0 = X$ and $[A]_1 = core(A)$.

Remark 2.1. For a fuzzy number, the $\alpha-$cut will be a closed interval.

Definition 2.3 Let $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$ be the $\alpha-$cut of the fuzzy number $\tilde{A}$, then $\tilde{A}$ is said to be symmetric if the mid-point $m_\alpha(\tilde{A}) = \dfrac{a_1^\alpha + a_2^\alpha}{2}$ is constant $\forall \alpha \in [0,1]$.

Remark 2.2. The spread $S_\alpha(\tilde{A}) = \dfrac{a_2^\alpha - a_1^\alpha}{2}$ is non-negative and a non-increasing function of α. It is the factor that determines the fuzziness of the quantity measured. As a particular case, when the spread is zero, the quantity reduces to a crisp quantity.

Remark 2.3. The $\alpha-$cut $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$ of the symmetric fuzzy numbers $\tilde{A}$ can also be represented as $[\tilde{A}]_\alpha = \dfrac{a_1^\alpha + a_2^\alpha}{2} + \dfrac{a_2^\alpha - a_1^\alpha}{2}[-1,1]$.

2.1 Ranking Technique For two symmetric fuzzy numbers $\tilde{A}$ and $\tilde{B}$ with $\alpha-$cuts $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$ and $[\tilde{B}]_\alpha = [b_1^\alpha, b_2^\alpha]$, define $\tilde{A} \prec \tilde{B}$ if either $\dfrac{a_1^\alpha + a_2^\alpha}{2} < \dfrac{b_1^\alpha + b_2^\alpha}{2}$ or $\dfrac{a_1^\alpha + a_2^\alpha}{2} = \dfrac{b_1^\alpha + b_2^\alpha}{2}$ and $\dfrac{a_2^\alpha - a_1^\alpha}{2} < \dfrac{b_2^\alpha - b_1^\alpha}{2}$. If $\dfrac{a_1^\alpha + a_2^\alpha}{2} = \dfrac{b_1^\alpha + b_2^\alpha}{2}$ and $\dfrac{a_2^\alpha - a_1^\alpha}{2} = \dfrac{b_2^\alpha - b_1^\alpha}{2}$, then $\tilde{A} = \tilde{B}$

2.2 Arithmetic Operations

Definition 2.4 Let $\tilde{A}$ and $\tilde{B}$ be two symmetric fuzzy numbers with $\alpha-$cuts $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$, $[\tilde{B}]_\alpha = [b_1^\alpha, b_2^\alpha]$ and $\lambda \in \mathbb{R}$, then the $\alpha-$cut of the arithmetic operations are defined as follows:

Sum

$$[\tilde{A} + \tilde{B}]_\alpha = \frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1] \quad (2.1)$$

Difference

$$[\tilde{A}-\tilde{B}]_\alpha = \frac{a_1^\alpha + a_2^\alpha}{2} - \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1] \tag{2.2}$$

Scalar Multiplication

$$[\lambda\tilde{A}]_\alpha = \lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \tag{2.3}$$

Product

$$\begin{aligned}[\tilde{A}.\tilde{B}]_\alpha &= \frac{a_1^\alpha + a_2^\alpha}{2}.\frac{b_1^\alpha + b_2^\alpha}{2} + \qquad (2.4)\\ &\left(\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]\end{aligned}$$

Inverse

$$\left[\frac{1}{\tilde{A}}\right]_\alpha = \frac{2}{a_1^\alpha + a_2^\alpha} + \frac{1}{2}\left(\frac{1}{a_1^\alpha} - \frac{1}{a_2^\alpha}\right)[-1,1] \tag{2.5}$$

here either $a_1^0 > 0$ or $a_2^0 < 0$

Proposition 2.1. *Let $\tilde{A}$ be a symmteric fuzzy number with $\alpha-$cut $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$, then for $\alpha < \beta$, $[a_1^\alpha, a_2^\alpha] \supseteq [a_1^\beta, a_2^\beta]$.*

Proof. $\tilde{A}$ is a symmetric fuzzy number $\implies$ the mid-point $m_\alpha(\tilde{A})$ is constant and the spread $S_\alpha(\tilde{A})$ is a non-increasing function of α.
Thus $\alpha < \beta \implies \frac{a_1^\alpha + a_2^\alpha}{2} = \frac{a_1^\beta + a_2^\beta}{2}$ and $\frac{a_2^\alpha - a_1^\alpha}{2} \geq \frac{a_2^\beta - a_1^\beta}{2}$
$\implies [a_1^\alpha, a_2^\alpha] \supseteq [a_1^\beta, a_2^\beta]$ □

Remark 2.4. Proposition 2.1 proves that the symmetric fuzzy number is convex.

Proposition 2.2. *If $\tilde{A}$ and $\tilde{B}$ are symmetric fuzzy numbers, then so are $\tilde{A}+\tilde{B}, \tilde{A}-\tilde{B}, \lambda\tilde{A}(\lambda \in \mathbb{R}), \tilde{A}.\tilde{B}, \frac{1}{\tilde{A}}$.*

Proof. Let the $\alpha-$cuts of $\tilde{A}$ and $\tilde{B}$ be $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$ and $[\tilde{B}]_\alpha = [b_1^\alpha, b_2^\alpha]$ respectively, then we know that for $\alpha < \alpha'$,
the mid-points

$$\frac{a_1^\alpha + a_2^\alpha}{2} = \frac{a_1^{\alpha'} + a_2^{\alpha'}}{2} \tag{2.6}$$

$$\frac{b_1^\alpha + b_2^\alpha}{2} = \frac{b_1^{\alpha'} + b_2^{\alpha'}}{2} \tag{2.7}$$

the spreads

$$\frac{a_2^\alpha - a_1^\alpha}{2} \geq \frac{a_2^{\alpha'} - a_1^{\alpha'}}{2} \tag{2.8}$$

$$\frac{b_2^\alpha - b_1^\alpha}{2} \geq \frac{b_2^{\alpha'} - b_1^{\alpha'}}{2} \tag{2.9}$$

Sum $[\tilde{A}+\tilde{B}]_\alpha = \frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]$

Adding equations 2.6 and 2.7, we get the mid-point of $\tilde{A}+\tilde{B}$ to be constant and adding 2.8 and 2.9 we see the spread of $\tilde{A}+\tilde{B}$ to be non-increasing. Thus $\tilde{A}+\tilde{B}$ is symmetric.

Difference $[\tilde{A}-\tilde{B}]_\alpha = \frac{a_1^\alpha + a_2^\alpha}{2} - \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]$

Subtracting equations 2.6 and 2.7, we get the mid-point of $\tilde{A}-\tilde{B}$ to be constant and adding 2.8 and 2.9 we see the spread of $\tilde{A}-\tilde{B}$ to be non-increasing. Thus $\tilde{A}-\tilde{B}$ is symmetric.

Scalar Multiplication $[\lambda\tilde{A}]_\alpha = \lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1]$

By the definition, it is clear that $\lambda\tilde{A}$ is a symmetric fuzzy number.

Product $[\tilde{A}.\tilde{B}]_\alpha = \frac{a_1^\alpha + a_2^\alpha}{2}.\frac{b_1^\alpha + b_2^\alpha}{2} +$
$\left(\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]$

Product of two constants is also a constant $\Longrightarrow$ the mid-point of $\tilde{A}.\tilde{B} = \frac{a_1^\alpha + a_2^\alpha}{2}.\frac{b_1^\alpha + b_2^\alpha}{2}$ is constant.
For $\alpha < \alpha'$,

$$\begin{aligned}
S_\alpha(\tilde{A}.\tilde{B}) &= \frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2} \\
&= \frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^{\alpha'} + b_2^{\alpha'}}{2}\right| + \left|\frac{a_1^{\alpha'} + a_2^{\alpha'}}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2} \\
&\qquad \text{using 2.6 and 2.7} \\
&\geq \frac{a_2^{\alpha'} - a_1^{\alpha'}}{2}\left|\frac{b_1^{\alpha'} + b_2^{\alpha'}}{2}\right| + \left|\frac{a_1^{\alpha'} + a_2^{\alpha'}}{2}\right|\frac{b_2^{\alpha'} - b_1^{\alpha'}}{2} + \frac{a_2^{\alpha'} - a_1^{\alpha'}}{2}\frac{b_2^{\alpha'} - b_1^{\alpha'}}{2} \\
&\qquad \text{using 2.8 and 2.9 and the fact that } \frac{a_2^{\alpha'} - a_1^{\alpha'}}{2}, \frac{b_2^{\alpha'} - b_1^{\alpha'}}{2} \\
&\qquad \text{are non-negative} \\
&= S_{\alpha'}(\tilde{A}.\tilde{B})
\end{aligned}$$

Thus $\tilde{A}.\tilde{B}$ is symmetric.

Inverse $\left[\frac{1}{\tilde{A}}\right]_\alpha = \frac{2}{a_1^\alpha + a_2^\alpha} + \frac{1}{2}\left(\frac{1}{a_1^\alpha} - \frac{1}{a_2^\alpha}\right)[-1,1]$

It is clear that $m_\alpha\left(\frac{1}{\tilde{A}}\right)$ is constant. To prove $S_\alpha\left(\frac{1}{\tilde{A}}\right)$ is non-increasing.
Let $\alpha < \alpha'$, then

$$\begin{aligned}
S_\alpha\left(\frac{1}{\tilde{A}}\right) &= \frac{1}{2}\left(\frac{1}{a_1^\alpha} - \frac{1}{a_2^\alpha}\right) \\
&= \frac{1}{2}\left(\frac{a_2^\alpha - a_1^\alpha}{a_1^\alpha a_2^\alpha}\right)
\end{aligned}$$

$$= \frac{S_\alpha(\tilde{A})}{a_1^\alpha a_2^\alpha} \tag{2.10}$$

We have $a_1^\alpha \le a_1^{\alpha'} \le a_2^{\alpha'} \le a_2^\alpha$. Since $\tilde{A}$ is symmetric, $a_1^{\alpha'} - a_1^\alpha = a_2^\alpha - a_2^{\alpha'} = k(say)$, let $a_2^{\alpha'} - a_1^{\alpha'} = c$, i.e

$$\begin{aligned}
a_1^{\alpha'} &= a_1^\alpha + k \\
a_2^{\alpha'} &= a_1^\alpha + k + c \\
a_2^\alpha &= a_1^\alpha + k + c + k \\
\text{Thus } a_1^\alpha a_2^\alpha - a_1^{\alpha'} a_2^{\alpha'} &= a_1^\alpha(a_1^\alpha + k + c + k) - (a_1^\alpha + k)(a_1^\alpha + k + c) \\
&= -k^2 - kc \\
&\le 0 \text{ as } k \ge 0, c \ge 0 \\
\implies \frac{1}{a_1^\alpha a_2^\alpha} &\ge \frac{1}{a_1^{\alpha'} a_2^{\alpha'}} \qquad (2.11) \\
\tilde{A} \text{ is symmetric} \implies S_\alpha(\tilde{A}) &\ge S_{\alpha'}(\tilde{A}) \qquad (2.12)
\end{aligned}$$

From the definition of the inverse all the terms appearing in equations 2.11 and 2.12 are positive, hence multiplying equations 2.11 and 2.12 and applying it in equation 2.10, we get $S_\alpha\left(\frac{1}{\tilde{A}}\right) \ge S_{\alpha'}\left(\frac{1}{\tilde{A}}\right)$.

Thus $\frac{1}{\tilde{A}}$ is symmetric.

□

Theorem 2.1. *[Properties of Arithmetic Operators] Let $\tilde{A}, \tilde{B}, \tilde{C}$ be symmetric fuzzy numbers, then the following properties hold:*

1. $\tilde{A} + \tilde{B} = \tilde{B} + \tilde{A}$ *(commutative)*
2. $\tilde{A}.\tilde{B} = \tilde{B}.\tilde{A}$ *(commutative)*
3. $(\tilde{A} + \tilde{B}) + \tilde{C} = \tilde{A} + (\tilde{B} + \tilde{C})$ *(associative)*
4. $(\tilde{A}.\tilde{B}).\tilde{C} = \tilde{A}.(\tilde{B}.\tilde{C})$ *(associative)*
5. $\tilde{A} + \tilde{0} = \tilde{0} + \tilde{A} = \tilde{A}$ *(identity)*
6. $\tilde{A}.\tilde{1} = \tilde{1}.\tilde{A} = \tilde{A}$ *(identity)*
7. $\tilde{A} + \tilde{B} = \tilde{A} + \tilde{C} \implies \tilde{B} = \tilde{C}$ *(cancellation)*
8. $\tilde{A}.\tilde{B} = \tilde{A}.\tilde{C} \implies \tilde{B} = \tilde{C}$ *(cancellation)*
9. *Scalar multiplication by non-negative real scalars satisfies:*
   (a) $\lambda(A + B) = \lambda A + \lambda B$
   (b) $(\lambda + \mu)A = \lambda A + \mu A$
   (c) $(\lambda\mu)A = \lambda(\mu A)$
10. $A.\tilde{}\ (B\tilde{} + C\tilde{}) \preceq A.\tilde{}\ B\tilde{} + A.\tilde{}\ C\tilde{}$ *(sub-distributive)*

11. $(\tilde{A}+\tilde{B})-\tilde{C}=\tilde{A}+(\tilde{B}-\tilde{C})$

12. $(\tilde{A}+\tilde{B})-\tilde{B}\neq\tilde{A}$

13. $\tilde{A}\preceq\tilde{C}$ *and* $\tilde{B}\preceq\tilde{D} \implies \tilde{A}+\tilde{B}\preceq\tilde{C}+\tilde{D}$ *and* $\tilde{A}-\tilde{B}\preceq\tilde{C}-\tilde{D}$ *(inclusion monotonicity)*

Proof. Let the $\alpha-$cut of the given symmetric fuzzy numbers be $[\tilde{A}]_\alpha = [a_1^\alpha, a_2^\alpha]$, $[\tilde{B}]_\alpha = [b_1^\alpha, b_2^\alpha]$, $[\tilde{C}]_\alpha = [c_1^\alpha, c_2^\alpha]$ and $[\tilde{0}]_\alpha = [0,0]$, $[\tilde{1}]_\alpha = [1,1]$ and thus in the mid-point and spread notation

$$[\tilde{A}]_\alpha = \frac{a_1^\alpha + a_2^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}[-1,1]$$
$$[\tilde{B}]_\alpha = \frac{b_1^\alpha + b_2^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}[-1,1]$$
$$[\tilde{C}]_\alpha = \frac{c_1^\alpha + c_2^\alpha}{2} + \frac{c_2^\alpha - c_1^\alpha}{2}[-1,1]$$

1. To prove $\tilde{A}+\tilde{B}=\tilde{B}+\tilde{A}$

$$\begin{aligned}
[\tilde{A}+\tilde{B}]_\alpha &= \frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1] \\
&= \frac{b_1^\alpha + b_2^\alpha}{2} + \frac{a_1^\alpha + a_2^\alpha}{2} + \left(\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \\
&= [\tilde{B}+\tilde{A}]_\alpha
\end{aligned}$$

2. To prove $\tilde{A}.\tilde{B}=\tilde{B}.\tilde{A}$

$$\begin{aligned}
[\tilde{A}.\tilde{B}]_\alpha &= \frac{a_1^\alpha + a_2^\alpha}{2}.\frac{b_1^\alpha + b_2^\alpha}{2} + \\
&\quad \left(\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1] \\
&= \frac{b_1^\alpha + b_2^\alpha}{2}.\frac{a_1^\alpha + a_2^\alpha}{2} + \\
&\quad \left(\frac{b_2^\alpha - b_1^\alpha}{2}\left|\frac{a_1^\alpha + a_2^\alpha}{2}\right| + \left|\frac{b_1^\alpha + b_2^\alpha}{2}\right|\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \\
&= [\tilde{B}.\tilde{A}]_\alpha
\end{aligned}$$

3. To prove $(\tilde{A}+\tilde{B})+\tilde{C}=\tilde{A}+(\tilde{B}+\tilde{C})$

$$\begin{aligned}
[(\tilde{A}+\tilde{B})+\tilde{C}]_\alpha &= \left(\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2}\right) + \frac{c_1^\alpha + c_2^\alpha}{2} \\
&\quad + \left\{\left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right) + \frac{c_2^\alpha - c_1^\alpha}{2}\right\}[-1,1] \\
&= \frac{a_1^\alpha + a_2^\alpha}{2} + \left(\frac{b_1^\alpha + b_2^\alpha}{2} + \frac{c_1^\alpha + c_2^\alpha}{2}\right) \\
&\quad + \left\{\frac{a_2^\alpha - a_1^\alpha}{2} + \left(\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{c_2^\alpha - c_1^\alpha}{2}\right)\right\}[-1,1] \\
&= [\tilde{A}+(\tilde{B}+\tilde{C})]_\alpha
\end{aligned}$$

4. To prove $(\tilde{A}.\tilde{B}).\tilde{C} = \tilde{A}.(\tilde{B}.\tilde{C})$
$[(\tilde{A}.\tilde{B}).\tilde{C}]_\alpha$

$$\begin{aligned}
&= \left\{ \frac{a_1^\alpha + a_2^\alpha}{2} . \frac{b_1^\alpha + b_2^\alpha}{2} + \right. \\
&\left. \left(\frac{a_2^\alpha - a_1^\alpha}{2} \left| \frac{b_1^\alpha + b_2^\alpha}{2} \right| + \left| \frac{a_1^\alpha + a_2^\alpha}{2} \right| \frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2} \frac{b_2^\alpha - b_1^\alpha}{2} \right) [-1,1] \right\} \\
&\quad . \left\{ \frac{c_1^\alpha + c_2^\alpha}{2} + \frac{c_2^\alpha - c_1^\alpha}{2} [-1,1] \right\} \\
&= \left(\frac{a_1^\alpha + a_2^\alpha}{2} . \frac{b_1^\alpha + b_2^\alpha}{2} \right) \left(\frac{c_1^\alpha + c_2^\alpha}{2} \right) \\
&\quad + \left\{ \left(\frac{a_2^\alpha - a_1^\alpha}{2} \left| \frac{b_1^\alpha + b_2^\alpha}{2} \right| + \left| \frac{a_1^\alpha + a_2^\alpha}{2} \right| \frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2} \frac{b_2^\alpha - b_1^\alpha}{2} \right) \left| \frac{c_1^\alpha + c_2^\alpha}{2} \right| \right. \\
&\quad + \left| \frac{a_1^\alpha + a_2^\alpha}{2} . \frac{b_1^\alpha + b_2^\alpha}{2} \right| \frac{c_2^\alpha - c_1^\alpha}{2} \\
&\quad + \left(\frac{a_2^\alpha - a_1^\alpha}{2} \left| \frac{b_1^\alpha + b_2^\alpha}{2} \right| + \left| \frac{a_1^\alpha + a_2^\alpha}{2} \right| \frac{b_2^\alpha - b_1^\alpha}{2} \right. \\
&\qquad \left. \left. + \frac{a_2^\alpha - a_1^\alpha}{2} \frac{b_2^\alpha - b_1^\alpha}{2} \right) \frac{c_2^\alpha - c_1^\alpha}{2} \right\} [-1,1] \\
&= \frac{a_1^\alpha + a_2^\alpha}{2} \left(\frac{b_1^\alpha + b_2^\alpha}{2} \frac{c_1^\alpha + c_2^\alpha}{2} \right) \\
&\quad + \left\{ \frac{a_2^\alpha - a_1^\alpha}{2} \left| \frac{b_1^\alpha + b_2^\alpha}{2} \right| \left| \frac{c_1^\alpha + c_2^\alpha}{2} \right| + \left| \frac{a_1^\alpha + a_2^\alpha}{2} \right| \frac{b_2^\alpha - b_1^\alpha}{2} \frac{c_1^\alpha + c_2^\alpha}{2} \right. \\
&\quad + \frac{a_2^\alpha - a_1^\alpha}{2} \frac{b_2^\alpha - b_1^\alpha}{2} \left| \frac{c_1^\alpha + c_2^\alpha}{2} \right| + \left| \frac{a_1^\alpha + a_2^\alpha}{2} \right| \left| \frac{b_1^\alpha + b_2^\alpha}{2} \right| \frac{c_2^\alpha - c_1^\alpha}{2} \\
&\quad + \frac{a_2^\alpha - a_1^\alpha}{2} \left| \frac{b_1^\alpha + b_2^\alpha}{2} \right| \frac{c_2^\alpha - c_1^\alpha}{2} + \left| \frac{a_1^\alpha + a_2^\alpha}{2} \right| \frac{b_2^\alpha - b_1^\alpha}{2} \frac{c_2^\alpha - c_1^\alpha}{2} \\
&\quad \left. + \frac{a_2^\alpha - a_1^\alpha}{2} \frac{b_2^\alpha - b_1^\alpha}{2} \frac{c_2^\alpha - c_1^\alpha}{2} \right\} [-1,1] \\
&= \frac{a_1^\alpha + a_2^\alpha}{2} \left(\frac{b_1^\alpha + b_2^\alpha}{2} \frac{c_1^\alpha + c_2^\alpha}{2} \right) \\
&\quad + \left\{ \frac{a_2^\alpha - a_1^\alpha}{2} \left| \frac{b_1^\alpha + b_2^\alpha}{2} \frac{c_1^\alpha + c_2^\alpha}{2} \right| \right. \\
&\quad + \left| \frac{a_1^\alpha + a_2^\alpha}{2} \right| \left(\frac{b_2^\alpha - b_1^\alpha}{2} \left| \frac{c_1^\alpha + c_2^\alpha}{2} \right| + \left| \frac{b_1^\alpha + b_2^\alpha}{2} \right| \frac{c_2^\alpha - c_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2} \frac{c_2^\alpha - c_1^\alpha}{2} \right) \\
&\quad \left. + \frac{a_2^\alpha - a_1^\alpha}{2} \left(\frac{b_2^\alpha - b_1^\alpha}{2} \left| \frac{c_1^\alpha + c_2^\alpha}{2} \right| + \left| \frac{b_1^\alpha + b_2^\alpha}{2} \right| \frac{c_2^\alpha - c_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2} \frac{c_2^\alpha - c_1^\alpha}{2} \right) \right\} \\
&\qquad [-1,1] \\
&= [\tilde{A}.(\tilde{B}.\tilde{C})]_\alpha
\end{aligned}$$

5. To prove $\tilde{A} + \tilde{0} = \tilde{0} + \tilde{A} = \tilde{A}$

$$\begin{aligned}
[\tilde{A} + \tilde{0}]_\alpha &= \frac{a_1^\alpha + a_2^\alpha}{2} + \frac{0+0}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{0-0}{2} \right) [-1,1] \\
&= \frac{a_1^\alpha + a_2^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2} [-1,1] \\
&= [\tilde{A}]_\alpha
\end{aligned}$$

Similarly, $\tilde{0}+\tilde{A}=\tilde{A}$.

6. To prove $\tilde{A}.\tilde{1}=\tilde{1}.\tilde{A}=\tilde{A}$

$$\begin{aligned}
[\tilde{A}.\tilde{1}]_\alpha &= \frac{a_1^\alpha+a_2^\alpha}{2}.\frac{1+1}{2} \\
&\quad+\left(\frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{1+1}{2}\right|+\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{1-1}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}\frac{1-1}{2}\right)[-1,1] \\
&= \frac{a_1^\alpha+a_2^\alpha}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}[-1,1] \\
&= [\tilde{A}]_\alpha
\end{aligned}$$

Similarly, $\tilde{1}.\tilde{A}=\tilde{A}$.

7. $\tilde{A}+\tilde{B}=\tilde{A}+\tilde{C} \implies \tilde{B}=\tilde{C}$ (cancellation)

$$\begin{aligned}
& [\tilde{A}+\tilde{B}]_\alpha=[\tilde{A}+\tilde{C}]_\alpha \\
\implies\ & \frac{a_1^\alpha+a_2^\alpha}{2}+\frac{b_1^\alpha+b_2^\alpha}{2}+\left(\frac{a_2^\alpha-a_1^\alpha}{2}+\frac{b_2^\alpha-b_1^\alpha}{2}\right)[-1,1] \\
& \quad =\frac{a_1^\alpha+a_2^\alpha}{2}+\frac{c_1^\alpha+c_2^\alpha}{2}+\left(\frac{a_2^\alpha-a_1^\alpha}{2}+\frac{c_2^\alpha-c_1^\alpha}{2}\right)[-1,1] \\
\implies\ & \frac{a_1^\alpha+a_2^\alpha}{2}+\frac{b_1^\alpha+b_2^\alpha}{2}=\frac{a_1^\alpha+a_2^\alpha}{2}+\frac{c_1^\alpha+c_2^\alpha}{2} \\
& \text{and } \frac{a_2^\alpha-a_1^\alpha}{2}+\frac{b_2^\alpha-b_1^\alpha}{2}=\frac{a_2^\alpha-a_1^\alpha}{2}+\frac{c_2^\alpha-c_1^\alpha}{2} \\
\implies\ & \frac{b_1^\alpha+b_2^\alpha}{2}=\frac{c_1^\alpha+c_2^\alpha}{2} \\
& \text{and } \frac{b_2^\alpha-b_1^\alpha}{2}=\frac{c_2^\alpha-c_1^\alpha}{2} \\
\implies\ & [\tilde{B}]_\alpha=[\tilde{C}]_\alpha \\
\implies\ & \tilde{B}=\tilde{C}
\end{aligned}$$

8. $\tilde{A}.\tilde{B}=\tilde{A}.\tilde{C} \implies \tilde{B}=\tilde{C}$ if $\tilde{A}\neq 0$ (cancellation)

$$\begin{aligned}
& \tilde{A}.\tilde{B}=\tilde{A}.\tilde{C} \\
\implies\ & [\tilde{A}.\tilde{B}]_\alpha=[\tilde{A}.\tilde{B}]_\alpha \\
\implies\ & \frac{a_1^\alpha+a_2^\alpha}{2}.\frac{b_1^\alpha+b_2^\alpha}{2}=\frac{a_1^\alpha+a_2^\alpha}{2}.\frac{c_1^\alpha+c_2^\alpha}{2} \\
& \text{and } \frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{b_1^\alpha+b_2^\alpha}{2}\right|+\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{b_2^\alpha-b_1^\alpha}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}\frac{b_2^\alpha-b_1^\alpha}{2} \\
& \quad =\frac{a_2^\alpha-a_1^\alpha}{2}\left|\frac{c_1^\alpha+c_2^\alpha}{2}\right|+\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{c_2^\alpha-c_1^\alpha}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}\frac{c_2^\alpha-c_1^\alpha}{2} \\
\implies\ & \frac{b_1^\alpha+b_2^\alpha}{2}=\frac{c_1^\alpha+c_2^\alpha}{2} \\
& \text{and } \left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{b_2^\alpha-b_1^\alpha}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}\frac{b_2^\alpha-b_1^\alpha}{2} \\
& \quad =\left|\frac{a_1^\alpha+a_2^\alpha}{2}\right|\frac{c_2^\alpha-c_1^\alpha}{2}+\frac{a_2^\alpha-a_1^\alpha}{2}\frac{c_2^\alpha-c_1^\alpha}{2}
\end{aligned}$$

$$\implies \left(\left|\frac{a_1^\alpha + a_2^\alpha}{2}\right| + \frac{a_2^\alpha - a_1^\alpha}{2}\right)\left(\frac{b_2^\alpha - b_1^\alpha}{2} - \frac{c_2^\alpha - c_1^\alpha}{2}\right) = 0$$
$$\implies \frac{b_2^\alpha - b_1^\alpha}{2} = \frac{c_2^\alpha - c_1^\alpha}{2}$$

as $\left|\frac{a_1^\alpha + a_2^\alpha}{2}\right| + \frac{a_2^\alpha - a_1^\alpha}{2} = 0$ would mean $\left|\frac{a_1^\alpha + a_2^\alpha}{2}\right| = 0$ and $\frac{a_2^\alpha - a_1^\alpha}{2} = 0$ $\implies \tilde{A} = 0$

9. Scalar multiplication by non-negative real scalars satisfies:

(a) $\lambda(A + B) = \lambda A + \lambda B$ for $\lambda \geq 0$

$$\begin{aligned}
[\lambda(A+B)]_\alpha &= \lambda\left\{\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]\right\} \\
&= \lambda\left\{\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2}\right\} \\
&\quad + \left\{|\lambda|\left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)\right\}[-1,1] \\
&= \left\{\lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \lambda\frac{b_1^\alpha + b_2^\alpha}{2}\right\} \\
&\quad + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2} + |\lambda|\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1] \\
&= \lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \\
&\quad + \lambda\frac{b_1^\alpha + b_2^\alpha}{2} + \left(|\lambda|\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1] \\
&= [\lambda A]_\alpha + [\lambda B]_\alpha
\end{aligned}$$

(b) $(\lambda + \mu)A = \lambda A + \mu A$ for $\lambda, \mu \geq 0$

$$\begin{aligned}
[(\lambda+\mu)A]_\alpha &= (\lambda+\mu)\left\{\frac{a_1^\alpha + a_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1]\right\} \\
&= (\lambda+\mu)\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda+\mu|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \\
&= \lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \mu\frac{a_1^\alpha + a_2^\alpha}{2} \\
&\quad + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2} + |\mu|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \text{ as } \lambda, \mu \geq 0 \\
&= \lambda\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \\
&\quad + \mu\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\mu|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1] \\
&= [\lambda A]_\alpha + [\mu A]_\alpha
\end{aligned}$$

(c) $(\lambda\mu)A = \lambda(\mu A)$ for $\lambda, \mu \geq 0$

$$[(\lambda\mu)A]_\alpha = (\lambda\mu)\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\lambda\mu|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1]$$

$$= \lambda\left\{\mu\frac{a_1^\alpha + a_2^\alpha}{2} + \left(|\mu|\frac{a_2^\alpha - a_1^\alpha}{2}\right)[-1,1]\right\} \text{ as } \lambda \geq 0$$
$$= [\lambda(\mu A)]_\alpha$$

10. $\tilde{A}.(\tilde{B}+\tilde{C}) \preceq \tilde{A}.\tilde{B}+\tilde{A}.\tilde{C}$ (sub-distributive)

$$\begin{aligned}
[\tilde{A}.(\tilde{B}+\tilde{C})]_\alpha &= \left\{\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}[-1,1]\right\}.\left\{\frac{b_1^\alpha + b_2^\alpha}{2} + \frac{c_1^\alpha + c_2^\alpha}{2}\right. \\
&\qquad \left. + \left(\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{c_2^\alpha - c_1^\alpha}{2}\right)[-1,1]\right\} \\
&= \frac{a_1^\alpha + a_2^\alpha}{2}.\left(\frac{b_1^\alpha + b_2^\alpha}{2} + \frac{c_1^\alpha + c_2^\alpha}{2}\right) \\
&\quad + \left\{\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2} + \frac{c_1^\alpha + c_2^\alpha}{2}\right|\right. \\
&\qquad + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\left(\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{c_2^\alpha - c_1^\alpha}{2}\right) \\
&\qquad\quad \left. + \frac{a_2^\alpha - a_1^\alpha}{2}\left(\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{c_2^\alpha - c_1^\alpha}{2}\right)\right\}[-1,1] \\
&\preceq \frac{a_1^\alpha + a_2^\alpha}{2}\frac{b_1^\alpha + b_2^\alpha}{2} + \frac{a_1^\alpha + a_2^\alpha}{2}\frac{c_1^\alpha + c_2^\alpha}{2} + \\
&\quad \left\{\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{c_1^\alpha + c_2^\alpha}{2}\right|\right. \\
&\qquad + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2} + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{c_2^\alpha - c_1^\alpha}{2} \\
&\qquad\quad \left. + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{c_2^\alpha - c_1^\alpha}{2}\right\}[-1,1] \\
&= \frac{a_1^\alpha + a_2^\alpha}{2}\frac{b_1^\alpha + b_2^\alpha}{2} + \left\{\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{b_1^\alpha + b_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{b_2^\alpha - b_1^\alpha}{2}\right. \\
&\qquad \left. + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{b_2^\alpha - b_1^\alpha}{2}\right\}[-1,1] \\
&\quad + \frac{a_1^\alpha + a_2^\alpha}{2}\frac{c_1^\alpha + c_2^\alpha}{2} + \left\{\frac{a_2^\alpha - a_1^\alpha}{2}\left|\frac{c_1^\alpha + c_2^\alpha}{2}\right| + \left|\frac{a_1^\alpha + a_2^\alpha}{2}\right|\frac{c_2^\alpha - c_1^\alpha}{2}\right. \\
&\qquad \left. + \frac{a_2^\alpha - a_1^\alpha}{2}\frac{c_2^\alpha - c_1^\alpha}{2}\right\}[-1,1] \\
&= [\tilde{A}.\tilde{B}]_\alpha + [\tilde{A}.\tilde{C}]_\alpha
\end{aligned}$$

11. $(\tilde{A}+\tilde{B})-\tilde{C} = \tilde{A}+(\tilde{B}-\tilde{C})$

$$\begin{aligned}
[(\tilde{A}+\tilde{B})-\tilde{C}]_\alpha &= \left\{\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]\right\} \\
&\qquad - \left\{\frac{c_1^\alpha + c_2^\alpha}{2} + \left(\frac{c_2^\alpha - c_1^\alpha}{2}\right)[-1,1]\right\} \\
&= \left(\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2}\right) - \frac{c_1^\alpha + c_2^\alpha}{2} \\
&\quad + \left\{\left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right) + \frac{c_2^\alpha - c_1^\alpha}{2}\right\}[-1,1]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a_1^\alpha + a_2^\alpha}{2} + \left(\frac{b_1^\alpha + b_2^\alpha}{2} - \frac{c_1^\alpha + c_2^\alpha}{2}\right) \\
&\quad + \left\{\frac{a_2^\alpha - a_1^\alpha}{2} + \left(\frac{b_2^\alpha - b_1^\alpha}{2} + \frac{c_2^\alpha - c_1^\alpha}{2}\right)\right\}[-1,1] \\
&= [\tilde{A} + (\tilde{B} - \tilde{C})]_\alpha
\end{aligned}
$$

12. $(\tilde{A} + \tilde{B}) - \tilde{B} \neq \tilde{A}$

$$
\begin{aligned}
[(\tilde{A} + \tilde{B}) - \tilde{B}]_\alpha &= \left\{\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]\right\} \\
&\quad - \left\{\frac{b_1^\alpha + b_2^\alpha}{2} + \left(\frac{b_2^\alpha - b_1^\alpha}{2}\right)[-1,1]\right\} \\
&= \left(\frac{a_1^\alpha + a_2^\alpha}{2} + \frac{b_1^\alpha + b_2^\alpha}{2}\right) - \frac{b_1^\alpha + b_2^\alpha}{2} \\
&\quad + \left\{\left(\frac{a_2^\alpha - a_1^\alpha}{2} + \frac{b_2^\alpha - b_1^\alpha}{2}\right) + \frac{b_2^\alpha - b_1^\alpha}{2}\right\}[-1,1] \\
&= \frac{a_1^\alpha + a_2^\alpha}{2} + \left\{\frac{a_2^\alpha - a_1^\alpha}{2} + b_2^\alpha - b_1^\alpha\right\}[-1,1] \\
&\neq [\tilde{A}]_\alpha
\end{aligned}
$$

13. $\tilde{A} \preceq \tilde{C}$ and $\tilde{B} \preceq \tilde{D} \implies \tilde{A} + \tilde{B} \preceq \tilde{C} + \tilde{D}$ (inclusion monotonicity)
$\tilde{A} \preceq \tilde{C} \implies m_\alpha(\tilde{A}) < m_\alpha(\tilde{C})$ or $\left[m_\alpha(\tilde{A}) = m_\alpha(\tilde{C}) \text{ and } S_\alpha(A) \leq S_\alpha(\tilde{C})\right]$
Similarly,
$\tilde{B} \preceq \tilde{D} \implies m_\alpha(\tilde{B}) < m_\alpha(\tilde{D})$ or $\left[m_\alpha(\tilde{B}) = m_\alpha(\tilde{D}) \text{ and } S_\alpha(B) \leq S_\alpha(\tilde{D})\right]$

Case i: $m_\alpha(\tilde{A}) < m_\alpha(\tilde{C})$ and $m_\alpha(\tilde{B}) < m_\alpha(\tilde{D})$
$m_\alpha(\tilde{A} + \tilde{B}) = m_\alpha(\tilde{A}) + m_\alpha(\tilde{B}) < m_\alpha(\tilde{C}) + m_\alpha(\tilde{D}) = m_\alpha(\tilde{C} + \tilde{D})$
$\implies \tilde{A} + \tilde{B} \preceq \tilde{C} + \tilde{D}$

Case ii: $m_\alpha(\tilde{A}) < m_\alpha(\tilde{C})$ and $\left[m_\alpha(\tilde{B}) = m_\alpha(\tilde{D}) \text{ and } S_\alpha(B) \leq S_\alpha(\tilde{D})\right]$
$m_\alpha(\tilde{A} + \tilde{B}) = m_\alpha(\tilde{A}) + m_\alpha(\tilde{B}) < m_\alpha(\tilde{C}) + m_\alpha(\tilde{D}) = m_\alpha(\tilde{C} + \tilde{D})$
$\implies \tilde{A} + \tilde{B} \preceq \tilde{C} + \tilde{D}$

Case iii: $\left[m_\alpha(\tilde{A}) = m_\alpha(\tilde{C}) \text{ and } S_\alpha(A) \leq S_\alpha(\tilde{C})\right]$ and $m_\alpha(\tilde{B}) < m_\alpha(\tilde{D})$
Similar to Case ii

Case iv: $\left[m_\alpha(\tilde{A}) = m_\alpha(\tilde{C}) \text{ and } S_\alpha(A) \leq S_\alpha(\tilde{C})\right]$ and $\left[m_\alpha(\tilde{B}) = m_\alpha(\tilde{D}) \text{ and } S_\alpha(B) \leq S_\alpha(\tilde{D})\right]$
$m_\alpha(\tilde{A} + \tilde{B}) = m_\alpha(\tilde{A}) + m_\alpha(\tilde{B}) = m_\alpha(\tilde{C}) + m_\alpha(\tilde{D}) = m_\alpha(\tilde{C} + \tilde{D})$
$S_\alpha(\tilde{A} + \tilde{B}) = S_\alpha(\tilde{A}) + S_\alpha(\tilde{B}) \leq S_\alpha(\tilde{C}) + S_\alpha(\tilde{D}) = S_\alpha(\tilde{C} + \tilde{D})$
$\implies \tilde{A} + \tilde{B} \preceq \tilde{C} + \tilde{D}$

□

3 Embedding To extend the concepts of coherent prevision and probability in a fuzzy ambit, it is necessary to obtain a structure of vector space based on fuzzy numbers. But, whatever definition of sum is utilized, the sum of two fuzzy numbers has left and right spreads greater than the spreads of the individual fuzzy numbers. Then we cannot have the additive inverse of a non degenerate fuzzy number and fuzzy numbers are neither a group nor a vector space.

In this section we prove that we can overcome this obstacle by introducing a suitable equivalence relation $\sim$ on the set $\mathcal{SF}$ of fuzzy numbers and by considering the quotient set $\mathcal{SF}/\sim$ and the induced structures. In fact, in this case we obtain a vector space.

Theorem 3.1. *[5]*

A. *Let M be a commutative semigroup in which the law of cancellation holds. That is, For $A, B, C \in M$, if*

   1. $(A + B) + C = A + (B + C)$
   2. $A + B = B + A$
   3. $A + C = B + C \implies A = B$

   then M can be embedded in a group N. Furthermore N can be chosen so as to be minimal in the following sense: If G is any group in which M is embedded, then N is isomorphic to a subgroup of G containing M.

B. *If a multiplication by non-negative real scalars satisfying:*

   4. $\lambda(A + B) = \lambda A + \lambda B$
   5. $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$
   6. $\lambda_1(\lambda_2)A = \lambda_1\lambda_2 A$
   7. $1A = A$

   is defined on M, then a multiplication be real scalars can be defined on N so as to make N a vector space and so that for $\lambda \geq 0$ and $A \in M$ the product λA coincides with the one given on M.

Theorem 2.1 shows that the collection of symmetric fuzzy numbers $\mathcal{SF}$ satisfy conditions 1 to 7 of theorem 3.1. Hence $\mathcal{SF}$ can be embedded into $\mathcal{SFN}$ which will be a group and a vector space. According to the proof of theorem 3.1 in [5], the class $\mathcal{SFN}$ consists of equivalence classes of pairs $(\tilde{A}, \tilde{B})$ of elements of $\mathcal{SF}$. The equivalence relation, $\sim$ is defined by $(\tilde{A}, \tilde{B}) \sim (\tilde{C}, \tilde{D})$ if and only if $\tilde{A} + \tilde{D} = \tilde{B} + \tilde{C}$ i.e. $m_\alpha(\tilde{A} + \tilde{D}) = m_\alpha(\tilde{B} + \tilde{C})$ and $S_\alpha(\tilde{A} + \tilde{D}) = S_\alpha(\tilde{B} + \tilde{C})$. The equivalence class containing the pair (A, B) is denoted by $[A, B]$.
Define addition on $\mathcal{SFN}$ as

$$[\tilde{A}, \tilde{B}] + [\tilde{C}, \tilde{D}] = [\tilde{A} + \tilde{C}, \tilde{B} + \tilde{D}]$$

and scalar multiplication as

$$c[\tilde{A}, \tilde{B}] = \begin{cases} [c\tilde{A}, c\tilde{B}] & \text{if } c \in \mathbb{R}_+ \\ [-c\tilde{B}, -c\tilde{A}] & \text{otherwise} \end{cases}$$

and the order relation may be defined on $\mathcal{SFN}$ as $[\tilde{A}, \tilde{B}] \preceq [\tilde{C}, \tilde{D}]$ if $\tilde{A} + \tilde{D} \preceq \tilde{B} + \tilde{C}$ holds.
The zero element in $\mathcal{SFN}$ will be $[\tilde{0}, \tilde{0}]$ and the inverse of $[A, B]$ will be $[B, A]$.
The element $\tilde{A} \in \mathcal{SF}$ will be identified with the class $[\tilde{A}, \tilde{0}] \in \mathcal{SFN}$, where $\tilde{0}$ is the zero element in $\mathcal{SF}$.

4 Conclusion In this paper, a special class of fuzzy numbers is considered, the symmetric fuzzy numbers whose shape is symmetric with respect to a vertical line. We introduced the necessary arithmetic operations on these numbers and also verified that they belong to the same class. When studying the structure of the class, we see that it forms a commutative semi-group with the cancellation property. Also it satisfies certain other properties that are required in the Radstorm embedding theorem. Hence using Radstorm embedding theorem, the class of symmetric fuzzy numbers are embedded into a class of equivalent pairs of symmetric fuzzy numbers which form a group and a vector space.

References

[1] Dong Qiu, Chongxia Lu, Wei Zhang, Yaoyao Lan, Algebraic Properties and topological properties of the quotient space of fuzzy numbers based on Mares equivalence relation, Fuzzy Sets and Systems, 245,(16), 63-82 (2014).

[2] Klaus D Schmidt, Embedding theorems for classes of convex sets in a hypernormed vector space, Analysis 6, 57-96 (1986).

[3] Mares M, Addition of fuzzy quantities: disjunction-conjunctionapproach, Kybernetika 25, 104-116 (1989).

[4] Maturo A, On some structures of fuzzy numbers, Iranian Journal of Fuzzy Systems, 6 49-59 (2009).

[5] Radstrom H, An embedding theorem for spaces of convex sets, Proc. Amer. Math. Soc., 3, 165-169 (1952).

[6] R. Yager, A characterization of the extension principle, Fuzzy Sets and Systems, 18(3), 205-217 (1986).

[7] L. Zadeh, Fuzzy sets, Inf. Control, 8, 338-353 (1965).

[8] L. Zadeh, Probability measures of fuzzy events, J. Math. Anal. Appl., 23, 421-427 (1968).