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A VANISHING THEOREM OF ADDITIVE HIGHER CHOW GROUPS

Toshiro Hiranouchi

Abstract. We show that the additive higher Chow group of the form $\mathrm{TCH}^{\dim(X)+q}(X,q;m)$ becomes 0 for some scheme X over a perfect field of positive characteristic and for $q \geqslant 2$. This is an analogy of Akhtar's theorem on the higher Chow groups: $\mathrm{CH}^{\dim(X)+q}(X,q) = 0$ for $q \geqslant 2$.

1 Introduction As a continuation[1] of [5], we study an analogy between

$$\mathrm{CH}^a(X,b) \quad \longleftrightarrow \quad \mathrm{TCH}^a(X,b;m).$$

Here, $\mathrm{CH}^a(X,b)$ is the higher Chow group of an appropriate scheme X over a field k and $\mathrm{TCH}^a(X,b;m)$ is the additive higher Chow group of X (see Sect. 2 for the definitions). An objective of this note is to show the following theorem:

Theorem 1.1 (Thm. 3.5)**.** *Let X be a projective smooth variety over a perfect field k with positive characteristic. Then, for $q \geqslant 2$,*

$$\mathrm{TCH}^{d+q}(X,q;m) = 0,$$

where $d = \dim(X)$ is the dimension of X.

This is an additive version of Akhtar's theorem ([1], Cor. 7.1) on the higher Chow group: For $q \geqslant 2$,

$$\mathrm{CH}^{d+q}(X,q) = 0,$$

when X is a smooth quasi-projective variety of $d = \dim(X)$ over a *finite field*.

Our motivation is to define an additive variant of Somekawa type K-groups. Recall that a Mackey functor over a field k is a contravariant functor from the category of étale schemes over k to that of abelian groups equipped with a covariant structure for finite morphisms satisfying some conditions (for the precise definition, see Def. 3.1). The higher Chow group $\mathrm{CH}^a(X,b)$ defines a Mackey functor

$$\mathscr{C}\mathrm{H}^a(X,b) : k'/k \mapsto \mathrm{CH}^a(X_{k'},b),$$

where k' is a finite field extension of k and $X_{k'} = X \otimes_k k'$. For some schemes X, X' over k with $d = \dim(X)$ and $d' = \dim(X')$, the Milnor type K-group

$$K(k;\mathscr{C}\mathrm{H}^{d+a}(X,a),\mathscr{C}\mathrm{H}^{d'+a'}(X',a'))$$

introduced by Raskind and Spiess ([12], Def. 2.1.1, see also Rem. 2.4.2) is defined by the quotient

$$\left(\bigoplus_{k'/k:\,\text{finite}} \mathscr{C}\mathrm{H}^{d+a}(X,a)(k') \otimes_{\mathbb{Z}} \mathscr{C}\mathrm{H}^{d'+a'}(X',a')(k')\right) \Big/ \ \textbf{(PF)}\ \&\ \textbf{(Rec)}, \tag{1}$$

where "**(PF)** & **(Rec)**" stands for the subgroup generated by elements of the following form: Put $\mathscr{M} := \mathscr{C}\mathrm{H}^{d+a}(X,a)$ and $\mathscr{M}' := \mathscr{C}\mathrm{H}^{d'+a'}(X',a')$.

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[1] This short note is taken from the preprint [4], Sect. 5 which has been deleted before publication ([5]).

(PF) Let $k \subset k_1 \subset k_2$ be finite field extensions and $j = j_{k_2/k_1} : \mathrm{Spec}(k_2) \to \mathrm{Spec}(k_1)$ the canonical map. The elements are of the form

$$\begin{aligned} &j^*(x) \otimes x' - x \otimes j_*(x') \quad \text{for } x \in \mathscr{M}(k_2) \text{ and } x' \in \mathscr{M}'(k_1), \text{ and} \\ &x \otimes j^*(x') - j_*(x) \otimes x' \quad \text{for } x \in \mathscr{M}(k_1) \text{ and } x' \in \mathscr{M}'(k_2). \end{aligned}$$

(Rec) Let F be a function field in one variable over k and $f \in F^\times, g \in \mathscr{M}(F), g' \in \mathscr{M}'(F)$. The required elements are of the form

$$\sum_v \partial_v (f \otimes g \otimes g'),$$

where the sum is taken over all places v of F/k, and

$$\partial_v : F^\times \otimes_{\mathbb{Z}} \mathscr{M}(F) \otimes_{\mathbb{Z}} \mathscr{M}'(F) \to \mathscr{M}(k(v)) \otimes_{\mathbb{Z}} \mathscr{M}'(k(v))$$

is the *local symbol*. This is given by using the connecting map in the localization sequence of higher Chow groups[2].

Using this, it is known the following expressions:

- $K(k; \mathscr{C}\mathrm{H}^1(k,1), \mathscr{C}\mathrm{H}^1(k,1)) \simeq K(k; \mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$, where the right side is Somekawa's K-group associated to the multiplicative groups $\mathbb{G}_{\mathrm{m}}$ [14], and
- $K(k; \mathscr{C}\mathrm{H}^{d+a}(X,a), \mathscr{C}\mathrm{H}^{d'+a'}(X',a')) \simeq \mathrm{CH}^{d+d'+a+a'}(X \times X', a+a')$ (*cf.* Thm. 3.3).

In our previous work [4], we introduced an additive variant of Somekawa's K-group of the form

$$K(k; \mathbb{W}_m, \mathbb{G}_{\mathrm{m}}),$$

where $\mathbb{W}_m$ is the Witt group scheme of lenth $m \in \mathbb{Z}_{>0}$. We *expect* to define the group of the form

$$K(k; \mathscr{T}\mathrm{CH}^{d+a}(X,a;m), \mathscr{C}\mathrm{H}^{d'+a'}(X',a'))$$

which gives

- $K(k; \mathscr{T}\mathrm{CH}^1(k,1;m), \mathscr{C}\mathrm{H}^1(k,1)) \simeq K(k; \mathbb{W}_m, \mathbb{G}_{\mathrm{m}})$, and
- $K(k; \mathscr{T}\mathrm{CH}^{d+a}(X,a;m), \mathscr{C}\mathrm{H}^{d'+a'}(X',a') \simeq \mathrm{TCH}^{d+d'+a+a'}(X \times X', a+a';m)$.

However, the localization property to define the condition corresponding to **(Rec)** above is not known on the additive higher Chow groups (due to lack of homotopy invariance). Instead of Somekawa type K-group, we consider the Mackey product

$$\left(\mathscr{T}\mathrm{CH}^{d+a}(X,a;m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X',a')\right)(k)$$

which is defined using the "projection formula" only as follows:

$$\left(\bigoplus_{k'/k} \mathscr{T}\mathrm{CH}^{d+a}(X,a;m)(k') \otimes_{\mathbb{Z}} \mathscr{C}\mathrm{H}^{d'+a'}(X',a')(k')\right) \Big/ \textbf{(PF)},$$

where **(PF)** is the subgroup defined similarly as in (1) (for the precise definition, see Def. 3.2). In this note, we present the following surjective homomorphism on 0-cycles (Thm. 3.4):

$$\left(\mathscr{T}\mathrm{CH}^{d+a}(X,a;m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X',a')\right)(k) \twoheadrightarrow \mathrm{TCH}^{d+d'+a+a'}(X \times X', a+a';m).$$

It is easy to show that the Mackey product on the left hand side becomes trivial when k has positive characteristic so that we obtain the main theorem noted above (Thm. 3.5).

[2] Although the precise definition of **(Rec)** is not given in [12], but we do not mention about the local symbol more on this. About this topic, see [6] and [1].

Notation In this note, a **variety** over a field k we mean an integral and separated scheme of finite type over $\mathrm{Spec}(k)$. For a field k, we use

- $\mathrm{char}(k)$: the characteristic of k.

For varieties X and Y over a field k, we denote by

- $\dim(X)$: the dimension of X,
- $X_{k'} := X \otimes_k k' := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$: the base change of X for an extension field k'/k, and
- $X \times Y := X \times_{\mathrm{Spec}(k)} Y$.

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2 (Additive) higher Chow groups of schemes In this section, we recall the definitions of (additive) higher Chow groups following [2],[10], and [3]. Throughout this section, we use the following notation:

- k : a field as a base field,
- $\square^q := (\mathbb{P}^1 \smallsetminus \{ 1 \})^q$ and we use the coordinates $(y_1, \dots, y_q)$ on $\square^q$, and
- X : a scheme of finite type over k.

Higher Chow groups The subscheme of $\square^q$ defined by equations $y_{i_1} = \varepsilon_1, \dots, y_{i_s} = \varepsilon_s$ for $\varepsilon_j \in \{ 0, \infty \}$ is called a **face** of $\square^q$. For $\varepsilon \in \{ 0, \infty \}$ and $i = 1, \dots, q-1$, let $\iota_{q,i,\varepsilon} : \square^{q-1} \to \square^q$ be the inclusion defined by $(y_1, \dots, y_{q-1}) \mapsto (y_1, \dots, y_{i-1}, \varepsilon, y_i, \dots, y_{q-1})$.

Definition 2.1. Let $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geqslant 0}$.

(i) We denote by $\underline{\mathrm{z}}_p(X, q)$ the free abelian group on integral closed subschemes Z of $X \times \square^q$ of dimension $p + q$ that intersect all faces of $\square^q$ properly.

(ii) For each $1 \leqslant i \leqslant q$ and $\varepsilon \in \{0, \infty\}$, let $\partial_i^\varepsilon := \mathrm{Id}_X \times \iota_{q,i,\varepsilon}^*$, where $\mathrm{Id}_X : X \to X$ is the identity morphism. The abelian groups $\underline{\mathrm{z}}_p(X, \bullet) = \{ \underline{\mathrm{z}}_p(X, q) \}_{q \geqslant 0}$ form a complex with boundary map

$$\sum_{i=1}^{q} (-1)^i (\partial_i^\infty - \partial_i^0) : \underline{\mathrm{z}}_p(X, q) \to \underline{\mathrm{z}}_p(X, q-1).$$

The **higher Chow complex** $\mathrm{z}_p(X, \bullet)$ is $\underline{\mathrm{z}}_p(X, \bullet)$ modulo the complex consists of the degenerate cycles, that is, the cycles on $X \times \square^q$ pulled back from cycles on $X \times \square^{q-1}$ by a projection $X \times \square^q \to X \times \square^{q-1}$ of the form $(x, y_1, \dots, y_q) \mapsto (x, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_q)$ for some index j. The homology group

$$\mathrm{CH}_p(X, q) := H_q(\mathrm{z}_p(X, \bullet))$$

is called the **higher Chow group** of X.

If the scheme X is equidimensional of $d = \dim(X)$ over k, we write

$$\mathrm{z}^p(X, q) := \mathrm{z}_{d-p}(X, q), \quad \text{and} \quad \mathrm{CH}^p(X, q) := H_q(\mathrm{z}^p(X, \bullet)) = \mathrm{CH}_{d-p}(X, q).$$

The higher Chow groups have functorial properties induced from the proper push-forward, and the flat pull-back of cycles. In particular, for a finite field extension k'/k, the projection $j = j_{k'/k} : X_{k'} = X \otimes_k k' \to X$ induces

$$N_{k'/k} := j_* : \mathrm{CH}^p(X_{k'}, q) \to \mathrm{CH}^p(X, q). \tag{2}$$

These functorial properties enable us to give the structure of $\mathrm{CH}^p(X,q)$ a Mackey functor as follows:

$$\mathscr{C}\mathrm{H}^p(X,q) : k' \mapsto \mathrm{CH}^p(X_{k'},q), \tag{3}$$

where $X_{k'} = X \otimes_k k'$, for a finite field extension k' of k. For two schemes X, Y of finite type over k, one can construct

$$\boxtimes : \mathrm{z}_p(X,\bullet) \otimes_{\mathbb{Z}} \mathrm{z}_r(Y,\bullet) \to \mathrm{z}_{p+r}(X \times Y,\bullet).$$

On integral cycles, it is defined by $Z \boxtimes W := \tau_*(Z \times W)$, where $\tau : X \times \square^p \times Y \times \square^r \to X \times Y \times \square^{p+r}$ is the exchange of factors (*cf.* [8], Sect. 1.3). On homology groups, $\boxtimes$ induces the **external product**

$$\boxtimes : \mathrm{CH}_p(X,q) \otimes_{\mathbb{Z}} \mathrm{CH}_r(Y,s) \to \mathrm{CH}_{p+r}(X \times Y, q+s). \tag{4}$$

If X is smooth over k, then pulling back of $\boxtimes$ along the diagonal $\Delta : X \to X \times X$, we have the **intersection product**

$$\cap : \mathrm{CH}^p(X,q) \otimes_{\mathbb{Z}} \mathrm{CH}^r(X,s) \to \mathrm{CH}^{p+r}(X, q+s). \tag{5}$$

We list some relevant calculations of higher Chow groups: There is a natural isomorphism $\mathrm{CH}^p(X,0) \simeq \mathrm{CH}^p(X)$, where the latter is the ordinary Chow group. In the case of $p = q$, we have the following theorem:

Theorem 2.2 ([11], [15])**.** *There is a canonical isomorphism*

$$\phi : \mathrm{CH}^q(k,q) \xrightarrow{\simeq} K_q^M(k),$$

where the latter group is the Milnor K-group of the field k.

In particular, in the case of $q = 1$, we have

$$\mathrm{CH}^1(k,1) \simeq k^\times = \mathbb{G}_{\mathrm{m}}(k),$$

where $\mathbb{G}_{\mathrm{m}}$ is the multiplicative group scheme. This extends to an isomorphism

$$\mathscr{C}\mathrm{H}^1(k,1) \simeq \mathbb{G}_{\mathrm{m}} \tag{6}$$

of Mackey functors. Here, we refer the construction of the map ϕ in Thm. 2.2. By the very definition, $\mathrm{CH}^q(k,q)$ is generated by classes $[P]$ represented by a closed point $P : \operatorname{Spec} k(P) \to \square^q$. It is determined by the maps $y_i(P) : \operatorname{Spec} k(P) \to \square^q \xrightarrow{y_i} \square$ for $i = 1, \dots, q$ and they give $y_i(P) \in k(P)^\times$ for each i. The map ϕ is defined by

$$\phi([P]) := N_{k(P)/k}\{\, y_1(P), \dots, y_q(P) \,\},$$

where $N_{k(P)/k} : K_q^M(k(P)) \to K_q^M(k)$ is the norm map of the Milnor K-groups and $\{\, y_1(P), \dots, y_q(P) \,\}$ is the element in $K_q^M(k(P))$ represented by $y_1(P) \otimes \cdots \otimes y_q(P) \in k(P)^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} k(P)^\times$.

Additive higher Chow groups The additive higher Chow groups are defined similarly to the higher Chow groups using B_q below instead of cubes $\square^q$. Let

- $B_q := \mathbb{A}^1 \times \square^{q-1}$, and
- $\overline{B}_q := \mathbb{A}^1 \times (\mathbb{P}^1)^{q-1} \supset B_q$. We use the coordinates $(t, y_1, \ldots, y_{q-1})$ on $\overline{B}_q$.

The subscheme of B_q defined by equations $y_{i_1} = \varepsilon_1, \ldots, y_{i_s} = \varepsilon_s$ for $\varepsilon_j \in \{0, \infty\}$ is called a **face** of B_q. For $\varepsilon \in \{0, \infty\}$ and $i = 1, \ldots, q-1$, let $\iota_{q,i,\varepsilon} : B_{q-1} \to B_q$ be the inclusion defined by $(t, y_1, \ldots, y_{q-2}) \mapsto (t, y_1, \ldots, y_{i-1}, \varepsilon, y_i, \ldots, y_{q-2})$. On $\overline{B}_q$, let $F^1_{q,i}$ be the Cartier divisor defined by $y_i = 1$ and $F_{q,0}$ the Cartier divisor defined by $t = 0$.

Definition 2.3. Let $p \in \mathbb{Z}$, and $q, m \in \mathbb{Z}_{>0}$.

(i) Define $\underline{\mathrm{Tz}}_p(X, 1; m)$ to be the free abelian group on integral closed subschemes Z of $X \times \mathbb{A}^1$ of dimension p satisfying $Z \cap (X \times \{0\}) = \varnothing$ and the modulus condition defined below. For the integer $q > 1$, $\underline{\mathrm{Tz}}_p(X, q; m)$ is the free abelian group on integral closed subschemes Z of $X \times B_q$ of dimension $p + q - 1$ satisfying the following two conditions:

(Good position) For each face F of B_q, Z intersects $X \times F$ properly.

(Modulus condition) Let $\pi : \overline{Z}^N \to Z \subset X \times \overline{B}_q$ be the normalization of the closure $\overline{Z}$ of Z in $X \times \overline{B}_q$. Then

$$(m+1)\pi^*(X \times F_{q,0}) \leqslant \pi^*(X \times F^1_q)$$

as Weil divisors, where $F^1_q := \sum_{i=1}^{q-1} F_{q,i}$.

(Here, we adapt the modulus condition M_{sum} in Def. 2.1 in [9]. For the other similar conditions on modulus and their relations, see [9], Sect. 2).

(ii) For each $1 \leqslant i \leqslant q-1$ and $\varepsilon \in \{0, \infty\}$, let $\partial^\varepsilon_i := \mathrm{Id}_X \times \iota^*_{q,i,\varepsilon}$. The boundary map of $\underline{\mathrm{Tz}}_p(X, \bullet; m)$ is given by

$$\sum_{i=1}^{q-1} (-1)^i (\partial^\infty_i - \partial^0_i) : \underline{\mathrm{Tz}}_p(X, q; m) \to \underline{\mathrm{Tz}}_p(X, q-1; m).$$

The **additive cycle complex** $\mathrm{Tz}_p(X, \bullet; m)$ is the nondegenerate complex associated to $\underline{\mathrm{Tz}}_p(X, \bullet; m)$. Its homology group

$$\mathrm{TCH}_p(X, q; m) := H_q(\mathrm{Tz}_p(X, \bullet; m))$$

is called the **additive higher Chow group** of X with modulus m.

If the scheme X is equidimensional of $d = \dim(X)$ over k, we write

$$\mathrm{Tz}^p(X, q; m) := \mathrm{Tz}_{d+1-p}(X, q; m), \quad \text{and} \quad \mathrm{TCH}^p(X, q; m) := H_q(\mathrm{Tz}^p(X, \bullet; m)).$$

The additive higher Chow groups have also functorial properties as projective push-forward, and the flat pull-back. For a finite field extension k'/k with the projection $j = j_{k'/k} : X_{k'} := X \otimes_k k' \to X$, we have

$$\mathrm{Tr}_{k'/k} := j_* : \mathrm{TCH}^p(X_{k'}, q; m) \to \mathrm{TCH}^p(X, q; m). \tag{7}$$

The assignment

$$\mathscr{T}\mathrm{CH}^p(X, q; m) : k' \mapsto \mathrm{TCH}^p(X \otimes_k k', q; m) \tag{8}$$

gives a structure of Mackey functors.

For two equidimensional schemes X, Y of finite type over k, one can construct the product

$$\boxtimes : \mathrm{z}_p(X, \bullet) \otimes_{\mathbb{Z}} \mathrm{Tz}_r(Y, \bullet; m) \to \mathrm{Tz}_{p+r}(X \times Y, \bullet; m).$$

On integral cycles it is defined by $Z \boxtimes W := \tau_*(Z \times W)$ where $\tau : X \times \square^p \times Y \times B_r \to X \times Y \times B_{p+r}$ is the exchange of factors (*cf.* [8], Sect. 4.1). On homology groups, $\boxtimes$ induces the **external product**

$$\boxtimes : \mathrm{CH}_p(X, q) \otimes_{\mathbb{Z}} \mathrm{TCH}_r(Y, s; m) \to \mathrm{TCH}_{p+r}(X \times Y, q + s; m).$$

If we assume that X is a *smooth and projective* variety over k, we obtain the **intersection product**

$$\cap : \mathrm{CH}_p(X, q) \otimes_{\mathbb{Z}} \mathrm{TCH}_r(X, s; m) \to \mathrm{TCH}_{p+r}(X, q + s; m). \tag{9}$$

Essentially, this product is defined by the pullback of $\boxtimes$ along the diagonal map $\Delta : X \to X \times X$ (see [8], Thm. 4.10 for the precise construction). The intersection product is natural with flat pull-back, and satisfying the projection formula:

$$f_*(f^*(x) \cap y) = x \cap f_*(y) \tag{10}$$

for a morphism $f : X \to Y$ of smooth projective varieties over k. If f is flat, we also have

$$f_*(x \cap f^*(y)) = f_*(x) \cap y. \tag{11}$$

Putting $\mathrm{TCH}^p(k, q; m) := \mathrm{TCH}^p(\mathrm{Spec}(k), q; m)$ we also have the following theorem:

Theorem 2.4 ([13], Thm. 3.20)**.** *For a field k with characteristic $\neq 2$, there is a canonical isomorphism*

$$\phi : \mathrm{TCH}^q(k, q; m) \xrightarrow{\simeq} \mathbb{W}_m\Omega_k^{q-1},$$

where the latter group is the generalized de Rham-Witt group.

In particular, in the case of $q = 1$, we have

$$\mathrm{TCH}^1(k, 1; m) \simeq \mathbb{W}_m(k), \quad \text{and hence} \quad \mathscr{T}\mathrm{CH}^1(k, 1; m) \simeq \mathbb{W}_m, \tag{12}$$

where $\mathbb{W}_m$ is the Witt group scheme. Recall the construction of the map ϕ in Thm. 2.4. The additive higher Chow group $\mathrm{TCH}^q(k, q; m)$ is generated by classes $[P]$ represented by a closed point $P : \mathrm{Spec}\, k(P) \to B_q$. It is determined by the maps $t(P) : \mathrm{Spec}\, k(P) \to B_q \xrightarrow{t} \mathbb{A}^1$ and $y_i(P) : \mathrm{Spec}\, k(P) \to \square^q \xrightarrow{y_i} \square$. They give $t(P) \in k(P), y_i(P) \in k(P)^\times$. The map ϕ is defined by

$$\phi([P]) := \mathrm{Tr}_{k(P)/k}\left([t(P)^{-1}]\, \mathrm{dlog}[y_1(P)] \cdots \mathrm{dlog}[y_{q-1}(P)]\right),$$

where $[-]$ is the Teichmüller lift. Note that the modulus condition assures $t(P) \neq 0$.

3 Mackey product and additive higher Chow groups

In this section, we assume

- k : a *perfect* field.

Mackey product We recall the definition of the Mackey functor.

Definition 3.1 (*cf.* [12], Sect. 3)**.** A **Mackey functor** $\mathscr{A}$ (over k) is a contravariant functor from the category of étale schemes over k to the category of abelian groups equipped with a covariant structure for finite morphisms such that $\mathscr{A}(X_1 \sqcup X_2) = \mathscr{A}(X_1) \oplus \mathscr{A}(X_2)$ and if

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a Cartesian diagram, then the induced diagram

$$\begin{array}{ccc} \mathscr{A}(X') & \xrightarrow{g'_*} & \mathscr{A}(X) \\ {\scriptstyle f'^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathscr{A}(Y') & \xrightarrow{g_*} & \mathscr{A}(Y) \end{array}$$

commutes.

For a Mackey functor $\mathscr{A}$, we denote by $\mathscr{A}(k')$ its value $\mathscr{A}(\mathrm{Spec}(k'))$ for a field extension k' of k.

Definition 3.2 (*cf.* [7])**.** For Mackey functors $\mathscr{A}_1, \dots, \mathscr{A}_q$, their **Mackey product** $\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q$ is defined as follows: For any finite field extension k'/k,

$$\left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k') := \left(\bigoplus_{k''/k' \colon \text{finite}} \mathscr{A}_1(k'') \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathscr{A}_q(k'')\right) \Big/ R, \tag{13}$$

where R is the subgroup generated by elements of the following form:

(PF) For finite field extensions $k' \subset k'_1 \subset k'_2$, and if $x_{i_0} \in \mathscr{A}_{i_0}(k'_2)$ and $x_i \in \mathscr{A}_i(k'_1)$ for all $i \neq i_0$, then

$$j^*(x_1) \otimes \cdots \otimes x_{i_0} \otimes \cdots \otimes j^*(x_q) - x_1 \otimes \cdots \otimes j_*(x_{i_0}) \otimes \cdots \otimes x_q,$$

where $j = j_{k'_2/k'_1} : \mathrm{Spec}(k'_2) \to \mathrm{Spec}(k'_1)$ is the canonical map.

For the Mackey product $\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q$, we write $\{x_1, \dots, x_q\}_{k'/k}$ for the image of $x_1 \otimes \cdots \otimes x_q \in \mathscr{A}_1(k') \otimes \cdots \otimes \mathscr{A}_q(k')$ in the product $\left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k)$. For any field extension k'/k, the canonical map $j = j_{k'/k} : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ induces the pull-back

$$\mathrm{Res}_{k'/k} := j^* : \left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k) \longrightarrow \left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k').$$

If the extension k'/k is finite, then the push-forward

$$N_{k'/k} := j_* : \left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k') \longrightarrow \left(\mathscr{A}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{A}_q\right)(k) \tag{14}$$

is given by $N_{k'/k}(\{x_1, \dots, x_q\}_{k''/k'}) = \{x_1, \dots, x_q\}_{k''/k}$.

Main theorem In the rest of this section, we use

- X, X': smooth projective varieties over k, and
- $d = \dim(X), d' = \dim(X')$.

Theorem 3.3 ([12], Thm. 2.2, (2.4.4)). *For $a, a' \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\psi : K(k; \mathscr{C}\mathrm{H}^{d+a}(X, a), \mathscr{C}\mathrm{H}^{d'+a'}(X', a')) \xrightarrow{\simeq} \mathrm{CH}^{d+d'+a+a'}(X \times X', a + a').$$

Recall the definition of ψ. We denote by $\{ x, x' \}_{k'/k}$ the image of $x \otimes x' \in \mathrm{CH}^{d+a}(X_{k'}, a) \otimes_{\mathbb{Z}} \mathrm{CH}^{d'+a'}(X_{k'}, a)$ in $K(k; \mathscr{C}\mathrm{H}^{d+a}(X_{k'}, a), \mathscr{C}\mathrm{H}^{d'+a'}(X_{k'}, a))$ (*cf.* (1)). Define

$$\psi(\{ x, x' \}_{k'/k}) := N_{k'/k}(p^*(x) \cap (p')^*(x')),$$

where $\cap$ is the intersection product (5), $N_{k'/k} = j_*$ is the push-forward along $X_{k'} \to X$ (*cf.* (2)), and $p : (X \times X')_{k'} \to X_{k'}$ and $p' : (X \times X')_{k'} \to (X')_{k'}$ are the projections.

As we explained in Introduction, for $a, m \in \mathbb{Z}_{>0}, a' \in \mathbb{Z}_{\geqslant 0}$, we consider the Mackey product (*cf.* Def. 3.2)

$$\left(\mathscr{T}\mathrm{CH}^{d+a}(X, a; m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X', a'; m) \right)(k).$$

Define a homomorphism

$$\psi : \left(\mathscr{T}\mathrm{CH}^{d+a}(X, a; m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X', a'; m) \right)(k) \to \mathrm{TCH}^{d+d'+a+a'}(X \times X', a + a'; m)$$

by the intersection product (9) (*cf.* [12], Proof of Thm. 2.2) as

$$\psi(\{ x, x' \}_{k'/k}) := \mathrm{Tr}_{k'/k}((p')^*(x') \cap p^*(x)),$$

for any finite extension field k'/k, where $\mathrm{Tr}_{k'/k} = j_*$ is the push-forward along $j : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ and $p : (X \times X')_{k'} \to X_{k'}$ and $p' : (X \times X')_{k'} \to (X')_{k'}$ are the projections. From the projection formula of the intersection product ((10) and (11)), the map ψ is well-defined.

Theorem 3.4. *For $a, m \in \mathbb{Z}_{>0}, a' \in \mathbb{Z}_{\geqslant 0}$, the map*

$$\psi : \left(\mathscr{T}\mathrm{CH}^{d+a}(X, a; m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X', a') \right)(k) \to \mathrm{TCH}^{d+d'+a+a'}(X \times X', a + a'; m)$$

is surjective.

Proof. Put

- $\mathscr{X} := X \times X'$,
- $\alpha = a + a'$, and
- $\delta = d + d'$.

By the very definition (Def. 2.3), the group $\mathrm{TCH}^{\delta+\alpha}(\mathscr{X}, \alpha; m)$ consists of 0-cycles on $\mathscr{X} \times B_\alpha$. Take a closed point $P : \mathrm{Spec}(k(P)) \to \mathscr{X} \times B_\alpha$ as a generator and it is enough to show the cycle $[P]$ associated to P is in the image of ψ. By the definition of ψ, the trace map on the additive Chow groups and the norm map on the Mackey products are compatible as in the following commutative diagram:

$$\begin{array}{ccc}
\left(\mathscr{T}\mathrm{CH}^{d+a}(X_{k(P)}, a; m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}({X'}_{k(P)}, a') \right)(k(P)) & \xrightarrow{\psi} & \mathrm{TCH}^{\delta+\alpha}(\mathscr{X}_{k(P)}, \alpha; m) \\
\downarrow N_{k(P)/k} & & \downarrow \mathrm{Tr}_{k(P)/k} \\
\left(\mathscr{T}\mathrm{CH}^{d+a}(X, a; m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d'+a'}(X', a') \right)(k) & \xrightarrow{\psi} & \mathrm{TCH}^{\delta+\alpha}(\mathscr{X}, \alpha; m).
\end{array}$$

Thus, to show the assertion that $[P]$ is in the image of ψ we may assume that P is a k-rational point, that is, $k(P) = k$. The point P is determined by the maps $P_X : \mathrm{Spec}(k) \to X \times B_a$ and $P_{X'} : \mathrm{Spec}(k) \to X' \times \square^{a'}$ satisfying $\tau_*(P_X \times P_{X'}) = P$, where $\tau : (X \times B_a) \times (X' \times \square^{a'}) \to \mathscr{X} \times B_\alpha$ is the exchange of factors. This gives cycles $[P_X]$ on $\mathrm{TCH}^{d+a}(X, a; m)$ and $[P_{X'}]$ on $\mathrm{CH}^{d'+a'}(X', a')$. Therefore, denoting by $p : \mathscr{X} \to X$ and $p' : \mathscr{X} \to X'$ the projection maps, we have

$$\psi(\{\,[P_X], [P_{X'}]\,\}_{k/k}) = (p')^*([P_{X'}]) \cap p^*([P_X]) = [P],$$

where the last equality follows from the very definition of the intersection product. The assertion follows from this. $\square$

Theorem 3.5. *Let X be a projective smooth variety of dimension d over a perfect field k with* $\mathrm{char}(k) > 0$. *Then,*

$$\mathrm{TCH}^{d+q}(X, q; m) = 0, \quad \text{for } q \geqslant 2.$$

Proof. There are isomorphisms $\mathscr{C}\mathrm{H}^1(k, 1) \simeq \mathbb{G}_{\mathrm{m}}$ (from (6)) and $\mathscr{T}\mathrm{CH}^1(k, 1; m) \simeq \mathbb{W}_m$ (from (12)) as Mackey functors. By Theorem 3.3 and Theorem 3.4, we have surjective homomorphisms

$$\Big(\mathbb{W}_m \overset{M}{\otimes} \overbrace{\mathbb{G}_{\mathrm{m}} \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathbb{G}_{\mathrm{m}}}^{q-1} \overset{M}{\otimes} \mathscr{C}\mathrm{H}^d(X)\Big)(k) \twoheadrightarrow \Big(\mathscr{T}\mathrm{CH}^1(k, 1; m) \overset{M}{\otimes} \mathscr{C}\mathrm{H}^{d+q-1}(X, q-1)\Big)(k)$$
$$\overset{\psi}{\twoheadrightarrow} \mathrm{TCH}^{d+q}(X, q; m)$$

for $q \geqslant 2$. The far left vanishes from the lemma below and the assertion follows. $\square$

Lemma 3.6 ([5], Lem. 2.2)**.** *Let G be a unipotent smooth and commutative algebraic group over a field F and A a semi-abelian variety over F. If F is a perfect field of* $\mathrm{char}(F) > 0$, *we have $G \overset{M}{\otimes} A = 0$.*

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Department of Basic Sciences
Graduate School of Engineering
Kyushu Institute of Technology
1-1 Sensui-cho, Tobata-ku, Kitakyushu-shi,
Fukuoka 804-8550 JAPAN
email: `hira@mns.kyutech.ac.jp`