OPEN MAPPING THEOREMS WITH FINITE FIBRES FOR C-SPACES

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Abstract

In this paper we study theorems for C-spaces and finite C-spaces on dimension-raising open mappings and dimension-lowering open mappings with finite fibres.

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1 Introduction

In this paper we assume that all spaces are normal and all mappings are continuous.

A space X is *A-weakly infinite-dimensional* or *Alexandroff weakly infinite-dimensional* if for every collection $\{(A_i, B_i) : i < \omega\}$ of pairs of disjoint closed subsets of X there exists a collection $\{L_i : i < \omega\}$ of closed subsets of X such that L_i is a partition in X between A_i and B_i for every $i < \omega$, and $\bigcap_{i<\omega} L_i = \emptyset$.

A space X is a *C-space* [1] if for every countable collection $\{\mathcal{G}_i : i < \omega\}$ of open covers of X there exists a countable collection $\{\mathcal{H}_i : i < \omega\}$ of collections of pairwise disjoint open subsets of X such that $\mathcal{H}_i$ is a refinement of $\mathcal{G}_i$ for every $i < \omega$ and $\bigcup_{i<\omega} \bigcup\{H : H \in \mathcal{H}_i\} = X$.

It is easily seen that every C-space is A-weakly infinite-dimensional. However, it is still unknown whether every compact A-weakly infinite-dimensional metrizable space is a C-space.

A space X is S-*weakly infinite-dimensional* or *Smirnov weakly infinite-dimensional* if for every collection $\{(A_i, B_i) : i < \omega\}$ of pairs of disjoint closed subsets of X there exists a collection $\{L_i : i < \omega\}$ of closed subsets of X such that L_i is a partition in X between A_i and B_i for every $i < \omega$, and $\bigcap_{i \leq n} L_i = \emptyset$ for some $n < \omega$. It directly follows from the definition that every S-weakly infinite dimensional space is A-weakly infinite dimensional, and every compact A-weakly infinite dimensional space is S-weakly infinite dimensional.

A space X is a *finite C-space* [2] if for every collection $\{\mathcal{G}_i : i < \omega\}$ of finite open covers of X there exists a collection $\{\mathcal{H}_i : i < \omega\}$ of collections of pairwise disjoint open subsets of X such that $\mathcal{H}_i$ is a refinement of $\mathcal{G}_i$ for every $i < \omega$ and $\bigcup_{i \leq n} \bigcup\{H : H \in \mathcal{H}_i\} = X$ for some $n < \omega$. It is well-known [2] that every finite C-space is S-weakly infinite-dimensional. There exists a C-space which is not a finite C-space (see [1, Example 2.15]). However, every compact C-space is a finite C-space.

For paracompact spaces, Gutev and Valov [5] proved the countable sum theorem for C-spaces. For countably paracompact and collectionwise normal spaces, the author proved the countable sum theorem for C-spaces (cf. [6, Corollary 3.2]). Addis and Gresham [1] proved that every finite-dimensional, paracompact space is a C-space. By the same proof, we can show that every finite-dimensional space is a finite C-spaces. The following two Lemmas will play a important role in the proof of our main theorems.

Lemma A *If there exists a closed subset K of a countably paracompact collectionwise normal space X satisfying the following conditions (1) and (2), then X is a C-space.*

(1) K is a C-space,

(2) for every closed subset F of X with $F \cap K = \emptyset$, F is a C-space.

Proof. Let $\{\mathcal{G}_i : i < \omega\}$, where $\mathcal{G}_i = \{G_\lambda : \lambda \in \Lambda_i\}$, be a collection of open covers of X. Since K is a countably paracompact C-space, by [6, Lemma 2.1], there exists a collection $\{\mathcal{U}_{2i} : i < \omega\}$, where $\mathcal{U}_{2i} = \{U_\lambda : \lambda \in \Lambda_{2i}\}$, of discrete collections of open subsets of K such that $U_\lambda \subset G_\lambda \cap K$ and $\bigcup_{i<\omega} \bigcup\{U_\lambda : \lambda \in \Lambda_{2i}\} = K$. Since X is collectionwise normal, by [6, Lemma 2.2], there exists a collection $\{\mathcal{H}_{2i} : i < \omega\}$, where $\mathcal{H}_{2i} = \{H_\lambda : \lambda \in \Lambda_{2i}\}$, of discrete collections of open subsets of X such that $H_\lambda \cap K = U_\lambda$ and $H_\lambda \subset G_\lambda$. Let us set $F = X - \bigcup_{i<\omega} \bigcup\{H : H \in \mathcal{H}_{2i}\}$. Similarly there exsits a collection $\{\mathcal{H}_{2i+1} : i < \omega\}$ of discrete collections of open subsets of X such that $\mathcal{H}_{2i+1}$ is a refinement of $\mathcal{G}_{2i+1}$ for every $i < \omega$ and $\bigcup_{i<\omega} \bigcup\{H : H \in \mathcal{H}_{2i+1}\} \supset F$. We get the required collection $\{\mathcal{H}_i : i < \omega\}$.

Lemma B *If there exists a closed subset K of a space X satisfying the following conditions (1) and (2), then X is a finite C-space.*

(1) K is a finite C-space,

(2) for every closed subset F of X with $F \cap K = \emptyset$, F is a finite C-space.

Proof. Let $\{\mathcal{G}_i : i < \omega\}$, where $\mathcal{G}_i = \{G_\lambda : \lambda \in \Lambda_i\}$, be a collection of finite open covers of X. Since K is a finite C-space, there exists a collection $\{\mathcal{U}_{2i} : i < \omega\}$, where $\mathcal{U}_{2i} = \{U_\lambda : \lambda \in \Lambda_{2i}\}$, of finite collections of pairwise disjoint open subsets of K such that $U_\lambda \subset G_\lambda \cap K$ and $\bigcup_{i=1}^{n} \bigcup\{U_\lambda : \lambda \in \Lambda_{2i}\} = K$ for some $n < \omega$. Since K is normal, there exists $\{\mathcal{F}_{2i} : i \leq n\}$, where $\mathcal{F}_{2i} = \{F_\lambda : \lambda \in \Lambda_{2i}\}$, of collections of closed subsets of K such that $F_\lambda \subset U_\lambda$ and $\bigcup_{i=1}^{n} \bigcup\{F_\lambda : \lambda \in \Lambda_{2i}\} = K$. There exists a collection $\{\mathcal{H}_{2i} : i < \omega\}$, where $\mathcal{H}_{2i} = \{H_\lambda : \lambda \in \Lambda_{2i}\}$, of finite collections of pairwise disjoint open subsets of X for every $i < \omega$ such that $F_\lambda \subset H_\lambda \subset G_\lambda$ and $\bigcup_{i=1}^{n} \bigcup\{H_\lambda : \lambda \in \Lambda_{2i}\} \supset K$. For every $i > n$ we let $\mathcal{H}_{2i} = \{\emptyset\}$. Let us set $F = X - \bigcup_{i\leq n} \bigcup\{H : H \in \mathcal{H}_{2i}\}$. For a space F repeating above procedure we obtain the required collection $\{\mathcal{H}_i : i < \omega\}$.

2 Dimension-raising mappings

Polkowski [8] proved the following theorem.

Theorem [8]. *If $f : X \longrightarrow Y$ is an open mapping of an A-weakly infinite-dimensional space X onto a countably paracompact space Y such that $|f^{-1}(y)| < \omega$ for every $y \in Y$, then Y is A-weakly infinite-dimensional.*

We shall prove the following theorem. This is an analogy of the above Polkowski's theorem.

2.1. Theorem *If $f : X \longrightarrow Y$ is an open mapping of a C-space X onto a countably paracompact and collectionwise normal Y such that $|f^{-1}(y)| < \omega$ for every $y \in Y$, then Y is a C-space.*

To prove Theorem 2.1 we need the following theorem and lemma.

2.2. Theorem(cf.[4, Lemma 6.7]) *If $f : X \longrightarrow Y$ is a closed mapping of a countably paracompact C-space X onto a space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then Y is a C-space.*

2.3. Lemma([3, Lemma 6.3.12]) *If all fibres of an open mapping $f : X \longrightarrow Y$ defined on a space X are finite and have the same cardinality, then f is closed.*

2.4 Proof of theorem 2.1. Let $K_j = \{y \in Y : |f^{-1}(y)| = j\}$ for every $j \in \mathbb{N}$. It is easy to see that the union $\bigcup_{j \leq i} K_j$ is closed in Y for every $i \in \mathbb{N}$. Inductively, we show that the union $\bigcup_{j \leq i} K_j$ is a C-space for every $i \in \mathbb{N}$. To this end, it suffices to show that every closed subspace Z of Y contained in K_i is a C-space, cf. Lemma A. By Lemma 2.3, the restriction $f|_{f^{-1}(Z)} : f^{-1}(Z) \longrightarrow Z$ is perfect. As the inverse image of a countably paracompact space under a perfect mapping is countably paracompact, then $f^{-1}(Z)$ is countably paracompact. By Theorem 2.2, Z is a C-space. Thus the union $\bigcup_{j \leq i} K_j$ is a closed C-space for every $i \in \mathbb{N}$. By countable sum theorem, Y is a C-space.

The following theorem is a counterpart for finite C-spaces of Polkowski's result.

2.5. Theorem *If $f : X \longrightarrow Y$ is an open mapping of a weakly paracompact finite C-space X onto a space Y such that $|f^{-1}(y)| < \omega$ for every $y \in Y$, then Y is a finite C-space.*

To prove Theorem 2.5 we need the following theorem and lemma.

2.6. Theorem([4, Theorem 6.4]) *If $f : X \longrightarrow Y$ is a mapping of a compact C-space X onto a space Y such that $|f^{-1}(y)| < \mathfrak{c}$ for every $y \in Y$, then Y is a C-space.*

For each space X and $n < \omega$ we let

$$G_n(X) = \bigcup\{U \subset X : U \text{ is open and } \dim \mathrm{Cl}\, \mathrm{U} \leq \mathrm{n}\}$$

and

$$S(X) = X - \bigcup_{n<\omega} G_n(X).$$

Sklyarenko ([9, Theorem 3]) proved the following lemma in the case when X is S-weakly infinite dimensional.

2.7. Lemma *A weakly paracompact space X is a finite C-space if and only if $S(X)$ is a compact finite C-space and every closed subspace $F \subset X$ disjoint from $S(X)$ is finite dimensional.*

Proof. Assume that the space X is a finite C-space. We shall show that $S(X)$ is compact. Suppose $S(X)$ is not compact. Since $S(X)$ is weakly paracompact, $S(X)$ is not pseudocompact. Thus there exists a countable discrete closed subspace F of $S(X)$. Let us set $F = \{x_i : i < \omega\}$. We can take a discrete collection $\{U_i : i < \omega\}$ of open subsets of X with $x_i \in U_i$ for every $i < \omega$. Thus we have $\dim \mathrm{Cl}\, U_i > i$ for every $i < \omega$. Let us set $Y = \bigcup\{\mathrm{Cl}\, U_i : i < \omega\}$. Since $\bigcup\{\mathrm{Cl}\, U_i : i < \omega\}$ is homeomorphic to $\bigoplus\{\mathrm{Cl}\, U_i : i < \omega\}$, Y is not a S-weakly infinite dimensional subspace of X. Thus Y is not a finite C-space. The contradiction shows that $S(X)$ is compact. Let F be a closed subset of X disjoint from $S(X)$. First, we shall show that $F \subset G_n(X)$ for some $n < \omega$. Suppose that for every $n < \omega$, $F \not\subset G_n(X)$. Since $F \setminus G_n(X)$ is infinite for every $n < \omega$, inductively, we choose points $x_1, x_2, \cdots$ such that $x_n \in F \setminus (G_n(X) \cup \{x_1, x_2, \cdots, x_{n-1}\})$ for every $n < \omega$. The space $E = \{x_n : n < \omega\}$ is a closed discrete subspace of F. For a space $E = \{x_n : n < \omega\}$ repeating above procedure we obtain a contradiction. Thus $F \subset G_n(X)$ for some $n < \omega$. Since X is weakly paracompact, by the point finite sum theorem, $\dim F \leq n$. By Lemma B, the converse holds. Lemma 2.7 has been proved.

2.8 Proof of Theorem 2.5. By Lemma 2.7, $S(X)$ is compact. Applying Theorem 2.6 to $f|_{S(X)}$, $f(S(X))$ is a finite C-space. For each closed subspace $F \subset Y$ disjoint from $f(S(X))$, as $f^{-1}(F) \cap S(X) = \emptyset$, by Lemma 2.7, we take an integer n with $\dim f^{-1}(F) \leq n$. As the restriction $f|_{f^{-1}(F)} : f^{-1}(F) \longrightarrow F$ is open, by Nagami [7] (cf.

[3, 3.3.G]), $\dim F = \dim f^{-1}(F) \leq n$. Thus F is a finite C-space. By Lemma B, Y is a finite C-space.

3 Dimension-lowering mappings

The following theorem is a counterpart for C-spaces of Polkowski's result, which was proved in the case when A-weakly infinite-dimensional (see [8, Theorem 3.3 (ii)]).

3.1. Theorem *If $f : X \longrightarrow Y$ is an open mapping of a paracompact space X onto a C-space Y such that $|f^{-1}(y)| < \omega$ for every $y \in Y$, then X is a C-space.*

To prove Theorem 3.1 we need the following lemma.

3.2. Lemma([7], cf [8, Lemma B]) *If $f : X \longrightarrow Y$ is an open mapping of a space X to a space Y and there exists an integer $n \geq 1$ such that $|f^{-1}(y)| = n$ for every $y \in Y$, then f is a local homeomorphism.*

3.3 Proof of Theorem 3.1. For every $n \in \mathbb{N}$ we set

$$Y_n = \{y \in Y : |f^{-1}(y)| = n\} \text{ and } X_n = f^{-1}(Y_n).$$

It is easy to see that the union $Y'_n = \bigcup_{k \leq n} Y_k$ is closed in Y for every $n \in \mathbb{N}$, therefore the union $X'_n = \bigcup_{k \leq n} X_k$ is also closed in X. Since X is the union of countable collection $\{X'_n : n \in \mathbb{N}\}$ of closed subsets of X, by the countable sum theorem for C-spaces, we only prove that X'_n is a C-space for every $n \in \mathbb{N}$. Let $f_n : X_n \longrightarrow Y_n$ be the mapping defined by $f_n(x) = f(x)$ for every $x \in X_n$.

Obviously, X'_1 is a C-space, because f_1 is a homeomorphism. Assume that X'_{n-1} is a C-space. To prove that X'_n is a C-space, it suffices to show that every closed subset Z of X'_n contained in X_n is a C-space.

By Lemma 3.2, the mapping f_n is a local homeomorphism. Thus for every $x \in X_n$ we can take a neighborhood U_x of x in X_n such that the restriction $f_n|_{U_x} : U_x \longrightarrow Y_n$ is an embedding. Since X_n is open in X'_n, U_x is open in X'_n. We may assume that U_x is an F_σ-set of X'_n. Let $U_x = \cup\{A(x, m) : m \in \mathbb{N}\}$, where $A(x, m)$ is closed in X'_n. For every $y \in Y_n$ let us set $f^{-1}(y) = \{x(y,1), x(y,2), \dots, x(y,n)\}$. Then the intersection $\bigcap_{i=1}^{n} f(U_{x(y,i)})$ is a neighborhood of y in Y_n'. Take an open F_σ-set V_y of y in Y_n' such that $y \in V_y \subset \bigcap_{i=1}^{n} f(U_{x(y,i)})$. Let $V_y = \cup\{B(y,\ell) : \ell \in \mathbb{N}\}$, where $B(y,\ell)$ is closed in Y_n'. The set $W(y,i) = U_{x(y,i)} \cap f^{-1}(V_y)$ is homeomorphic to $f(W(y,i))$. We have

$$W(y,i) = \bigcup\{A(x(y,i),m) \cap f^{-1}(B(y,\ell)) : m, \ell \in \mathbb{N}\}.$$

We shall prove that $A(x(y,i),m) \cap f^{-1}(B(y,\ell))$ is a C-space. Since $f_n|_{U_{x(y,i)}}$ is an embedding, $A(x(y,i),m) \cap f^{-1}(B(y,\ell))$ is homeomorphic to $f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell)))$.

By Lemma 2.3, f_n is closed, therefore $f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell)))$ is closed in Y_n. Since $f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell))) \subset B(y,\ell) \subset Y_n$, $f_n(A(x(y,i),m) \cap f^{-1}(B(y,\ell)))$ is closed in $B(y,\ell)$. As $B(y,\ell)$ is a C-space, $f_n(A(x(y,i),m) \cap f^{-1}(b(y,\ell)))$ is a C-space. Thus $A(x(y,i),m) \cap f^{-1}(b(y,\ell))$ is a C-space. By the countable sum theorem for C-spaces, $W(y,i)$ is a C-space. Since Z is paracompact, the open cover $\mathcal{W} = \{W(y,i) \cap Z : y \in Y_n, 1 \le i \le n\}$ of Z has a locally-finite closed refinement $\mathcal{F}$. Since every member of $\mathcal{F}$ is a C-space, by the locally finite sum theorem for C-spaces (cf. [6, Theorem 1.1(i)]), Z is a C-space. Theorem 3.1 has been proved.

3.4. Theorem *If $f : X \longrightarrow Y$ is a closed-and-open mapping of a space X onto a weakly paracompact finite C-space Y such that $|f^{-1}(y)| < \omega$ for every $y \in Y$, then X is a finite C-space.*

Proof. Since for every $y \in Y$ $|f^{-1}(y)| < \omega$, the closed mapping $f : X \longrightarrow Y$ is perfect. As $S(Y)$ is compact, $f^{-1}(S(Y))$ is compact. By Theorem 3.1, $f^{-1}(S(Y))$ is a finite C-space. For each closed subset $F \subset X$ disjoint from $f^{-1}(S(Y))$, as $f(F) \cap S(Y) = \emptyset$, by Lemmma 2.7, we take integer n with $\dim f(F) \le n$. As $f|_F : F \longrightarrow f(F)$ is closed, by [3, Theorem 3.3.10], $\dim F \le \dim f(F) \le n$. Thus F is a finite C-space, by Lemma B, X is a finite C-space.

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